Original Research Article

Some Fixed Point results of Rational Type-Contraction Mapping in S-Metric Space

Abstract:

In this paper, we demonstratethe existence ofsome fixed points ofrational type contraction in context of S-metric space and we examine the T-stability of the P-property for some mapping. Also, we present few examples to illustrate the validity of the results obtained in the paper.

Keywords: Fixed point,rational type contraction, S-metric space.
MSC: 54H25, 47H10

1. Introduction and Preliminaries

Fixed point theory is anactive area of research with various application in real life.One of the main approaches used in this theory to demonstrate the existence and uniqueness of fixed point is contraction.In1989,Bakhtin[3]wasthefirstwhointroducedtheconceptofb-metricspace. In1993,Czerwik[6]extendedtheresultsofBakhtin[7]andgavegeneralizationofBanachfixedpointtheoreminb-metricspaces. In 2012, the idea of S- metric space was established by Sedghi *et al.* [14], who also proved fixed point theorems there in. Manoj K. *et al.* [12], proved fixed point theorem by using altering distance function in S-metric space. More well-known results in the direction of S-metric space are involved in (refer [15]-[18]).

Theorem 1.1 [12]:"Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping on a complete S-metric space $(\mathcal{X}, \mathcal{S})$ such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda\, \mathcal{S}(u,u,v) + \eta\, \frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)},$$

Comment [a1]: "," is missed

for all $u, v \in X$, $\lambda, \eta > 0$, $\lambda + \eta < 1$. Then $\mathcal{T}$ possess a fixed point $w \in \mathcal{X}$ which is unique."

Theorem 1.2 [12]:"Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping on a complete S-metric space $(\mathcal{X}, \mathcal{S})$ such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda\, \mathcal{S}(u,u,v) + \eta\, \frac{\mathcal{S}(v,v,\mathcal{T}v)[1 + \mathcal{S}(u,u,\mathcal{T}u)]}{1 + \mathcal{S}(u,u,v)},$$

for all $u, v \,\epsilon\, X,\ \lambda, \eta > 0,\ \lambda + \eta < 1$. Then $\mathcal{T}$ possess a fixed point $w \,\epsilon\, \mathcal{X}$ which is unique.”

Furthermore, we proceed by reviewing some important definitions and key terms that would be used throughout our discussion.

Definition 1.3 [14]: “Let $\mathcal{X}$ be a non-empty set. An S-metric on $\mathcal{X}$ is a mapping $\mathcal{S}: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ which satisfies the following condition:

Comment [a2]: Bring before Theorem 1.1

$(\mathcal{S}_1)$ $\mathcal{S}(u, v, w) = 0$ if and only if $u = v = w = 0$;

$(\mathcal{S}_2)$ $\mathcal{S}(u, v, w) \le \mathcal{S}(u, u, a) + \mathcal{S}(v, v, a) + \mathcal{S}(w, w, a)$, for all $u, v, w, a \,\epsilon\, \mathcal{X}$.

The pair $(\mathcal{X}, \mathcal{S})$ is called an S-metric space.”

Example 1.4 [14]: “Let $\mathcal{X} = \mathbb{R}$. Then $\mathcal{S}(u, v, w)$is an S-metric on $\mathbb{R}$ given by $\mathcal{S}(u, v, w) = |u - w| + |v - w|$, which is known as usual S-metric space on $\mathcal{X}$.”

Comment [a3]: After definition 1.3 and before theorem 1.1

Lemma 1.5 [14]: “If $(\mathcal{X}, \mathcal{S})$ is an S-metric space on a non-empty set $\mathcal{X}$, then $(\mathcal{X}, \mathcal{S})$ satisfy the symmetric condition, that is$\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$, for all $u, v \,\epsilon\, \mathcal{X}$.”

Definition 1.6 [14] “Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space. For $r > 0$ and $u \,\epsilon\, \mathcal{X}$ we define the open ball $B_s(u, r)$ and closed ball and $B_s[u, r]$ with a center u and radius r as follows:

$$B_s(u, r) = \{v \,\epsilon\, \mathcal{X}: \mathcal{S}(v, v, u) < r\}$$

$B_s[u, r] = \{v \,\epsilon\, \mathcal{X}: \mathcal{S}(v, v, u) \le r\}$.”

Definition 1.7 [15]: “A sequence $\{u_n\}$ in $(\mathcal{X}, \mathcal{S})$ is said to be convergent to some point $u \,\epsilon\, \mathcal{X}$, if $\mathcal{S}(u_n, u_n, u) \to 0$ as $n \to \infty$.”

Definition 1.8 [15]: “A sequence $\{u_n\}$ in $(\mathcal{X}, \mathcal{S})$ is said to be Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$.”

Definition 1.9 [15]: “An S -metric space $(\mathcal{X}, \mathcal{S})$ is said to be complete if every Cauchy sequence in X is convergent in $\mathcal{X}$.”

Lemma 1.10 [15]:“Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space. If $u_n \to u$ and $v_n \to v$ then $\mathcal{S}(u_n, u_n, v_n) \to \mathcal{S}(u, u, v)$.”

Lemma 1.11 [16]: “Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space and $\{u_n\}$is a convergent sequence in $\mathcal{X}$. Then $\lim_{n\to\infty} u_n$ is unique.”

Definition 1.12 [16]:"Let$(\mathcal{X},\mathcal{S})$ be S-metric pace. A map $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is said to be contraction if there exists a constant $k\in[0.1)$ such that

Comment [a4]: k\in[0,1)

$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)\leq\lambda\,\mathcal{S}(u,u,v)$, for all $u,v\ \in\mathcal{X}$."

Lemma 1.13 [16]: "If $\{u_n\}$ is a sequence of elements from S-metric space $(\mathcal{X},\mathcal{S})$ satisfying the following property$\mathcal{S}(u_n\,,u_n\,,u_{n+1})\ \leq\ k\,\mathcal{S}(u_{n-1}\,,u_{n-1}\,,u_n)$, for each $k\,\epsilon\,[0,\,1)$ where $n\,\epsilon\,\mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence."

2. Main Results

In this section, weestablish fixed points of rational typecontractions in the context of S-metric spaces and demonstrates that the P property is T-stable for some mappings. In order to show the relevance of the conclusions drawn in this work, we also provide a few examples.

Theorem 2.1: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$be a mapping such that

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)\leq a_1\mathcal{S}(u,u,v)+a_2\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u)+\mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}u)},$$

(2.1)

for all $u,v\in\mathcal{X}$ and $a_1,a_2\geq 0\ ,\mathcal{S}(u,u,\mathcal{T}v)+\ \mathcal{S}(v,v,\mathcal{T}u)\neq 0$ with $a_1+a_2<1$. Then, $\mathcal{T}$has a unique fixed point $\mathcal{X}$.

Proof: Letu_0 be an arbitrary in $\mathcal{X}$, we define a sequence$\{u_n\}$ in $\mathcal{X}$such that $\mathcal{T}u_n=u_{n+1}$ for all $n=1,2,\ldots.$ From condition (2.1) with $u=u_n$and $v=u_{n-1}$, Therefore

$$\mathcal{S}(u_n,u_n,u_{n+1})=\mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)$$

$$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$+a_2\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}$$

$$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$+a_2\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_n)+\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})+\mathcal{S}(u_n,u_n,u_n)}$$

$$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n)+a_2\mathcal{S}(u_n,u_n,u_{n+1}).$$

It follows that

$(1 - a_2)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$

(2.2)

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1}{1 - a_2}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \left(\frac{a_1}{1-a_2}\right)$. In view of $a_1 + a_2 < 1$, then $0 \leq \lambda < 1$. Thus, by Lemma 1.13, $\{u_n\}$ is a Cauchy sequence in $\mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

By (2.2), it is easy to see that

$\mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$

(2.3)

$$\leq a_1\mathcal{S}(u_n, u_n, u^*) + a_2\frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}$$

$$\leq a_1\mathcal{S}(u_n, u_n, u^*) + a_2\frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})}.$$

(2.4)

Taking the limit as $n \to \infty$ on both side of (2.4), we have $\lim_{n\to\infty} \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = 0$.

That is, $u_n \to \mathcal{T}u^*$. Hence, $\mathcal{T}u^* = u^*$, u^* is afixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (2.1), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_2\frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_2\frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(u^*, u^*, v^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)}$$

$\mathcal{S}(u^*, u^*, v^*) \leq a_1\mathcal{S}(u^*, u^*, v^*)$.

(2.5)

Since $a_1 + a_2 < 1$implies $p < 1$.

Comment [a5]: what'sp?

Therefore, we obtain that $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence the fixed point is unique.

This completes theproof. ■

Example 2.2: Let$\mathcal{X} = [0.1]$ be equipped with complete S-metric space define by

$$\mathcal{S}(u, v, w) = (|u - v| + |u - w| + |v - w|)^2.$$

Consider a mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$defined by

$$\mathcal{T}(u) = \frac{1}{36} u^2 e^{-u^2},$$

for all $u, v, w \in \mathcal{X}$.

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) = (|\mathcal{T}u - \mathcal{T}u| + |\mathcal{T}u - \mathcal{T}w| + |\mathcal{T}u - \mathcal{T}w|)^2$$

$$= (2\ |\mathcal{T}u - \mathcal{T}w|)^2$$

$$= 4 \left| \frac{1}{36} u^2 e^{-u^2} - \frac{1}{36} w^2 e^{-w^2} \right|^2 = \left| \frac{1}{18} u^2 e^{-u^2} - \frac{1}{18} w^2 e^{-w^2} \right|^2$$

$$\leq \frac{1}{9} \left| u^2 e^{-u^2} - w^2 e^{-w^2} \right|^2$$

$$\leq \frac{4}{9} |u - v|^2 = \frac{1}{9} |2(u - v)|^2$$

$$\leq \frac{1}{3} \mathcal{S}(u, v, w)$$

$$\leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u) + \mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u)}.$$

Clearly by taking $a_2 = \frac{1}{2}$, we have $a_1 + a_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1$. Then, from Theorem 2.1 we conclude that, $\mathcal{T}$ has a unique fixed point.Also, 0 is the only fixed point of$\mathcal{T}$.

Theorem 2.3: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$be a mapping such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}v) + \ \mathcal{S}(v, v, \mathcal{T}u)}$$

$$+a_3 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u)+\mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}u)}, \quad (2.6)$$

where $a_1, a_2, a_3$are non-negative constant with $a_1 + a_2 + a_3 < 1$. Then, $\mathcal{T}$has a unique fixed point $\mathcal{X}$.

Proof: Choose$u_0 \in X$and construct a Picard iterative sequence$\{u_n\}$as $\mathcal{T}u_n = u_{n+1}$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} = u_{n_0+1} = \mathcal{T}u_{n_0}$, i.e., u_{n_0} is a fixed point of $\mathcal{T}$. Next, without loss of generality, let $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, Using (2.6), we get

$$\mathcal{S}(u_n, u_n, u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n)$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$+a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}$$

$$+\ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$+ a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)}$$

$$+ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)}$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_3 \mathcal{S}(u_n, u_n, u_{n+1}).$$

It follows that

$(1 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (a_1 + a_2)\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$
(2.7)

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1 + a_2}{1 - a_3}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \frac{a_1+a_2}{1-a_3}$. In view of $a_1 + a_2 + a_3 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.6) it is easy to see that

$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)$

(2.8)

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$$

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*)$$

$$+\, a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}$$

$$+\, a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}$$

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*)$$

$$+a_2 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,u_{n+1})}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,u_{n+1})} \tag{2.9}$$

$$+a_3 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,u_{n+1})+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,u_{n+1})}.$$

Taking the limit as $n \to \infty$ on both side of (2.9), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$it follows thatu^* is a fixed point of $\mathcal{T}$.

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.6), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}$$

$$+\, a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)}$$

$$+a_3 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(u^*, u^*, v^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)}$$

$\mathcal{S}(u^*, u^*, v^*) \leq a_1 \mathcal{S}(u^*, u^*, v^*)$. (2.10)

Since$a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$, we obtain that $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence the fixed point is unique.

This completes the proof. ■

Theorem 2.4: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space. Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$be a mapping satisfying

$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + a_3 \frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}$,

(2.11) for all $u, v \in \mathcal{X}$ and $a_1, a_2, a_3$are non-negative constant with $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$has a unique fixed point $\mathcal{X}$.

Comment [a6]: next line

Proof: Choose$u_0 \in X$. Construct a sequence$\{u_n\}$in $\mathcal{X}$ by$\mathcal{T}u_n = u_{n+1}$. For all $n \in \mathbb{N}$, from condition (2.11) with $u = u_n$ and $v = u_{n-1}$, we have

$$\mathcal{S}(u_n, u_n, u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n)$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$+a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)[1 + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$+a_3 \frac{\mathcal{S}(u_n, u_n, u_{n+1})[1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)]}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1}) + a_3 \mathcal{S}(u_n, u_n, u_{n+1}).$$

It follows that

$(1 - a_2 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$

(2.12)

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1}{1 - a_2 - a_3}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \frac{a_1}{1-a_2-a_3}$.In view of $a_1 + a_2 + a_3 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.11) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)$$

(2.13)

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$$

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*) + a_2\frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}$$

$$+a_3\frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)[1 + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)]}{1 + \mathcal{S}(u_n, u_n, u^*)}$$

$$\leq 2\,\mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*) + a_2\frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}$$

$+ a_3\frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)[1+\mathcal{S}(u_n, u_n, \mathcal{T}u_{n+1})]}{1+\mathcal{S}(u_n, u_n, u^*)}$. (2.14) Taking the limit as $n \to \infty$ on both side of (2.14), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.11), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_1\frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_1\frac{\mathcal{S}(v^*, v^*, \mathcal{T}v^*)[1 + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)}$$

$$\leq a_1\mathcal{S}(u^*, u^*, v^*) + a_2\frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3\frac{\mathcal{S}(v^*, v^*, v^*)[1 + \mathcal{S}(u^*, u^*, u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq a_1\mathcal{S}(u^*, u^*, v^*).$$

Since $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$, thus, we obtain $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence, we proved that $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Example 2.5: Let $\mathcal{X} = [0.1]$ and $(\mathcal{X}, \mathcal{S})$ be a usual S-metric space which is complete, define by

$$\mathcal{S}(u, v, w) = |u - w| + |v - w|.$$

Consider a mapping $\mathcal{T}: X \to X$ be define as $(u) = \frac{u}{8}$, for all $u, v, w \in X$.

Obviously,

$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) = 2\,|\mathcal{T}u - \mathcal{T}v| = 2\,\left|\frac{u}{8} - \frac{v}{8}\right| = \frac{1}{4}\,|u - v|,$

$$\mathcal{S}(u, u, v) = 2\,|u - v|.$$

Also,

$$\mathcal{S}(u, u, \mathcal{T}u) = 2\,|u - \mathcal{T}u| = 2\,\left|u - \frac{u}{8}\right| = \frac{7u}{4},$$

$\mathcal{S}(v, v, \mathcal{T}v) = 2\,|v - \mathcal{T}v| = 2\,\left|v - \frac{v}{8}\right| = \frac{7v}{4},$

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) = \frac{1}{4}|u - v|$$

$$= \frac{1}{8}\,2\,|u - v|$$

$\leq \frac{1}{8}\mathcal{S}(u, u, v) + \frac{1}{4}\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + \frac{1}{7}\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}.$

It is clear that, $a_1 + a_2 + a_3 = \frac{1}{8} + \frac{1}{4} + \frac{1}{7} = \frac{29}{56} < 1$. Thus, we conclude that inequality (2.11) of Theorem 2.4 remains valid. Hence, $\mathcal{T}$ has a unique fixed point and the fixed point is 0.

Theorem 2.6: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping satisfying the following condition

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1\mathcal{S}(u, u, v) + a_2\frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + a_3\frac{\mathcal{S}(u, u, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, v)}$$
$$+ a_4[\mathcal{S}(u, u, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)] + a_5[\mathcal{S}(v, v, \mathcal{T}u) + \mathcal{S}(u, u, \mathcal{T}v)], \qquad (2.15)$$

Comment [a7]: Tu=...

Comment [a8]: Up line

for all $u, v \in \mathcal{X}$ and a_1, a_2, a_3, a_4, a_5 are non-negative constant $a_1 + a_2 + a_3 + 2\, a_4 + 3\, a_5 < 1$. Then, $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Comment [a9]: Next center line

Proof: Choose $u_0 \in X$. Construct a sequence $\{u_n\}$ in $\mathcal{X}$ by $\mathcal{T}u_n = u_{n+1}$.

For all $n \in \mathbb{N}$, from condition (2.15) with $u = u_n$ and $v = u_{n-1}$, we have

$$\mathcal{S}(u_n, u_n, u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n)$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$+a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$+a_4[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)]$$

$$+a_5[\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)]$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}$$

$$+a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} + a_4[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})]$$

$$+a_5[\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})]$$

$$\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1})$$

$$+a_4[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})]$$

$$+a_5[2\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})].$$

It follows that

$$(1 - a_2 - a_4 - a_5)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (a_1 + a_4 + 2a_5)\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) (2.16)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1 + a_4 + 2a_5}{1 - a_2 - a_4 - a_5}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \frac{a_1+a_4+2a_5}{1-a_2-a_4-a_5}$. In view of $a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13,$\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.15) it is easy to see that

$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)$
(2.17)

$$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$$

$$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}$$

$$+a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, u^*)} + a_4[\mathcal{S}(u_n, u_n, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]$$

$$+a_5[\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*)]$$

$$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}$$

$$+ a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, u_{n+1})}{1 + \mathcal{S}(u_n, u_n, u^*)} + a_4[\mathcal{S}(u_n, u_n, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]$$

$$+a_5[\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*) \qquad (2.18)$$

Taking the limit as $n \to \infty$ on both side of (2.18), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.15), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, v^*)}$$

$$+a_4\, [\mathcal{S}(u^*, u^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)] + a_5[\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}v^*)]$$

$$\leq a_1\mathcal{S}(u^*,u^*,v^*) + a_2\frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(v^*,v^*,v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_3\frac{\mathcal{S}(u^*,u^*,v^*)\mathcal{S}(v^*,v^*,u^*)}{\mathcal{S}(u^*,u^*,v^*)}$$

$$+a_4\,[\mathcal{S}(u^*,u^*,u^*) + \mathcal{S}(v^*,v^*,v^*)] + a_5[\mathcal{S}(v^*,v^*,u^*) + \mathcal{S}(u^*,u^*,v^*)]$$

$\mathcal{S}(u^*,u^*,v^*) \leq (a_1 + a_3 + 2a_5)\,\mathcal{S}(u^*,u^*,v^*)$.
(2.19)

Since $0 < a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1 \Rightarrow a_1 + a_3 + 2a_5 < 1$, thus, we obtain $\mathcal{S}(u^*,u^*,v^*) = 0$,which further implies$u^* = v^*$.

Therefore,$\mathcal{T}$have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Theorem 2.7: Let$(\mathcal{X},\mathcal{S})$ be a complete S-metric space and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$be a self mapthat satisfies the following inequality

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \leq a_1\mathcal{S}(u,u,v) + a_2\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + a_3\frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,v)}$$

$$+ a_4\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}, \qquad (2.20)$$

for all $u,v\in\mathcal{X}$ and a_1,a_2,a_3,a_4 are non-negative constant $a_1 + a_2 + a_3 + a_4 < 1$. Then, $\mathcal{T}$has a unique fixed point $\mathcal{X}$.

Proof: Choose$u_0 \in X$ and construct a Picard iterative sequence$\{u_n\}$as $\mathcal{T}u_n = u_{n+1}$.

For all $n\in\mathbb{N}$, from condition (2.20) with $u = u_n$ and $v = u_{n-1}$, we have

$$\mathcal{S}(u_n,u_n,u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)$$

$$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}$$

$$+a_3\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}$$

$$+a_4\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)[1+\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})]}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}$$

$$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}$$

$$+a_3\frac{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})\mathcal{S}(u_n,u_n,u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)} + a_4\frac{\mathcal{S}(u_n,u_n,u_{n+1})[1+\mathcal{S}(u_{n-1},u_{n-1},u_n)]}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}$$

$\leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\mathcal{S}(u_n,u_n,u_{n+1}) + a_4\mathcal{S}(u_n,u_n,u_{n+1}),$

which further implies,

$(1-a_2-a_4)\mathcal{S}(u_n,u_n,u_{n+1}) \leq a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n)$ (2.21)

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq \left(\frac{a_1}{1-a_2-a_4}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n).$$

Put $\lambda = \frac{a_1}{1-a_2-a_4}$. In view of $a_1+a_2+a_3+a_4<1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13, $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X},\mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.20) it is easy to see that

$\mathcal{S}(u^*,u^*,\mathcal{T}u^*) \leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(u_{n+1},u_{n+1},\mathcal{T}u^*)$
(2.22)

$$= 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}u^*)$$

$$\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\mathcal{S}(u_n,u_n,u^*) + a_2\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}$$

$$+a_3\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}u_n)}{\mathcal{S}(u_n,u_n,u^*)} + a_4\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)]}{1+\mathcal{S}(u_n,u_n,u^*)}$$

$$\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\mathcal{S}(u_n,u_n,u^*) + a_2\frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}$$

$+\,a_3\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,u_{n+1})}{1+\mathcal{S}(u_n,u_n,u^*)} + a_4\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,u_{n+1})]}{1+\mathcal{S}(u_n,u_n,u^*)}.$ (2.23)

Taking the limit as $n \to \infty$ on both side of (2.23), we have $\lim_{n\to\infty}\mathcal{S}(u^*,u^*,\mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.20), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_1 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_1 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, v^*)}$$

$$+a_4 \frac{\mathcal{S}(v^*, v^*, \mathcal{T}v^*)[1 + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)}$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(u^*, u^*, v^*)\mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*)}$$

$$+a_4 \frac{\mathcal{S}(v^*, v^*, v^*)[1 + \mathcal{S}(u^*, u^*, u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)}.$$

$$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_3)\, \mathcal{S}(u^*, u^*, v^*)$$

$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_2 + a_3 + a_4)\, \mathcal{S}(u^*, u^*, v^*)$,
(2.24)

a contradiction.

Thus, we obtain $\mathcal{S}(u^*, u^*, v^*) = 0$, which further implies$u^* = v^*$.

Therefore,$\mathcal{T}$have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Example 2.8: Let$\mathcal{X} = [0.1]$ be equipped with complete S-metric space define by

$$\mathcal{S}(u, v, w) = (|u - v| + |u - w| + |v - w|)^2.$$

Let the mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be defined by

$\mathcal{T}(u) = \frac{u}{5}$.

Then, for all $u, v, w \in \mathcal{X}$, we have,

$\mathcal{S}(u, u, v) = 4|u - v|^2$,

$$\mathcal{S}(u,u,\mathcal{T}u) = 4|u-\mathcal{T}u|^2 = 4\left|u-\frac{u}{5}\right|^2 = \frac{16}{25}u^2$$

$$\mathcal{S}(v,v,\mathcal{T}v) = 4|v-\mathcal{T}v|^2 = 4\left|v-\frac{v}{5}\right|^2 = \frac{16}{25}v^2$$

Also,

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) = (|\mathcal{T}u-\mathcal{T}u| + |\mathcal{T}u-\mathcal{T}w| + |\mathcal{T}u-\mathcal{T}w|)^2$$

$$= (2\,|\mathcal{T}u-\mathcal{T}w|)^2$$

$$= \frac{4}{25}|u-v|^2$$

$$\le \frac{1}{25}\mathcal{S}(u,v,w) + \frac{4}{25}\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + \frac{2}{5}\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}$$

Clearly, we have $a_1 + a_2 + a_3 = \frac{1}{25} + \frac{4}{25} + \frac{2}{5} = \frac{3}{5} < 1$. Then, from Theorem 2.7 we conclude that, $\mathcal{T}$ has a unique fixed point. Also, 0 is the only fixed point of $\mathcal{T}$.

Theorem 2.9: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space. Let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$be a mapping satisfying

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \le a_1\mathcal{S}(u,u,v) + a_2\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u)}$$

$$+ a_3\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)\,,$$

(2.25)

for all $u,v\in\mathcal{X}$ and $a_1,a_2,a_3 \ge 0$. $\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u) \neq 0$ with $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$has a unique fixed point $\mathcal{X}$.

Proof: Choose$u_0$as an arbitrary point in X. We define a sequence $\{u_n\}$in $\mathcal{X}$ by $\mathcal{T}u_n = u_{n+1}$.Then or all $n\in\mathbb{N}$, from condition (2.25) with $u = u_n$ and $v = u_{n-1}$, we have

$$\mathcal{S}(u_n,u_n,u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)$$

$$\le a_1\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$+a_2\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}$$

$$+a_3\,\mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)$$

$$\leq a_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$+a_2\frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)}$$

$$+a_3\mathcal{S}(u_n, u_n, u_{n+1})$$

$$\leq a_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n)+a_2\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_3\mathcal{S}(u_n, u_n, u_{n+1}).$$

Which further implies

$(1 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (a_1+a_2)\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$
(2.26)

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1+a_2}{1 - a_3}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \frac{a_1+a_2}{1-a_3}$. In view of $a_1 + a_2 + a_3 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.25) it is easy to see that

$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)$
(2.27)

$$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$$

$$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*)$$

$$+a_2\frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}$$

$$+a_3\mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)$$

$\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\mathcal{S}(u_n, u_n, u^*)$

$$+a_2\frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, u_{n+1})}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})}$$

$+ a_3\, \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)$. (2.28) Taking the limit as $n \to \infty$ on both side of (2.28), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.25), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}$$

$$+ a_3 \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)}$$

$$+ a_3 \mathcal{S}(u^*, u^*, v^*)$$

$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_3)\, \mathcal{S}(u^*, u^*, v^*)$.
(2.29)

Since $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 + a_3 < 1$, thus, we obtain $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence, we proved that $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Remark 2.10.

1. If we put $a_2 = a_3 = a_4 = a_5 = 0$ in Theorem 2.6,we get the Banach Theorem [2].

Comment [a10]: Which S=?

2.If we put $a_1 = a_2 = a_3 = a_5 = 0$ in Theorem 2.6, we get the Kanan Theorem [11].

3. If we put $a_2 = a_3 = a_5 = 0$ in Theorem 2.6, we get the Fisher Theorem [9].

4. If we put $a_1 = a_2 = a_3 = a_4 = 0$ in Theorem 2.6,we get the result of Chaterjee Theorem [5].

5. If we put $a_2 = a_3 = 0$ in Theorem 2.9, we get the result ofDass and Gupta Theorem [8].

Theorem 2.11: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$be a mapping such that $F(\mathcal{T}) \neq \Phi$and that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2 u) \leq \lambda\, \mathcal{S}(u, u, \mathcal{T}u)\,, \tag{2.30}$$

for all $u \in \mathcal{X}$, where $0 \leq \lambda < 1$ is a constant. Then $\mathcal{T}$ has the P property.

Comment [a11]: Where is definition of P-property?

Proof: We always assume that $n > 1$. Since the statement for $n = 1$ is trivial. Let $w \in F(\mathcal{T}^n)$. By the hypotheses, we get

$$\mathcal{S}(w, w, \mathcal{T}w) = \mathcal{S}(\mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}^2\mathcal{T}^{n-1}w)$$

$$\leq \lambda\, \mathcal{S}(\mathcal{T}^{n-1}w, \mathcal{T}^{n-1}w, \mathcal{T}^n w)$$

$$\leq \lambda\, \mathcal{S}(\mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}^2\mathcal{T}^{n-2}w)$$

$$\leq \lambda^2\, \mathcal{S}(\mathcal{T}^{n-2}w, \mathcal{T}^{n-2}w, \mathcal{T}^{n-1}w)$$

$\leq \cdots \leq \lambda^n \mathcal{S}(w, w, \mathcal{T}w) \to 0$, as $n \to \infty$.

Hence $\mathcal{S}(w, w, \mathcal{T}w) = 0$, that is $\mathcal{T}w = w$.

Theorem 2.12: Under the condition of Theorem 2.3,$\mathcal{T}$ has the P property.

Proof: We have to prove that the mapping$\mathcal{T}$ satisfies (2.30). In fact, for any $u \in \mathcal{X}$, we have

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2 u) = \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)$$

$$\leq a_1 \mathcal{S}(u, u, \mathcal{T}u) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)}$$

$$+ a_3 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)}$$

$$\leq a_1 \mathcal{S}(u, u, \mathcal{T}u) + a_2\, \mathcal{S}(u, u, \mathcal{T}u) + a_3\, \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)$$

$$(1 - a_3)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2 u) \leq (a_1 + a_2)\mathcal{S}(u, u, \mathcal{T}u)$$

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2 u) \leq \frac{a_1 + a_2}{1 - a_3} \mathcal{S}(u, u, \mathcal{T}u)$$

Deduce that $\lambda = \frac{a_1 + a_2}{1 - a_3}$. Note that $a_1 + a_2 + a_3 < 1$, then $\lambda < 1$. Accordingly, (2.30) is satisfied. Consequently, by Theorem 2.3, $\mathcal{T}$ has the P property.

References

[1] Amini-Harandi A., Fixed point theorem for quasi-contraction maps in b-metric spaces, Fixed Point Theory. Vol. 5(2), pp 351-358, (2014).

[2] Banach S., Sur les opérationsdans les ensembles abstraits et leur application aux equations integrals, Fund. Math.Vol. 3, pp 133-181, (1922).

[3] Bakhtin I. .A., The contraction mapping principle in almost metric spaces. Funct. Anal. Vol. 30, pp 26-37, (1989).

[5] Chatterjee S. Fixed point theorems, DokladinaBolgarskataAkademiyanaNaukite. Vol. 25(6) pp727-730, (1989).

[6] Czerwik S., The contraction mapping in b - metric spaces. Acta Math. Inform. Univ. Ostrav. Vol. 1, pp 5- 11, (1993).

[7] Czerwik, S. Nonlinear set - valued contraction mapping in b -metric spaces. Atti Semin. Mat. Fis. Dell'Universita Modena Reeggio Emilia.Vol. 46, pp 263–276, (1998).

[8] Das B. K. and Gupta S., An extension of Banachcontraction principle through rational expression, Indian J. Pure Appl. Math. Vol. 6(12), pp 1455-1458, (1975).

[9] Fisher B., A fixed point theorem for compact metric spaces, Publ. Math. Debrecen.Vol. 25 (1978).

[10]Hussein M.M., Bin Saad M.G., Al-Sayad A. A., Fixed point results of relation type-Contractionmapping in b-metric spaces with an application, Earthline Journal of Mathematical science.Vol. 12(2), pp 141-164, (2023).

[11] Kannan R., Some results on fixed points, Bull. Cal. Math. Soc. Vol. 60, pp 71-76,(1968).

[12] Manoj K., Sushma D., Parul S., Fixed point theorems by using altering distance function in S-metric space, Communication in Mathematics and application. Vol. 13(2), pp 553-573, (2022).

[13] Naddler S. B. Jr, Multi-valued contraction mappings. Pac. J. Math. Vol. 30, pp 475-488, (1969).

[14] Sedghi S., Shobe N., Aliouche A., A generalization of fixed point theorem in S-metric

spaces. Mat. Vesnik. Vol. 64, pp 258-266, (2012).

[15] Sedghi S. and Dung N. V., Fixed point theorems on S-metric space, Mat. Vesnik, Vol. 66(1), pp 113–124, (2014).

[16] Sedghi S., Shobe N., Dosenovic T., Fixed Point Results In S-Metric Spaces, Nonlinear Funct. Anal. Appl., Vol. 20(1), pp 55–67, (2015).

[17] Sedghi S., Kyu Kim J., Gholidahneh A., Mahdi Rezaee M., Fixed point theorems in S-metric Spaces, East Asian Math. J. Vol.32(5), pp 677–684, (2016).

[18] Sedghi S., Gholidahneh A., Dosenovic T., Esfahani J., Radenovic S., Common fixed point of four maps in S-metric spaces, J. Linear Topol. Algebra. Vol.5(2), pp 93–104, (2016).

Please add the following references in text:

[19] $\boldsymbol{\alpha - \psi}$-Geraghty contractions on generalized metric spaces

M Asadi, E Karapınar, A Kumar, Journal of Inequalities and Applications 2014 (1), 423

[20] Results on Contractions of Reich Type in Graphical -Metric Spaces with Applications

M Younis, D Singh, M Asadi, V Josh, Filomat 33 (17), 5723–5735

[21] Discontinuity of Control Function in the (F, φ, θ)-Contraction in Metric Spaces

M Asadi, FILOMAT 31 (17), 5427-5433

[22] Some fixed point and common fixed point theorems on G-metric spaces

M Asadi, P Salimi, Nonlinear Functional Analysis and Applications 21 (3), 523-530

[23] New fixed point results on Branciari metric spaces

M ESHRAGHISAMANI, M Vaezpour, M Asadi, Journal of Mathematical Analysis 8 (6), 132-141

[24] $F(\psi, \theta)$-contractions for α-admissible mappings on metric spaces and related fixed point results

H Monfared, M Asadi, M Azhini, Communications in Nonlinear Analysis (CNA) 2 (1), 86-94

[25] Lower semi-continuity in a generalized metric space

M Asadi, A Khalesi, Advances in the Theory of Nonlinear Analysis and its Applications(ATNAA) 1 …