Original Research Article

A Study on the Norms of Toeplitz Matrices with the Generalized Mersenne Numbers

Abstract. In this article, we present results on Toeplitz matrices with Mersenne numbers. First, the Toeplitz matrices whose elements are the Mersenne numbers are created and then the euclidian, row and column norms of these matrices are found. Furthermore lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius (Euclidian) and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Mersenne numbers are calculated.

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1. Introduction

Numerous scholarly articles have been published that delve into the fascinating realm of special matrices, particularly focusing on Toeplitz matrices, each with distinct number sequences such as Fibonacci, Lucas, Pell, k-Fibonacci, k-Lucas, Jacobsthal, Jacobsthal-Lucas, and modified Pell numbers. Several notable contributors have significantly advanced in this field such as Solak [5], Akbulak and Bozkurt [1], Shen [4], Karpuz [3], Uygun [8] and Uygun [9]. Daşdemir [2] also made significant contributions to this field by investigating special norms of Toeplitz matrices, including those involving Pell, Pell-Lucas, and modified Pell numbers. Daşdemir and their colleagues further enriched the field by deriving comprehensive lower and upper bounds for the spectral norm.

The collective efforts of these researchers have significantly expanded our knowledge of special matrices, shedding light on their intricate properties and opening avenues for further exploration in this captivating field. Before giving some special norms of Toeplitz matrices with Mersenne numbers, we present information on generalized Mersenne sequence and its special cases.

A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.1) holds for all integer n.

The first few generalized Mersenne numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Mersenne numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{3}{2}W_0 - \frac{1}{2}W_1$
2	$3W_1 - 2W_0$	$\frac{7}{4}W_0 - \frac{3}{4}W_1$
3	$7W_1 - 6W_0$	$\frac{15}{8}W_0 - \frac{7}{8}W_1$
4	$15W_1 - 14W_0$	$\frac{31}{16}W_0 - \frac{15}{16}W_1$
5	$31W_1 - 30W_0$	$\frac{63}{32}W_0 - \frac{31}{32}W_1$
6	$63W_1 - 62W_0$	$\frac{127}{64}W_0 - \frac{63}{64}W_1$
7	$127W_1 - 126W_0$	$\frac{255}{128}W_0 - \frac{127}{128}W_1$
8	$255W_1 - 254W_0$	$\frac{511}{256}W_0 - \frac{255}{256}W_1$
9	$511W_1 - 510W_0$	$\frac{1023}{512}W_0 - \frac{511}{512}W_1$
10	$1023W_1 - 1022W_0$	$\frac{2047}{1024}W_0 - \frac{1023}{1024}W_1$
11	$2047W_1 - 2046W_0$	$\frac{4095}{2048}W_0 - \frac{2047}{2048}W_1$
12	$4095W_1 - 4094W_0$	$\frac{8191}{4096}W_0 - \frac{4095}{4096}W_1$

For more information on generalized Mersenne numbers, see for example, Soykan [6].

Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$M_n = 3M_{n-1} - 2M_{n-2} \qquad M_0 = 0, M_1 = 1, \tag{1.2}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \qquad H_0 = 2, H_1 = 3. \tag{1.3}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)},$$

$$H_{-n} = \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}.$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence equation (1.2), equation (1.3) hold for all integer n.

Next, we present the first few values of the Mersenne and Mersenne-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
M_n	0	1	3	7	15	31	63	127	255	511	1023	2047	4095
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$	$-\frac{2047}{2048}$	$-\frac{4095}{4096}$
H_n	2	3	5	9	17	33	65	129	257	513	1025	2049	4097
H_{-n}	2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$	$\frac{4097}{4096}$

Characteristic equation of generalized Mersenne sequence $\{W_n\}_{n\geq 0}$ is given as the quadratic equation

$$x^2 - 3x + 2 = 0,$$

whose roots are α, β and

$$\begin{aligned} \alpha &= 2 \\ \beta &= 1. \end{aligned}$$

Binet's formula of Generalized Mersenne sequence is given as

$$\begin{aligned} W_n &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \\ &= (W_1 - W_0)2^n - (W_1 - 2W_0). \end{aligned}$$

Binet's formulas of Mersenne and Mersenne-Lucas are

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1 \end{aligned}$$

and Binet's formulas of Mersenne numbers and Mersenne-Lucas at the negative index are

$$\begin{aligned} M_{-n} &= \frac{1}{\alpha^n} - \frac{1}{\beta^n} = \frac{-2^n + 1}{2^n}, \\ H_{-n} &= \frac{1}{\alpha^n} + \frac{1}{\beta^n} = \frac{2^n + 1}{2^n}. \end{aligned}$$

2. Preliminaries

A matrix $T = [t_{ij}] \in M_n(\mathbb{C})$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{pmatrix}.$$

Now, we give some preliminaries related to our study. Let $A = (a_{ij})$ be an $m \times n$ matrix. The ℓ_p norm of the matrix A is defined by

$$\|A\|_p = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \qquad (1 \le p < \infty).$$

If $p = \infty$, then $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p = \max_{i,j} |a_{ij}|$.

The well-known Frobenius (Euclidean) and spectral norms of the matrix A are defined respectively by

$$\|A\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

and

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} |\lambda_i|} \tag{2.1}$$

where the numbers λ_i are the eigenvalues of matrix $A^H A$ and the matrix A^H is the conjugate transpose of the matrix A. The following inequality between the Frobenius and spectral norms of A holds.

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F . \tag{2.2}$$

It follows that

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\, \|A\|_2 .$$

In literature, there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \tag{2.3}$$

and the maximum row sum matrix norm is

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| . \tag{2.4}$$

The maximum column lenght norm $c_1(.)$ and maximum row lenght norm $r_1(.)$ of on matrix of order $m \times n$ are defined as follows

$$c_1(A) \equiv \max_{1\le j\le n} \left(\sum_{i=1}^{m} |a_{ij}|^2\right)^{\frac{1}{2}} = \max_{1\le j\le n} \left\|[a_{ij}]_{i=1}^{m}\right\|_F \tag{2.5}$$

and

$$r_1(A) \equiv \max_{1\le i\le m}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} = \max_{1\le i\le m}\left\|[a_{ij}]_{j=1}^{n}\right\|_F \tag{2.6}$$

respectively.

For any $A, B \in M_{mn}(\mathbb{C})$, the Hadamard product of $A=(a_{ij})$ and $B=(b_{ij})$ is entrywise product and defined by $A \circ B = (a_{ij}b_{ij})$ and have the following properties

$$\|A \circ B\|_2 \le r_1(A)\, c_1(B), \tag{2.7}$$

and

$$\|A \circ B\|_2 \le \|A\|_2 \|B\|_2. \tag{2.8}$$

In addition,

$$\|A \circ B\|_F \le \|A\|_F \|B\|_F. \tag{2.9}$$

Let $A \in M_{mn}(\mathbb{C})$, and $B \in M_{mn}(\mathbb{C})$ be given, then the Kronecker product of A, B is defined by

$$\|A \otimes B\| = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

and have the following properties:

$$\begin{aligned} \|A \otimes B\|_2 &= \|A\|_2 \|B\|_2, \\ \|A \otimes B\|_F &= \|A\|_F \|B\|_F. \end{aligned} \tag{2.10}$$

In the following theorem, we give some formulas of generalized Mersenne of numbers.

Theorem 1. *For generalized Mersenne numbers, we have following sum formulas:*

(a): *[6, Proposition 22. a] If $2x^2-3x+1=0$, i.e., $x=1$ or $x=\frac{1}{2}$, then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(2(n+2)x-3(n+1))x^n W_n + 2(n+1)x^n W_{n-1} + (W_1-3W_0)}{4x-3}.$$

(b): *[7, Proposition 2.1. a] If $(2x-1)(4x-1)(x-1)=0$, i.e., $x=\frac{1}{4}$ or $x=\frac{1}{2}$ or $x=1$ then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(24x^2-28x+7)}$$

where

$\Psi = (n(2x-1)(4x-5)+24x^2-28x+5)x^n W_n^2 + 4((2x-1)n+4x-1)x^n W_{n-1}^2 + 2W_0^2 - (4x-1)(3W_0-W_1)^2 + 2(W_1^2+2W_0^2-3W_1W_0)(2^n(n+1)x^n-1)$.

(c): *[7, Proposition 2.1. d] If $(x-2)(x-1)(x-4)=0$, i.e., $x=1$ or $x=2$ or $x=4$ then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-14x+14)}$$

where

$\Psi = (n(x-2)+2(x-1))x^n W_{-n+1}^2 + (n(x-2)(x-5)+3x^2-14x+10)x^n W_{-n}^2 + 4W_0^2 - 2(x-1)W_1^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)x^n-1)$.

If we set $x=1$ in the last Theorem, we have the following corollary.

COROLLARY 2. *For generalized Mersenne numbers, we have following sum formulas:*

(a):

$$\sum_{k=0}^{n} W_k = (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0). \tag{2.11}$$

(b):

$$\sum_{k=0}^{n} W_k^2 = \frac{1}{3}((1-n)W_n^2 + 4(n+3)W_{n-1}^2 - 25W_0^2 + 18W_0W_1 - 3W_1^2 + 2(W_1-2W_0)(W_1-W_0)(2^n(n+1)-1)). \tag{2.12}$$

(c):

$$\sum_{k=0}^{n} W_{-k}^2 = \frac{1}{3}(-nW_{-n+1}^2 + (4n-1)W_{-n}^2 + 4W_0^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)-1)). \tag{2.13}$$

3. Main Results

In this paper we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized Mersenne numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{3.1}$$

For exclusive cases, we get

$$A = \begin{pmatrix} M_0 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ M_1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ M_2 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 3 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.2}$$

for the Toeplitz matrix $A = T(M_0, M_1, \cdots, M_{n-1})$ with Mersenne numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 3 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 5 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{3.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with Mersenne- Lucas numbers.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

THEOREM 3. *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized Mersenne numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A\|_1 = \|A\|_\infty = \begin{cases} nW_n - (2n+2)W_{n-1} + 3W_0 - W_1 & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \leq 0 \\ -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0) & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \geq 0 \end{cases}$$

where $k = i - j : i,\ j = 0, 1, \cdots, n-1;\ k \in N, -k \in N^-$.

Proof. Acknowledge $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (3.1). By the definitions of $1 - norm$ and $\infty - norm$, and equation (2.3), equation (2.4) and equation(2.11), we conclude that

(i): If $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \leq 0, k \in N$, then we get

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{i1}| \\ &= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1} |W_k| \\ &= -(\sum_{k=0}^{n-1} W_k) = -(\sum_{k=0}^{n} W_k - W_n) = -\sum_{k=0}^{n} W_k + W_n \\ &= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\ &= (n-1)W_n - (2n+2)W_{n-1} + (3W_0 - W_1) + W_n \\ &= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1 \end{aligned}$$

and if $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \geq 0, k \in N$, then we obtain

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{i1}| \\ &= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1} |W_k| \\ &= \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\ &= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\ &= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0). \end{aligned}$$

(ii): If $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \leq 0, k \in N$, then it follows that

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_k| \\
&= -(\sum_{k=0}^{n} W_k - W_n) = -\sum_{k=0}^{n} W_k + W_n \\
&= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\
&= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|$, $k \in N$ and $W_k \geq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n} W_k - W_n \\
&= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\
&= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0).
\end{aligned}$$

Thus, the proof is completed. □

Remark 4. *In the statement of the Theorem 3 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|A\|_1$ and $\|A\|_\infty$ norms of Mersenne, Mersenne-Lucas numbers. The other cases can be handled similarly.*

From the last Theorem 3, we have the following corollary which gives norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A with Mersenne numbers and Mersenne-Lucas numbers, respectively, (set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Corollary 5.

(a): For $A = T(M_0, M_1, \cdots, M_{n-1})$, the vaules of norms of Toeplitz matrices with Mersenne numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nM_n + (2n+2)M_{n-1} + 1$$

(b): For $A = T(H_0, H_1, \cdots, H_{n-1})$, the values of norms of Toeplitz matrices with Mersenne-Lucas numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nH_n + (2n+2)H_{n-1} - 3.$$

Next theorem presents the Frobenious (Euclidian) norm of a Toeplitz matrix A.

THEOREM 6. *Consider* $A = T(W_0, W_1, \cdots, W_{n-1})$ *which is given in (3.1), then the Frobenious (Euclidian) norm of matrix A is*

$$\|A\|_F = \sqrt{\Lambda_1}$$

where

$\Lambda_1 = (\frac{-3n^2+5n}{18})W_{-n+1}^2 + (\frac{6n^2-10n+4}{9})W_{-n}^2 - (\frac{144n+271}{18})W_0^2 + (6n+11)W_0W_1 - \frac{(6n+11)}{6}W_1^2 - (\frac{3n^2+23n+34}{18})W_n^2 + \frac{6n^2+46n+84}{9}W_{n-1}^2 - 2W_{-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0)$.

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad + (n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

and so

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_i^2) + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_{-i}^2) - 2(n-1)W_0^2 \\ &= (n - 2n + 2)W_0^2 + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_i^2) + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_{-i}^2) \\ &= (\frac{-3n^2+5n}{18})W_{-n+1}^2 + (\frac{6n^2-10n+4}{9})W_{-n}^2 - (\frac{144n+271}{18})W_0^2 \\ &\quad + (6n+11)W_0W_1 - \frac{(6n+11)}{6}W_1^2 - (\frac{3n^2+23n+34}{18})W_n^2 + \frac{6n^2+46n+84}{9}W_{n-1}^2 \\ &\quad - 2W_{-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) \\ &\quad + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0) \end{aligned}$$

Moreover, we use equation 2.12 and equation 2.13 in Corollary 2.

Therefore, we get

$\|A\|_F^2 = (\frac{-3n^2+5n}{18})W_{-n+1}^2 + (\frac{6n^2-10n+4}{9})W_{-n}^2 - (\frac{144n+271}{18})W_0^2 + (6n+11)W_0W_1 - \frac{(6n+11)}{6}W_1^2 - (\frac{3n^2+23n+34}{18})W_n^2 + \frac{6n^2+46n+84}{9}W_{n-1}^2 - 2W_{-1}^2$
$+ \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0)$.

This complates the proof. □

From the last Theorem 6, we have the following corollary which gives Frobenius norm formulas of Mersenne numbers and Mersenne-Lucas numbers, respectively, (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_{-1} = -\frac{1}{2}$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3, H_{-1} = \frac{3}{2}$, respectively).

Corollary 7. *For $n \geq 0$, Toeplitz matrices with the Mersenne and Mersenne-Lucas numbers, respectively have the following properties:*

(a): $\|A\|_F = \sqrt{\Lambda_2}$

where A is given as in (3.2)

$\Lambda_2 = (\frac{-3n^2+5n}{18})M_{-n+1}^2 + (\frac{6n^2-10n+4}{9})M_{-n}^2 - (\frac{3n^2+23n+34}{18})M_n^2 + \frac{6n^2+46n+84}{9}M_{n-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 27n - 7)$.

(b): $\|A\|_F = \sqrt{\Lambda_3}$

where A is given as in (3.3)

$\Lambda_3 = (\frac{-3n^2+5n}{18})H_{-n+1}^2 + (\frac{6n^2-10n+4}{9})H_{-n}^2 - (\frac{3n^2+23n+34}{18})H_n^2 + (\frac{6n^2+46n+84}{9})H_{n-1}^2 - \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) + 27n + 151)$.

In the following theorem, we find the lower and upper bounds for the spectral norm of the matrices with the Mersenne numbers, Mersenne-Lucas numbers, respectively, (take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Theorem 8.

(a): Consider $A = T(M_0, M_1, \cdots, M_{n-1})$ which is given as in (3.2). Let

$$C = \begin{pmatrix} 1 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ 1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ 1 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 1 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} M_0 & 1 & 1 & \cdots & 1 \\ M_1 & 1 & 1 & \cdots & 1 \\ M_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_2}$$

where Λ_2 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Lambda_4$$

where

$$\begin{aligned}\Lambda_4 &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 + 2^{n+1}(n+1) - 2))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 + 2^{n+1}(n+1) - 5))^{\frac{1}{2}}.\end{aligned}$$

(b): Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (3.3). Let

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 1 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 1 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 5 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_3}$$

where Λ_3 is as in Corollary 7

(ii):

$$\|A\|_2 \leq \Lambda_5$$

where

$$\begin{aligned}\Lambda_5 &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}.\end{aligned}$$

Proof.

(a): **(i):** We use equation (2.2).

(ii): We get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} M_k^2 + 1)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n} M_k^2 - M_n^2 - M_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 - 25M_0^2 + 18M_0M_1 \\ &\quad -3M_1^2 + 2(M_1 - 2M_0)(M_1 - M_0)(2^n(n+1) - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 + 2^{n+1}(n+1) - 2))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} W_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} W_k^2 - W_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 - 25M_0^2 + 18M_0M_1 \\ &\quad -3M_1^2 + 2(M_1 - 2M_0)(M_1 - M_0)(2^n(n+1) - 1)))^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 + 2^{n+1}(n+1) - 5))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Lambda_4 \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1) - 2))^{\frac{1}{2}} \times (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1) - 5))^{\frac{1}{2}}. \end{aligned}$$

(b): **(i):** We use equation (2.2).

(ii): By definition, we get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} H_k^2 + 1)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n} H_k^2 - H_n^2 - H_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\ &\quad +2(H_1 - 2H_0)(H_1 - H_0)(2^n(n+1) - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \end{aligned}$$

and

$$
\begin{aligned}
c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} = (\sum_{k=0}^{n-1} H_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} H_k^2 - H_n^2)^{\frac{1}{2}} \\
&= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\
&\quad +2(H_1 - 2H_0)(H_1 - H_0)(2^n(n+1) - 1)))^{\frac{1}{2}} \\
&= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}
\end{aligned}
$$

so, from inequality (2.7)

$$
\begin{aligned}
\|A\|_2 &\leq r_1(C)c_1(D) = \Lambda_5 \\
&= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \\
&\quad \times(\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}.
\end{aligned}
$$

This complates the proof. □

From the equation (2.10) and Corollary 7, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized Mersenne numbers.

COROLLARY 9. *Suppose that* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:.*

$$
\begin{aligned}
\|A \otimes B\|_F &= \|A\|_F \|B\|_F \\
&= \sqrt{\Lambda_2}\sqrt{\Lambda_3}
\end{aligned}
$$

where Λ_2 *and* Λ_3 *are as in Corollary 7 (a) and (b),*

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$ respectively).

Proof. It can be easily seen from equation (2.10) and Theorem 6 and Corollary 7. □

From the above inequality (2.9) and Theorem 6 and Corollary 7, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

COROLLARY 10. *Assume that* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$
\begin{aligned}
\|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\
&\leq \sqrt{\Lambda_2}\sqrt{\Lambda_3}
\end{aligned}
$$

where Λ_2 *and* Λ_3 *are as in Corollary 7 (a) and (b),* (set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Proof. For the proof see inequality (2.9) and Theorem 6. □

In the last inequality (2.8) and Theorem 8, we have the following corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

Corollary 11. *Given $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:*

$$\|A \circ B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 and Λ_5 are as in Theorem 8,

(take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Proof. See inequality (2.8) and Theorem 8. □

From the related equation (2.10) and Theorem 8, we have the following corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with different Mersenne sequences.

Corollary 12. *Let $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$\|A \otimes B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 and Λ_5 are as in Theorem 8,

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Proof. See equation (2.10) and Theorem 8. □

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