Odd Chen Distribution: Proposition, Estimations and Applications

ABSTRACT

In this paper, a new statistical distribution with three parameters called the Odd-Chen Exponential is introduced. The density shows different shapes, making it more flexible for analysing different forms of data. The hazard function also exhibits different shapes, including the well-known bathtub shape, which means that the distribution is flexible with real-life data. The created distribution's statistical properties, such as the quantile, moments, incomplete moments, moment-generating function, and mean residual life, were developed. To estimate the distribution parameters, ordinary least squares estimators, Cramér-von Mises estimators, and maximum likelihood estimators are derived. Their results are compared using a Monte Carlo simulation. Two-time datasets; one from the mining field and the other from survival analysis, were used to check the applicability of the proposed distribution.The results revealed that the OCE distribution performed better than the Odd Chen Weibull, Odd Chen Rayleigh, Rayleigh, Cauchy, Generalised Inverse Weibull and the Modified Extended Chen distributions.

Keywords: Odd Chen Exponential, Odd Chen Generator, Exponential, Hazard Function

1. INTRODUCTION

The concept of statistical distribution is widely used in many disciplines. It is used in the area of computer science, engineering, actuarial science, social science and health sciences due to its importance when it comes to modelling datasets generated from some random experiment [1]. But the problem researchers cannot escape is the fact that most of the existing distributions are not best suited for some existing and new datasets [2]. For this reason, there is the need to continuously research into old distributions and even create new ones with the aim of improving the goodness-of-fit of the existing distribution and as well as deducing more powerful distributions to help model datasets. As a result, several studies have developed a more flexible distributions. See [3, 4, 5, 6, 7, 8, and 9].

The Odd Chen G distribution (OCG) as defined by [10] is a generator distribution with only two parameters which happens to be shape parameters. The absence of a scale parameter in the OCG distribution makes it difficult to control the spread and the variability of the distribution. To prevent this deficiency, a modification of the OCG with a distribution that has a scale parameter will do.

There has been several modifications of the Odd Chen G distribution [10, 11, 13]. For instance [12] proposed a new family of distribution called the Odd Chen-G family of distribution using the T-X approach. [11], also proposed an exponentiated Odd Chen distribution capable of modellingcharacteristics of the data sets such as skewness, symmetric, kurtosis and various shapes of failure rate (increasing – decreasing – J – inverse) . But some of thesewell-known models are insufficient to model these kinds of data sets.

The exponential distribution happens to be one of the simple distributions that has a wide usage and also has the scale parameter present once the rate parameter is further parameterized. Modifying the OCG with such a distribution will provide control over the variability and also generate a new member of the Odd Chen family of distributions. Therefore, this article seeks to propose a new distribution namely, the Odd Chen Exponential, (OCE). The properties, estimations, and applications of the OCE will be covered as well.

2.FORMULATING THE ODD CHEN EXPONENTIAL DISTRIBUTION

2.1 Odd Chen G Distribution

The Odd Chen G distribution can be obtained by performing integration and differentiation on the Chen distribution. The cumulative density function (cdf) (denoted by F(t)) for Chen distribution is given by $F(t)=1-\exp\left(\lambda(1-e^{t^{\beta}})\right), t>0$ [7]. Suppose $G(x;\psi)$ is the baseline cdf of an arbitrary continuous random variable X on any continuous support say $(-\infty,\infty)$ and ψ is a (p x 1) vector of associated parameters, the cdf of the OC family of distributions is defined as

$$F(x)=\int_{0}^{\frac{G(x;\psi)}{1-G(x;\psi)}} f(t)\,dt=1-\exp\left(\lambda\left(1-e^{\frac{G(x;\psi)}{1-G(x;\psi)}}\right)\right), x>0, \lambda>0, \beta>0 \qquad (1)$$

where λ and β are extra shape parameters [10].

By differentiating the cdf in (1), the probability density function (pdf) of equation (1)is obtained as

$$f(x)=\lambda\beta g\left(x;\psi\right)G\left(x;\psi\right)^{\beta-1}\left[1-G\left(x;\psi\right)\right]^{-(\beta-1)}\exp\left(\frac{G\left(x;\psi\right)}{1-G\left(x;\psi\right)}\right)^{\beta}$$
$$\times\exp\left(\lambda\left(1-\exp\left(\frac{G\left(x;\psi\right)}{1-G\left(x;\psi\right)}\right)^{\beta}\right)\right),x>0. \tag{2}$$

The exponential distribution with location parameter, α has its cdf defined as;

$$M(x)=1-\exp\left[-\left(\frac{x}{\alpha}\right)\right] \tag{3}$$

The pdf of the exponential distribution;

$$m(x)=\frac{1}{\alpha}\exp\left[-\left(\frac{x}{\alpha}\right)\right] \tag{4}$$

Now, substituting equation (3) into equation (1) gives the cdf of the Odd Chen-Exponential distribution, thus;

$$F(x)=1-\exp\left[\lambda-\lambda\exp\left[e^{x/\alpha}-1\right]^{\beta}\right] \tag{5}$$

Substituting equations (4) and (3) into (2) gives the pdf for the Odd Chen-Exponential (OCE) distribution, thus;

$$f(x)=\frac{\lambda\beta}{\alpha}e^{\left(-x/\alpha\right)}\left(e^{\left(-x/\alpha\right)}\right)^{-(\beta+1)}\left(1-e^{\left(-x/\alpha\right)}\right)^{(\beta-1)}\phi_{(x,\alpha)} \tag{6}$$

where,

$$\phi_{(x,\alpha)}=\exp\left(\left(e^{x/\alpha}-1\right)^{\beta}\right)\exp\left(\lambda\left(1-\exp\left(\left(e^{x/\alpha}-1\right)^{\beta}\right)\right)\right) \tag{7}$$

Survival and failure rate functions are critical in reliability analysis and other fields. The survival (or reliability) function expresses the likelihood of successfully completing a specified task under specified conditions over a specified time period. Thus, reliability can be used to assess a system's ability to perform its function correctly. S(x) is the mathematical expression for the survival function and is given by;

$$S(x) = 1 - F(x) \quad (8)$$

Substituting the CDF of the OCG distribution into equation (8) yields the survival function of the distribution.

Thus,

$$S(x) = \exp\left[\lambda\left(1 - e^{\left(e^{x/\alpha} - 1\right)^{\beta}}\right)\right] \quad (9)$$

The hazard function, is a primary tool in the field of data analysis because the nature of the hazard could determine the type of distribution that could be used in modelling data sets. The failure rate or the hazard function, $h(x)$, is the instantaneousfailure rate and is mathematically expressed as;

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \quad (10)$$

Substituting the survival function, S(x) and the pdf of the OCE distribution into (10) gives the hazard function. Thus,

$$h(x) = \frac{\frac{\lambda\beta}{\alpha} e^{\left(-x/\alpha\right)}\left(e^{\left(-x/\alpha\right)}\right)^{-(\beta+1)}\left(1 - e^{\left(-x/\alpha\right)}\right)^{(\beta-1)} \exp\left(\left(e^{x/\alpha} - 1\right)^{\beta}\right)\exp\left(\lambda\left(1 - \exp\left(\left(e^{x/\alpha} - 1\right)^{\beta}\right)\right)\right)}{\exp\left[\lambda\left(1 - e^{\left(e^{x/\alpha} - 1\right)^{\beta}}\right)\right]}$$

Therefore, the hazard function is given by;

$$h(x) = \frac{\lambda\beta}{\alpha} e^{\left(-x/\alpha\right)}\left(e^{\left(-x/\alpha\right)}\right)^{-(\beta+1)}\left(1 - e^{\left(-x/\alpha\right)}\right)^{(\beta-1)} \exp\left(\left(e^{x/\alpha} - 1\right)^{\beta}\right) \quad (11)$$

2.2 Mixture Representation

The mixture representation of the pdf is essential in the derivation of the statistical properties of the OCE family of distributions. Analternative form for the OCE distribution's pdf is provided in this section. Through the concept of power series, the density function of the OCE distribution is expressed in a mixture form to obtain certain statistical features of the distribution.

Lemma 1. The density function of the OCE distribution can be expressed in a series representation as

$$f\left(x\right)=\frac{\lambda\beta}{\alpha}e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm}x^{m}e^{-kx/\alpha} \tag{12}$$

Where,

$$\omega_{ijkm}=\binom{\beta\left(j+1\right)-1}{k}\frac{\left(-1\right)^{i+k}\lambda^{i}}{i!}.\frac{\left(i+1\right)^{j}}{r!}.\frac{\beta^{m}\left(j+1\right)^{m}}{m!\alpha^{m}} \tag{13}$$

Proof. Given the density function

$$f(x)=\frac{\lambda\beta}{\alpha}e^{\left(-x/\alpha\right)}\left(e^{\left(-x/\alpha\right)}\right)^{-(\beta+1)}\left(1-e^{\left(-x/\alpha\right)}\right)^{(\beta-1)}\exp\left(\left(e^{x/\alpha}-1\right)^{\beta}\right)$$
$$\times\exp\left(\lambda\left(1-\exp\left(\left(e^{x/\alpha}-1\right)^{\beta}\right)\right)\right)\quad\alpha>0,\ \beta>0,\ \lambda>0 \tag{14}$$

Applying the Taylor series expansion yields;

$$f(x)=\frac{\lambda\beta}{\alpha}e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(-1\right)^{i}\lambda^{i}}{i!}.\frac{\left(i+1\right)^{j}}{j!}\left(1-e^{\left(-x/\alpha\right)}\right)^{\beta(j+1)-1}e^{\beta(j+1)x/\alpha} \tag{15}$$

Further applying the Taylor's series and binomial expansion yields;

$$f(x)=\frac{\lambda\beta}{\alpha}e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\beta\left(j+1\right)-1}{k}\frac{\left(-1\right)^{i}\lambda^{i}}{i!}.\frac{\left(i+1\right)^{j}}{j!}.\frac{\beta^{m}\left(j+1\right)^{m}}{m!\alpha^{m}}$$
$$\times x^{m}e^{-kx/\alpha}$$

This completes the proof.

2.3 Statistical Properties of the Odd Chen Exponential Distribution

In this section, the structural properties of the OCE distribution such as the quantile function, moments, moment-generating function, incomplete moment, entropy, order statistic, and inequality measures are derived and discussed.

2.3.1 Quantile Function

In probability and statistics, the **quantile function**, associated with a probability distribution of a random variable, specifies the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. The quantile function is the distribution function's inverse. The quantile function can be used to calculate random variable characteristics such as skewness, kurtosis, and median [12].

Proposition 1 The quantile function of the OCE distribution is given by;

$$Q(p)=\alpha\log\left\{1+\left[\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right]^{\frac{1}{\beta}}\right\},\quad 0<p<1 \tag{16}$$

Proof. Suppose the random variable p follows the standard uniform distribution then, $0<p<1$.

$$\text{Let } p=1-\exp\left[\lambda\left(1-\exp\left(e^{x/\alpha}-1\right)^{\beta}\right)\right] \tag{17}$$

$$\exp\left[\lambda\left(1-\exp\left(e^{x/\alpha}-1\right)^{\beta}\right)\right]=1-p$$

$$\lambda\left[1-\exp\left(e^{x/\alpha}-1\right)^{\beta}\right]=\log(1-p)$$

$$\exp\left(e^{x/\alpha}-1\right)^{\beta}=1-\frac{1}{\lambda}\log(1-p)$$

$$\left(e^{x/\alpha}-1\right)^{\beta}=\log\left(1-\frac{1}{\lambda}\log(1-p)\right)$$

$$x=\alpha\log\left\{1+\left[\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right]^{\frac{1}{\beta}}\right\} \tag{18}$$

By observations, we can tell that x is a function of p, hence the quantile function is a function of p. Therefore,

$$Q(p)=\alpha\log\left\{1+\left[\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right]^{\frac{1}{\beta}}\right\},\quad 0<p<1 \tag{19}$$

This completes the proof.

Also, the measures of skewness and kurtosis can be computed based on the quantile measures. The Bowley measure of skewness and the Moors measure of kurtosis are respectively defined as

$$\text{Skewness}=\frac{Q\left(1/4\right)+Q\left(3/4\right)-2Q\left(1/2\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \tag{20}$$

and

$$Kurtosis = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad (21)$$

2.3.2 Moment

The r^{th} moment of the OCE distributed random variable X is given as;

$$\mu'_r = \frac{\lambda\beta}{\alpha} e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} \int_0^{\infty} x^{m+r} e^{-kx/\alpha} dx \quad (22)$$

Proof. By definition, the r^{th} non-central moment is given by;

$$\mu'_r = \int_0^{\infty} x^r f(x) dx \quad (23)$$

Applying the series representation of the pdf, we derive;

$$\mu'_r = \int_0^{\infty} \frac{\lambda\beta}{\alpha} e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} x^{m+r} e^{-kx/\alpha} dx \quad (24)$$

$$\mu'_r = \frac{\lambda\beta}{\alpha} e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} \int_0^{\infty} x^{m+r} e^{-kx/\alpha} dx \quad (25)$$

This therefore completes the proof.

2.3.3 Incomplete Moment

The incomplete moment of the OCE distribution is given as;

$$M_r(x) = \int_0^{x} \frac{\lambda\beta}{\alpha} e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} y^{m+r} e^{-ky/\alpha} dy \quad (26)$$

Where, ω_{ijkm} is the same as it was in equation (13).

Proof. The incomplete moment of a random variable is given as;

$$M_r(x) = \int_0^{x} y^r f(y) dy \quad (27)$$

Thus, by replacing the series representation of the density function of the OCE distribution into the definition of the incomplete moment, we have;

$$M_r(x)=\int_0^x \frac{\lambda\beta}{\alpha}e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm}y^{m+r}e^{-ky/\alpha}dy \quad (28)$$

This therefore completes the proof.

2.3.4 Inequality Measures

Several fields like insurance, econometrics and reliability studies employ Lorenz and Bonferroni curves in the study of inequality measures like income and poverty.

2.3.4.1 Lorenz Curve

The Lorenz Curve is defined as;

$$L_f(x)=\frac{1}{\mu}\int_0^x yf(y)dy \quad (29)$$

Therefore, for the OCE distribution, it is obtained by substituting the mixture representation of the density in equation (13) into the definition of the Lorenz curve. Therefore, the Lorenz Curve for the OCE distribution is given by;

$$L_f(x)=\frac{\lambda\beta e^{\lambda}}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm}\int_0^x y^{m+1}e^{-ky/\alpha}dy \quad (30)$$

2.3.4.2 Bonferroni Curve

The Bonferroni Curve is defined as;

$$B_F(x)=\frac{L_f(x)}{F(x)} \quad (31)$$

hence, for the OCE distribution, it is obtained by substituting the Lorenz curve in equation (31) into the definition of the Bonferroni curve. Therefore, the Bonferroni curve for the OCE distribution is given by;

$$B_F(x)=\frac{\lambda\beta e^{\lambda}}{\mu F(x)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm}\int_0^x y^{m+1}e^{-ky/\alpha}dy \quad (32)$$

2.3.5 Entropy

The Renyi entropy of the OCE distribution is given by;

$$I_R(\psi)=\frac{1}{(1-\psi)}\log\left\{\int_0^{\infty}\left(\frac{\lambda\beta e^{\lambda}}{\alpha}\right)^{\psi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\phi_{ijkm}x^{m}e^{-kx/\alpha}dx\right\},\ \psi\neq 1,\ \psi>0 \quad (33)$$

Proof: Renyi entropy, which is denoted by $I_R(\psi)$ for the random variable, X is defined as;

$$I_R(\psi) = \frac{1}{(1-\psi)} \log\left\{\int_0^\infty f(x)^\psi dx\right\},\ \psi > 0 \tag{34}$$

Using the density function, f(x) as it is in equation (6),

$$f(x)^\psi = \left[\frac{\lambda\beta}{\alpha} e^{\left(-x/\alpha\right)} \left(e^{\left(-x/\alpha\right)}\right)^{-(\beta+1)} \left(1-e^{\left(-x/\alpha\right)}\right)^{(\beta-1)} \exp\left(\left(e^{x/\alpha}-1\right)^\beta\right) \exp\left(\lambda\left(1-\exp\left(\left(e^{x/\alpha}-1\right)^\beta\right)\right)\right)\right]^\psi$$

By some algebraic simplification;

$$f(x)^\psi = \left(\frac{\lambda\beta e^\lambda}{\alpha}\right)^\psi \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \phi_{ijkm} x^m e^{-kx/\alpha}$$

where,

$$\phi_{ijkm} = \binom{\beta(j+1)-\psi}{k} \frac{(-1)^{i+k} \lambda^i \psi^i}{i!} \cdot \frac{(i+\psi)^j}{r!} \cdot \frac{\beta^m (j+\psi)^m}{m!\alpha^m}$$

Hence, the Renyi entropy is becomes;

$$I_R(\psi) = \frac{1}{(1-\psi)} \log\left\{\int_0^\infty \left(\frac{\lambda\beta e^\lambda}{\alpha}\right)^\psi \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \phi_{ijkm} x^m e^{-kx/\alpha} dx\right\} \tag{35}$$

This therefore completes the proof.

2.4 The Moment Generating Function

The moment generating function (MGF) is a special function used to compute statistical measures such as the mean and variance of a given random variable. In this sub-section the moment generating function of the OCE distribution is derived. The MGF of a random variable X having the OCE distribution is given by;

$$M_x(t) = \lambda\beta e^\lambda \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \sum_{r=0}^\infty \omega_{ijkm} \frac{t^r}{r!} \int_0^\infty x^{m+r} e^{-kx/\alpha} dx \tag{36}$$

Proof. The MGF is defined as;

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x) dx \tag{37}$$

Using Taylor series expression, the MGF can be rewritten as;

$$M_x(t)=\int_0^\infty \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} f(x)\,dx \quad (38)$$

$$M_x(t)=\sum_{r=0}^{\infty} \frac{t^r x^r}{r!} \mu'_r \quad (39)$$

Substituting the r^{th} moment of the OCE distribution yields;

$$M_x(t)=\lambda\beta e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\omega_{ijkm}\frac{t^r}{r!}\int_0^\infty x^{m+r}e^{-kx/\alpha}dx \quad (40)$$

This therefore completes the proof.

2.5 Characteristic Function

Characteristic functions are particularly useful in handling heavy-tailed random variables for which the corresponding moment-generating functions do not exist. The characteristic function of the OCE distribution for a random variable X is given by;

$$C_x(t)=\lambda\beta e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r\,\omega_{ijkm}}{r!}\int_0^\infty x^{m+r}e^{-kx/\alpha}dx \quad (41)$$

Proof. The characteristic function of X is defined as;

$$C_x(t)=E\left(e^{itx}\right)=\int_0^\infty e^{itx} f(x)\,dx, \text{ where, } i=\sqrt{-1}$$

Using Taylor's expansion, the characteristic function can be rewritten as;

$$C_x(t)=\int_0^\infty \sum_{r=0}^{\infty}\frac{(it)^r x^r}{r!} f(x)\,dx \quad (42)$$

$$C_x(t)=\sum_{r=0}^{\infty}\frac{(it)^r x^r}{r!}\mu'_r \quad (43)$$

Substitute the r^{th} moment of the OCE distribution yields;

$$C_x(t)=\lambda\beta e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r\,\omega_{ijkm}}{r!}\int_0^\infty x^{m+r}e^{-kx/\alpha}dx \quad (44)$$

This completes the proof.

2.6 Order Statistics

The most common application of order statistics is in statistical theory. Order statistics is concerned with the properties and applications of these ordered random variables, as well as the functions that incorporate them. The order statistics for the OCE distribution is given by;

$$f_{p:n} = \lambda\beta \sum_{i=0}^{p-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} D_{ijklm} \frac{n!}{(p-1)!(n-p)!} x^m e^{-lx/\alpha} \qquad (45)$$

Where,

$$D_{ijklm} = \binom{\beta(k+1)-1}{l} \binom{p-1}{i} \frac{(-1)^{i+j+l} \left[\lambda(n-p+i+1)\right]^j (j+1)^k \beta^m (k+1)^m}{j!k!m!\alpha^{m+1}} e^{\lambda(n-p+i+1)} \qquad (46)$$

Proof. The pdf of the p^{th} order statistics is given by;

$$f_{p:n} = U_{r:n} \left[F(x)\right]^{p-1} \left[1-F(x)\right]^{n-p} f(x), \quad r = 1,2,3,...,n \quad ....(47)$$

where

$$U_{r:n} = \frac{n!}{(p-1)!(n-p)!} \quad (48)$$

By further simplification, equation (47) becomes;

$$f_{p:n} = U_{r:n} \left[F(x)\right]^{p-1} \left[S(x)\right]^{n-p} f(x) \qquad (49)$$

$$f_{p:n} = U_{r:n} \left[1-\exp\left[\lambda - \lambda \exp\left[e^{x/\alpha} - 1\right]^{\beta}\right]\right]^{p-1} \left[\exp\left[\lambda - \lambda \exp\left[e^{x/\alpha} - 1\right]^{\beta}\right]\right]^{n-p} f(x) \qquad (50)$$

Applying the binomial expansion to the above yields;

$$f_{p:n} = U_{r:n} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i \left[\exp\left[\lambda - \lambda \exp\left[e^{x/\alpha} - 1\right]^{\beta}\right]\right]^{n-p+i} f(x) \ ... \ (51)$$

$$f_{p:n} = U_{r:n} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i \left[S(x)\right]^{n-p+i} f(x) \ ... \ (52)$$

$$\left[S(x)\right]^{n-p+i} f(x) = \left[\exp\left[\lambda - \lambda \exp\left[e^{x/\alpha} - 1\right]^{\beta}\right]\right]^{n-p+i} \left[\frac{\lambda\beta}{\alpha} e^{(-x/\alpha)} \left(e^{(-x/\alpha)}\right)^{-(\beta+1)} \left(1-e^{(-x/\alpha)}\right)^{(\beta-1)} \phi_{(x,\alpha)}\right] \qquad (53)$$

Applying Taylor's and binomial expansions, we obtain;

$$\left[S(x)\right]^{n-p+i} f(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{\beta(k+1)-1}{l} \frac{(-1)^{j+l} \left[\lambda(n-p+i+1)\right]^j (j+1)^k \beta^m (k+1)^m}{e^{-\lambda(n-p+i+1)} j!k!m!\alpha^m} e^{-lx/\alpha} x^m$$

(54)

Substituting equation (54) into equation (52), equation (55) is obtained as;

$$f_{r:n} = \lambda\beta\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\beta(k+1)-1}{l}\binom{p-1}{i}\frac{n!(-1)^{i+j+l}\left[\lambda(n-p+i+1)\right]^{j}(j+1)^{k}\beta^{m}(k+1)^{m}}{e^{-\lambda(n-p+i+1)}(p-1)!(n-p)!\,j!\,k!\,m!\,\alpha^{m+1}}e^{-lx/\alpha}x^{m}$$

(55)

Hence the proof.

2.7 Mean Residual Life

The expected remaining life, $X-x$, given that the item has survived to time x, is the mean residual life (MRL). Thus, in life-testing situations, the MRL is the expected additional lifetime, given that a component has survived until time x. Because the MRL function represents the expected remaining life, x must be subtracted, resulting in;

$$h(x) = E[X-x \mid X>x] = \frac{\int_{x}^{\infty} yf(y)dy}{S(x)} - x \quad (56)$$

$$h(x) = \lambda\beta e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\omega_{ijkm}}{\exp\left[\lambda\left(1-\exp\left(e^{x/\alpha}-1\right)^{\beta}\right)\right]}\int_{x}^{\infty} y^{m+1}e^{-ky/\alpha}dy - x \quad (57)$$

2.8 Method of Estimations

In this section, we compute some estimations for the OCE. We will look at the maximum likelihood estimators, least square estimators, percentile estimators, maximum product of spacings estimators and the minimum distances estimators.

2.8.1 Maximum Likelihood Estimators

$$L(x;\alpha,\beta,\lambda) = \prod_{i=1}^{n} f(x) \quad (58)$$

$$L(x;\alpha,\beta,\lambda) = \left(\frac{\lambda\beta}{\alpha}\right)^{n} e^{\sum_{i=1}^{\infty}\beta x_i/\alpha}\sum_{i=1}^{n}\left(1-e^{-x_i/\alpha}\right)^{\beta-1} e^{\sum_{i=1}^{n}\left(e^{x_i/\alpha}-1\right)^{\beta}} \exp\left(\lambda\sum_{i=1}^{n}\left(1-\exp\left(e^{x_i/\alpha}-1\right)^{\beta}\right)\right)$$

(59)

$$l = \log\ L(x;\alpha,\beta,\lambda) = n\log\left(\frac{\lambda\beta}{\alpha}\right) + \sum_{i=1}^{\infty}\frac{\beta x_i}{\alpha} + (\beta-1)\sum_{i=1}^{n}\log\left(1-e^{-x_i/\alpha}\right) + \lambda\sum_{i=1}^{n}\left(1-\exp\left(e^{x_i/\alpha}-1\right)^{\beta}\right)$$

(60)

For ease of notation, the first partial derivative of equation (60) with respect to α, β, λ is denoted by $ml_\alpha, ml_\beta, ml_\lambda$ respectively. Now, equating the first partial derivatives to zero results in equation (61);

$$ml_\alpha = -\frac{n}{\alpha} - \sum_{i=1}^{n} \frac{\beta x_i}{\alpha^2} - (\beta - 1)\sum_{i=1}^{n} \frac{xe^{-x_i/\alpha}}{\alpha^2\left(1 - e^{-x_i/\alpha}\right)} + \lambda \sum_{i=1}^{n} \left(\frac{\left(e^{x_i/\alpha} - 1\right)^{\beta} \beta x e^{x_i/\alpha} \exp\left(\left(e^{x_i/\alpha} - 1\right)^{\beta}\right)}{\alpha^2\left(e^{x_i/\alpha} - 1\right)} \right) = 0 \quad (61)$$

$$ml_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left(1 - e^{-x_i/\alpha}\right) - \lambda \sum_{i=1}^{n} \left(e^{x_i/\alpha} - 1\right)^{\beta} \ln\left(e^{x_i/\alpha} - 1\right) \exp\left(\left(e^{x_i/\alpha} - 1\right)^{\beta}\right) = 0 \quad (62)$$

$$ml_\lambda = \frac{n}{\lambda} + \sum_{i=1}^{n} \left(1 - \exp\left(e^{x_i/\alpha} - 1\right)^{\beta}\right) = 0 \quad (63)$$

The maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ of α, β and λ can be obtained by solving the above nonlinear system equations.

2.8.2 Least Squares Estimators

The least-square estimators and weighted least-square estimators were proposed by [13] to estimate the parameters of Beta distributions. Suppose $F(x_i)$ denotes the OCE distribution function of the ordered random variables $X_1 < X_2 < ... < X_n$ where $\{X_1, X_2, \ldots, X_n\}$ is a random sample of size *n* from a distribution function $F(x; \alpha, \beta, \lambda)$. The least square estimators $\hat{\alpha}_{lse}, \hat{\beta}_{lse}$ and $\hat{\lambda}_{lse}$ of α, β and λ can be obtained by minimizing

$$LSE = \sum_{i=1}^{n} \left[F\left(x_{i:1} \mid \alpha, \beta, \lambda\right) - \frac{i}{n+1} \right]^2 \quad (64)$$

with respect to α, β and λ and equating them to zero where $F\left(x_{i:1} \mid \alpha, \beta, \lambda\right)$ is the cdf of the OCE. Consequently, they can be obtained from solving the following;

$$\frac{\partial LSE}{\partial \alpha} = -2\sum_{i=1}^{n} \left[1 - \exp\left(\lambda \left(1 - \exp\left(\left(e^{x_{i:n}/\alpha} - 1 \right)^{\beta} \right) \right) \right) - \frac{i}{n+1} \right] \mu_1\left(x_{i:n} \mid \alpha, \beta, \lambda\right) = 0$$

$$\frac{\partial LSE}{\partial \beta} = 2\sum_{i=1}^{n}\left[1-\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)-\frac{i}{n+1}\right]\mu_2\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=0$$

$$\frac{\partial LSE}{\partial \lambda} = -2\sum_{i=1}^{n}\left[1-\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)-\frac{i}{n+1}\right]\mu_3\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=0$$

where,

$$\mu_1\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=\frac{\lambda\beta x\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)}{\alpha^2\left(1-e^{-x_{i:n}/\alpha}\right)}$$

(65)

$$\mu_2\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=\lambda\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\ln\left(e^{x_{i:n}/\alpha}-1\right)\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)$$

(66)

$$\mu_3\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)$$

(67)

The weighted least squares estimators, on the other hand, can be obtained from minimising

$$WLSE=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{i:1}\mid\alpha,\beta,\lambda\right)-\frac{i}{n+1}\right]^2$$

(68)

with respect to α,β and λ. The estimators can be obtained by solving the system of equations in equation (69);

$$\frac{\partial LSE}{\partial \alpha} = -2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)-\frac{i}{n+1}\right]\mu_1\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=0$$

$$\frac{\partial LSE}{\partial \beta} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)-\frac{i}{n+1}\right]\mu_2\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=0$$

$$\frac{\partial LSE}{\partial \lambda} = -2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\exp\left(\lambda\left(1-\exp\left(\left(e^{x_{i:n}/\alpha}-1\right)^{\beta}\right)\right)\right)-\frac{i}{n+1}\right]\mu_3\left(x_{i:n}\mid\alpha,\beta,\lambda\right)=0$$

(69)

2.8.3 Percentile Estimators

If the data come from a closed-form distribution function, we can estimate the unknown parameters by fitting a straight line to the theoretical points from the distribution function and the sample percentile points. [14] proposed this method, which has been used for Weibull distribution and generalized exponential distribution. Let j be the j-th order statistics, that is, $X_1 < X_2 < ... < X_n$. If p_j denotes some estimates of $F\left(x_j;\alpha,\beta,\lambda\right)$ the estimates, $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$ can be obtained by minimizing with respect to α,β and λ and equate them to zero the equation below;

$$PE = \sum_{j=1}^{n}\left(x_j - \alpha\log\left[1+\left(\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right)^{1/\beta}\right]\right)^2 \quad (70)$$

The estimators can be obtained by solving the following system in equation (71);

$$\frac{\partial PE}{\partial \alpha} = -2\sum_{j=1}^{n}\left(x_j - \alpha\log\left[1+\left(\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right)^{1/\beta}\right]\right)^2\Omega_1\left(p;\alpha,\beta,\lambda\right)=0,$$

$$\frac{\partial PE}{\partial \beta} = 2\sum_{j=1}^{n}\left(x_j - \alpha\log\left[1+\left(\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right)^{1/\beta}\right]\right)^2\Omega_2\left(p;\alpha,\beta,\lambda\right)=0$$

and

$$\frac{\partial PE}{\partial \lambda} = -2\sum_{j=1}^{n}\left(x_j - \alpha\log\left[1+\left(\log\left(1-\frac{1}{\lambda}\log(1-p)\right)\right)^{1/\beta}\right]\right)^2\Omega_3\left(p;\alpha,\beta,\lambda\right)=0, \quad (72)$$

where

$$\Omega_1\left(p;\alpha,\beta,\lambda\right)=\ln\left(1+\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)^{1/\beta}\right),$$ and

$$\Omega_2(p;\alpha,\beta,\lambda)=\frac{\alpha\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)^{1/\beta}\ln\left(\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)\right)}{\beta^2\left(1+\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)^{1/\beta}\right)} \quad (73)$$

$$\Omega_3(p;\alpha,\beta,\lambda)=\frac{\alpha\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)^{1/\beta}\ln(1-p)}{\beta\lambda^2\left(1-\frac{\ln(1-p)}{\lambda}\right)\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)\left(1+\ln\left(1-\frac{\ln(1-p)}{\lambda}\right)^{1/\beta}\right)} \quad (74)$$

These estimators are referred to as percentile estimators.

2.9 Maximum Product of Spacings Estimators

[15] proposed the maximum product spacing (MPS) method as an alternative to MLE for estimating the unknown parameters of continuous univariate distributions. [8] independently developed the MPS method as an approximation to the Kullback-Leibler measure of information. [16] demonstrated that this method is as efficient as MLE estimators and consistent under more general conditions, which motivated our choice. Define the uniform spacings of a random sample from the OCE distribution as follows, using the same notation as in the percentile estimator:

$$D_i(\alpha,\beta,\lambda)=F(x_{i:n}\mid\alpha,\beta,\lambda)-F(x_{i-1:n}\mid\alpha,\beta,\lambda),\;\; i=1,2,...,n,$$

where $F(x_{0:n}\mid\alpha,\beta,\lambda)=0$ and $F(x_{n+1:n}\mid\alpha,\beta,\lambda)=1$. Clearly, $\sum_{i=1}^{n+1}D_i(\alpha,\beta,\lambda)=1$.

The maximum product of spacings estimator $\hat{\alpha}_{MPS},\hat{\beta}_{MPS}$ and $\hat{\lambda}_{MPS}$ of the parameters α,β and λ are obtained by maximizing with respect to α,β and λ, the geometric mean of spacings:

$$G(\alpha,\beta,\lambda)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\beta,\lambda)\right]^{\frac{1}{n+1}} \quad (75)$$

or, equivalently, by minimizing the function;

$$H(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\beta,\lambda) \quad (76)$$

The estimators can be obtained by solving the non-linear systemin equation (77);

$$\frac{\partial}{\partial \alpha} H(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}\left[\mu_1(x_{i:n}\,|\,\alpha,\beta,\lambda)-\mu_1(x_{i-1:n}\,|\,\alpha,\beta,\lambda)\right]=0$$
$$\frac{\partial}{\partial \beta} H(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}\left[\mu_2(x_{i:n}\,|\,\alpha,\beta,\lambda)-\mu_2(x_{i-1:n}\,|\,\alpha,\beta,\lambda)\right]=0 \quad (77)$$
$$\frac{\partial}{\partial \beta} H(\alpha,\beta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,\lambda)}\left[\mu_3(x_{i:n}\,|\,\alpha,\beta,\lambda)-\mu_3(x_{i-1:n}\,|\,\alpha,\beta,\lambda)\right]=0$$

2.10 Minimum Distances Estimators

Three minimization-based estimation methods for α,β and λ with respect to α,β and λ of the goodness-of-fit statistics are presented. This statistical class is based on the difference between the cumulative distribution function estimate and the empirical distribution function [10].

2.10.1 Cramer-Von-Mises

[17] provided empirical evidence that the bias of the estimator is smaller than that of the other minimum distance estimators, which motivated our choice of Cramér-von-Mises type minimum distance estimators. Thus, the Cramer-Von-Mises estimators, $\hat{\alpha}_{CVME},\hat{\beta}_{CVME}$ and $\hat{\lambda}_{CVME}$ of the parameters α,β and λ can be obtained by minimizing with respect to α,β and λ, the function;

$$C(\alpha,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:1}\,|\,\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2 \quad (78)$$

These estimators can also be obtained by solving the system of nonlinear equations below;

$$\frac{\partial}{\partial \alpha} C(\alpha,\beta,\lambda)=-2\sum_{i=1}^{n}\left[F(x_{i:1}\,|\,\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2\mu_1(x_{i:n}\,|\,\alpha,\beta,\lambda)=0$$
$$\frac{\partial}{\partial \beta} C(\alpha,\beta,\lambda)=2\sum_{i=1}^{n}\left[F(x_{i:1}\,|\,\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2\mu_2(x_{i:n}\,|\,\alpha,\beta,\lambda)=0$$
$$\frac{\partial}{\partial \lambda} C(\alpha,\beta,\lambda)=-2\sum_{i=1}^{n}\left[F(x_{i:1}\,|\,\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2\mu_3(x_{i:n}\,|\,\alpha,\beta,\lambda)=0$$

(78)

where, $\mu_1(x_{i:n}\,|\,\alpha,\beta,\lambda)$, $\mu_2(x_{i:n}\,|\,\alpha,\beta,\lambda)$ and $\mu_3(x_{i:n}\,|\,\alpha,\beta,\lambda)$ are the same as they have been through this work.

2.10.2 Anderson-Darling and right-tail Anderson-Darling Estimators

The Anderson-Darling test [18] is an alternative to other statistical tests for detecting deviations from normality in sample distributions. The AD test, in particular, converges very quickly toward the asymptote [18,19]. The Anderson-Darling Estimators, $\hat{\alpha}_{ADE}, \hat{\beta}_{ADE}$ and $\hat{\lambda}_{ADE}$ of the parameters α, β and λ can be obtained from minimizing with respect to α, β and λ the function;

$$A(\alpha,\beta,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{i:n} \mid \alpha,\beta,\lambda\right) + \log S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)\right] \tag{79}$$

These estimators can also be obtained by solving the nonlinear equations;

$$\begin{aligned}
&\sum_{i=1}^{n}(2i-1)\left[\frac{\mu_1\left(x_{i:n} \mid \alpha,\beta,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\beta,\lambda\right)} + \frac{\mu_1\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}\right] = 0,\\
&\sum_{i=1}^{n}(2i-1)\left[\frac{\mu_2\left(x_{i:n} \mid \alpha,\beta,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\beta,\lambda\right)} + \frac{\mu_2\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}\right] = 0,\\
&\sum_{i=1}^{n}(2i-1)\left[\frac{\mu_3\left(x_{i:n} \mid \alpha,\beta,\lambda\right)}{F\left(x_{i:n} \mid \alpha,\beta,\lambda\right)} + \frac{\mu_3\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}\right] = 0
\end{aligned} \tag{80}$$

The right-tail Anderson-Darling estimators $\hat{\alpha}_{RTADE}, \hat{\beta}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the parameters α, β and λ are obtained by minimizing, with respect to α, β and λ, the function;

$$R(\alpha,\beta,\lambda) = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{i:n} \mid \alpha,\beta,\lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right) \tag{81}$$

These estimators can also be obtained by solving the nonlinear equations;

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}R(\alpha,\beta,\lambda) = -2\sum_{i=1}^{n}\mu_1\left(x_{i:n} \mid \alpha,\beta,\lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\frac{\mu_1\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)} = 0,\\
&\frac{\partial}{\partial\beta}R(\alpha,\beta,\lambda) = \sum_{i=1}^{n}\mu_2\left(x_{i:n} \mid \alpha,\beta,\lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\frac{\mu_2\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)} = 0,\\
&\frac{\partial}{\partial\lambda}R(\alpha,\beta,\lambda) = -2\sum_{i=1}^{n}\mu_3\left(x_{i:n} \mid \alpha,\beta,\lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\frac{\mu_3\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)}{S\left(x_{n+1-i:n} \mid \alpha,\beta,\lambda\right)} = 0,
\end{aligned} \tag{82}$$

3. SPAPES OF CDF, DENSITY AND HAZARD FUNCTION

Fig. 1 shows the cdf of the OCE distribution for different parameter values. It can be observed that for some parameter values, as *x* gets closer to zero the CDF approaches zero and as *x* gets bigger the CDF approaches one. This satisfies the condition of a CDF, which states that the limit of a CDF as x approaches zero must be zero and the limit of CDF as x approaches infinity must be 1.

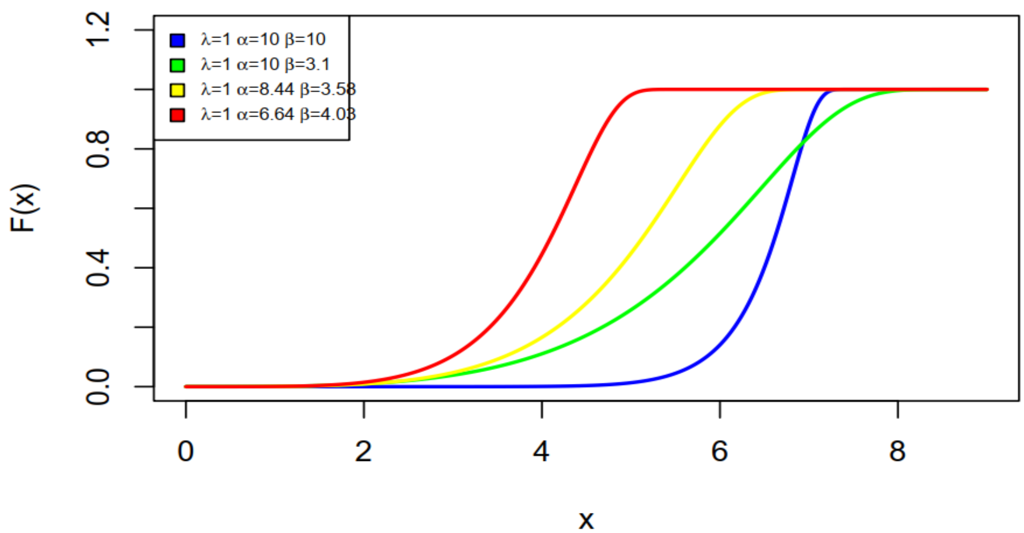

Fig.1 CDF of the OCE Distribution

Fig. 2 also shows the density plots of the OCG distribution. The density plots forvarious parameter values exhibit different kinds of shapes; most of the plots are uni-modalin shape with different degrees of kurtosis. For the various parameter values that wereused, it can be observed that some of the plots exhibit a right skewed shape, whereas a significant number also showed a left skewed shape. In addition,for some parameter values, the density plot exhibits a higher altitude.

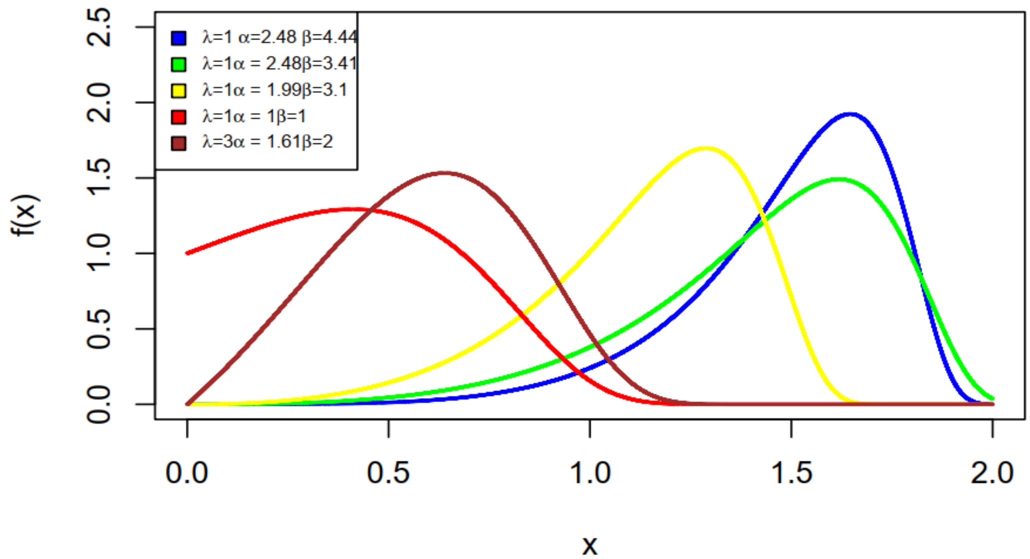

Fig. 2 PDF of the OCE Distribution

Fig. 3 depicts the various shapes of the survival function for the OCE distribution for various parameter values. It shows that the survival function decays very quickly as it approaches one for some parameter values, indicating low reliability. Furthermore, the survival function decays slowly as it approaches one for some parameter values, indicating high reliability.

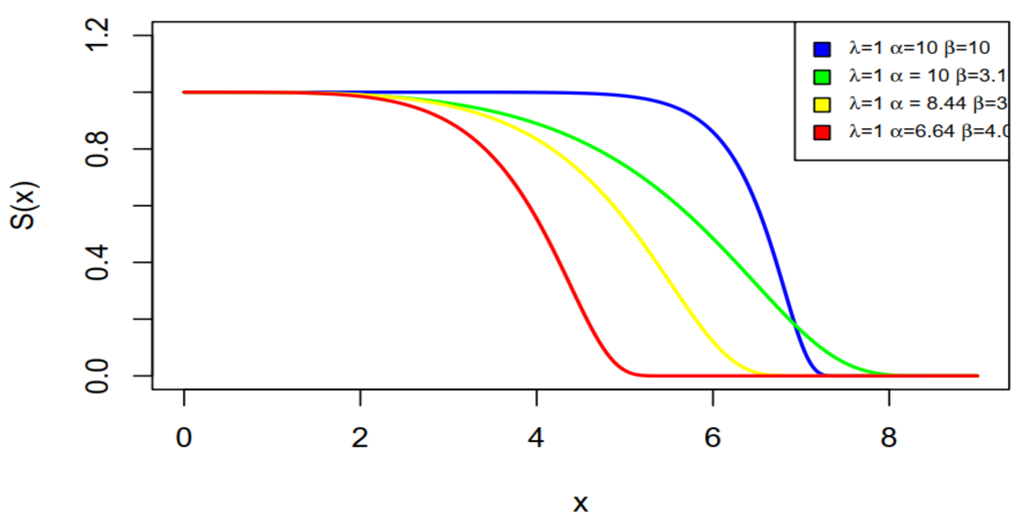

Fig. 3 Survival Function plots for the OCE Distribution

Fig.4 shows the plots of the hazard function of the OCE distribution for differentparameter values. We can observe that the hazard function can be a constant and can also increase exponentially. It can also be a bathtub and also decrease exponentially. Furthermore, for a few selectedparameter values, the hazard plot shows an upside-down bathtub shape, which is anindication that the OCE distribution is appropriate for modeling lifetime data.

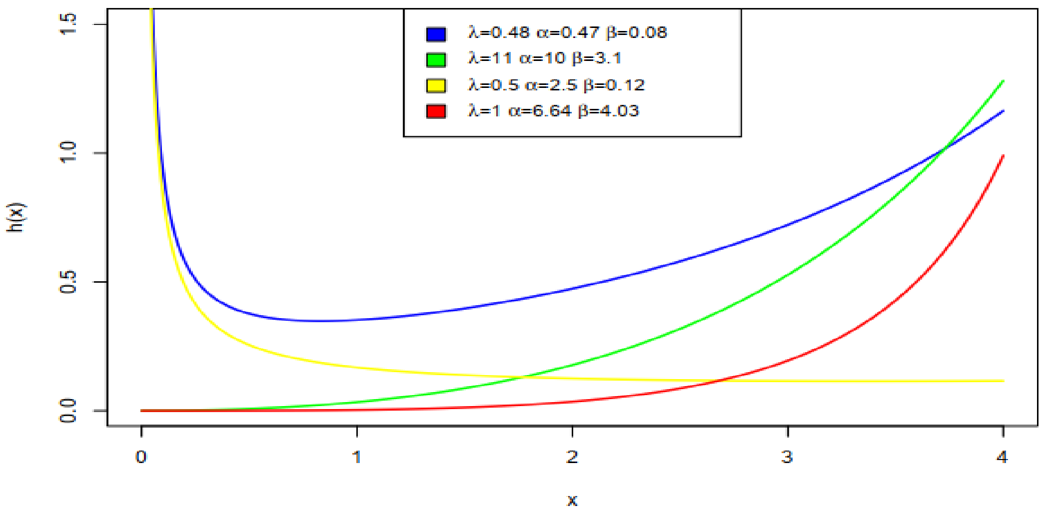

Fig. 4 Hazard Function Diagram for OCG Distribution

3.1 Data Analysis

The OCE was compared to the Odd Chen Weibull (OCW), Odd Chen Rayleigh (OCR), Rayleigh, Cauchy, Generalized Inverse Weibull and the Modified Extended Chen by applying them to real data and observing their AIC and the p-values for their parameter estimates.

Dataset I: This is the unscheduled downtime for milling processing at a mining company. This dataset is in hours and contains 29 data points.

7.6, 12.1, 20.7, 23.8, 53.7, 22.4, 2.91, 54.8, 27.1, 28, 30, 12.2, 28.7, 21.3, 7.1, 7.7, 8.2, 5.8, 14.9, 19.2, 36.3, 74.5, 22.4, 35.5, 13.3, 11.4, 9.1, 29.1, 26.9

Dataset II: Survival times of patients with acute myelogenous leukaemia. This dataset was picked from [14]. It contains 33 data points

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

Table 1: Descriptive statistics of the two above-mentioned datasets

Data	n	min.	Q_1	median	mean	Q_3	max	Skew.	Kurto.	Var	Sd
I	29	2.91	11.40	21.30	22.99	28.70	74.50	1.31	1.60	268	16.4
II	33	1	4.00	22.00	40.88	65.00	156.00	1.11	-0.06	2181	46.7

From Tables 2 and 3, it can be observed that the OCE distribution has the highest log likelihood of -116.3209 and -153.3405. It also has the smallest AIC value of 238.6419 in Table 2 and 312.6810 in Table 2 as compared to the other fitted distributions. The p-values of the OCE in most cases are lower than the other distributions, proving that the OCE can perform better than the other distributions.

Table 2: Comparison criteria for data set I

Distribution	**-L**	**AIC**	**CVM (P-value)**	**AD (P-value)**	**KS(P-value)**
OCE	**-116.3209**	**238.6419**	**0.0499**	**0.3586**	**0.1109**
			(0.8806)	**(0.8877)**	**(0.8683)**
OCW	-116.5975	241.1951	0.0579	0.4151	0.1220
			(0.8310)	(0.8325)	(0.7809)
OCR	-116.3662	238.7323	0.0513	0.3691	0.1132
			(0.8719)	(0.8778)	(0.8517)
Rayleigh	-118.4599	238.9199	0.2321	1.5678	0.1781
			(0.2133)	(0.1613)	(0.3166)
Cauchy	-123.2769	250.5539	0.0995	0.7613	0.1511
			(0.5905)	(0.5085)	(0.522)
GIW	-119.3165	244.6331	0.1304	0.8139	0.1566
			(0.4577)	(0.4700)	(0.476)
MEC	-116.9755	239.9510	0.0800	0.4572	0.1351
			(0.6958)	(0.7893)	(0.6651)

Table 3: Comparison Criteria for dataset II

Distribution	-L	AIC	CVM (P-Value)	AD (P-value)	KS (P-value)
OCE	**-153.3405**	**312.6810**	**0.0932**	**0.6345**	**0.1344**
			(0.6221)	**(0.6148)**	**(0.5907)**
OCW	-153.3508	314.7016	0.0932	0.6351	0.1342
			(0.6219)	(0.6142)	(0.5921)
OCR	-153.4761	312.9522	0.0946	0.6417	0.1354
			(0.615)	(0.6082)	(0.5804)
Rayleigh	-188.6356	379.2713	1.9215	23.5960	0.4255
			(1.14e-05)	(1.818e-05)	(1.296e-05)
Cauchy	-172.8878	349.7756	0.6688	4.2108	0.2931
			(0.0146)	(0.0070)	(0.0069)
GIW	-155.9985	317.9971	0.1444	0.8935	0.1490
			(0.4088)	(0.4174)	(0.4561)
MEC	-154.3179	314.6358	0.0908	0.6242	0.1326
			(0.6348)	(0.6242)	(0.6078)

4 Conclusion

In this study, a new continuous probability distribution with a flexible hazard rate is introduced and discussed for the Oden Chen Distribution. Its properties were investigated accordingly. The parameter estimates were carried out with the use of the maximum likelihood method. Two real life data were fitted with this distribution, and the results revealed that the OCE distribution performed better than the Odd Chen Weibull, Odd Chen Rayleigh, Rayleigh, Cauchy, Generalised Inverse Weibull and the Modified Extended Chen distributions.

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