

## Some Fixed Point results of Rational Type-Contraction Mapping in S-Metric Space

### Abstract:

In this paper, we demonstrate the existence of some fixed points of rational type contraction in context of S-metric space and we examine the T-stability of the P-property for some mapping. Also, we present few examples to illustrate the validity of the results obtained in the paper.

**Keywords:** Fixed point, rational type contraction, S-metric space.

**MSC:** 54H25, 47H10

### 1. Introduction and Preliminaries

Fixed point theory is an active area of research with various application in real life. One of the main approaches used in this theory to demonstrate the existence and uniqueness of fixed point is contraction. In 1989, Bakhtin [3] was the first who introduced the concept of b-metrics pace. In 1993, Czerwik [6] extended the results of Bakhtin [7] and gave generalization of Banach fixed point theorem in b-metric spaces. In 2012, the idea of S- metric space was established by Sedghi *et al.* [14], who also proved fixed point theorems there in. Manoj K. *et al.* [12], proved fixed point theorem by using altering distance function in S-metric space. More well-known results in the direction of S-metric space are involved in (refer [15]-[18]).

**Theorem 1.1 [12]:** “Let  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping on a complete S-metric space  $(\mathcal{X}, \mathcal{S})$  such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda \mathcal{S}(u, u, v) + \eta \frac{\mathcal{S}(uu, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)},$$

for all  $u, v \in \mathcal{X}$ ,  $\lambda, \eta > 0$ ,  $\lambda + \eta < 1$ . Then  $\mathcal{T}$  possess a fixed point  $w \in \mathcal{X}$  which is unique.”

**Theorem 1.2 [12]:** “Let  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping on a complete S-metric space  $(\mathcal{X}, \mathcal{S})$  such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda \mathcal{S}(u, u, v) + \eta \frac{\mathcal{S}(v, v, \mathcal{T}v)[1 + \mathcal{S}(u, u, \mathcal{T}u)]}{1 + \mathcal{S}(u, u, v)},$$

for all  $u, v \in X$ ,  $\lambda, \eta > 0$ ,  $\lambda + \eta < 1$ . Then  $\mathcal{T}$  possess a fixed point  $w \in X$  which is unique.”

Furthermore, we proceed by reviewing some important definitions and key terms that would be used throughout our discussion.

**Definition 1.3 [14]:** “Let  $X$  be a non-empty set. An S-metric on  $X$  is a mapping  $\mathcal{S}: X \times X \times X \rightarrow \mathbb{R}^+$  which satisfies the following condition:

$$(\mathcal{S}_1) \mathcal{S}(u, v, w) = 0 \text{ if and only if } u = v = w = 0;$$

$$(\mathcal{S}_2) \mathcal{S}(u, v, w) \leq \mathcal{S}(u, u, a) + \mathcal{S}(v, v, a) + \mathcal{S}(w, w, a), \text{ for all } u, v, w, a \in X.$$

The pair  $(X, \mathcal{S})$  is called an S-metric space.”

**Example 1.4 [14]:** “Let  $X = \mathbb{R}$ . Then  $\mathcal{S}(u, v, w)$  is an S-metric on  $\mathbb{R}$  given by  $\mathcal{S}(u, v, w) = |u - w| + |v - w|$ , which is known as usual S-metric space on  $X$ .”

**Lemma 1.5 [14]:** “If  $(X, \mathcal{S})$  is an S-metric space on a non-empty set  $X$ , then  $(X, \mathcal{S})$  satisfy the symmetric condition, that is  $\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$ , for all  $u, v \in X$ .”

**Definition 1.6 [14]** “Let  $(X, \mathcal{S})$  be an S-metric space. For  $r > 0$  and  $u \in X$  we define the open ball  $B_s(u, r)$  and closed ball and  $B_s[u, r]$  with a center  $u$  and radius  $r$  as follows:

$$B_s(u, r) = \{v \in X : \mathcal{S}(v, v, u) < r\}$$

$$B_s[u, r] = \{v \in X : \mathcal{S}(v, v, u) \leq r\}.”$$

**Definition 1.7 [15]:** “A sequence  $\{u_n\}$  in  $(X, \mathcal{S})$  is said to be convergent to some point  $u \in X$ , if  $\mathcal{S}(u_n, u_n, u) \rightarrow 0$  as  $n \rightarrow \infty$ .”

**Definition 1.8 [15]:** “A sequence  $\{u_n\}$  in  $(X, \mathcal{S})$  is said to be Cauchy sequence if  $\mathcal{S}(u_n, u_n, u_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ .”

**Definition 1.9 [15]:** “An S -metric space  $(X, \mathcal{S})$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .”

**Lemma 1.10 [15]:** “Let  $(X, \mathcal{S})$  be an S-metric space. If  $u_n \rightarrow u$  and  $v_n \rightarrow v$  then  $\mathcal{S}(u_n, u_n, v_n) \rightarrow \mathcal{S}(u, u, v)$ .”

**Lemma 1.11 [16]:** “Let  $(X, \mathcal{S})$  be an S-metric space and  $\{u_n\}$  is a convergent sequence in  $X$ . Then  $\lim_{n \rightarrow \infty} u_n$  is unique.”

**Definition 1.12 [16]:** “Let  $(\mathcal{X}, \mathcal{S})$  be S-metric pace. A map  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  is said to be contraction if there exists a constant  $k \in [0,1)$  such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda \mathcal{S}(u, u, v), \text{ for all } u, v \in \mathcal{X}.”$$

**Lemma 1.13 [16]:** “If  $\{u_n\}$  is a sequence of elements from S-metric space  $(\mathcal{X}, \mathcal{S})$  satisfying the following property  $\mathcal{S}(u_n, u_n, u_{n+1}) \leq k \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$ , for each  $k \in [0, 1)$  where  $n \in \mathbb{N}$ , then  $\{u_n\}$  is a Cauchy sequence.”

## 2. Main Results

In this section, we establish fixed points of rational type contractions in the context of S-metric spaces and demonstrates that the P property is T-stable for some mappings. In order to show the relevance of the conclusions drawn in this work, we also provide a few examples.

**Theorem 2.1:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space and  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(u, u, \mathcal{T}v)}{\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)}, \quad (2.1)$$

for all  $u, v \in \mathcal{X}$  and  $a_1, a_2 \geq 0$ ,  $\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u) \neq 0$  with  $a_1 + a_2 < 1$ . Then,  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Let  $u_0$  be an arbitrary in  $\mathcal{X}$ , we define a sequence  $\{u_n\}$  in  $\mathcal{X}$  such that  $\mathcal{T}u_n = u_{n+1}$  for all  $n = 1, 2, \dots$ . From condition (2.1) with  $u = u_n$  and  $v = u_{n-1}$ , Therefore

$$\begin{aligned} \mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\ &\quad + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\ &\quad + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1}). \end{aligned}$$

It follows that

$$(1 - a_2)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \quad (2.2)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1}{1-a_2}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \left(\frac{a_1}{1-a_2}\right)$ . In view of  $a_1 + a_2 < 1$ , then  $0 \leq \lambda < 1$ . Thus, by Lemma 1.13,  $\{u_n\}$  is a Cauchy sequence in  $\mathcal{X}$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

By (2.2), it is easy to see that

$$\mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \quad (2.3)$$

$$\begin{aligned} &\leq a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)} \\ &\leq a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})}. \end{aligned} \quad (2.4)$$

Taking the limit as  $n \rightarrow \infty$  on both side of (2.4), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = 0$ .

That is,  $u_n \rightarrow \mathcal{T}u^*$ . Hence,  $\mathcal{T}u^* = u^*$ ,  $u^*$  is a fixed point of  $\mathcal{T}$ .

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point  $v^*$ , then by (2.1), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)} \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(u^*, u^*, v^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)} \end{aligned}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq a_1 \mathcal{S}(u^*, u^*, v^*). \quad (2.5)$$

Since  $a_1 + a_2 < 1$  implies  $p < 1$ .

Therefore, we obtain that  $\mathcal{S}(u^*, u^*, v^*) = 0$ , i.e.,  $u^* = v^*$ .

Hence the fixed point is unique.

This completes the proof.  $\blacksquare$

**Example 2.2:** Let  $\mathcal{X} = [0.1]$  be equipped with complete S-metric space define by

$$\mathcal{S}(u, v, w) = (|u - v| + |u - w| + |v - w|)^2.$$

Consider a mapping  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  defined by

$$\mathcal{T}(u) = \frac{1}{36}u^2e^{-u^2},$$

for all  $u, v, w \in \mathcal{X}$ .

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &= (|\mathcal{T}u - \mathcal{T}u| + |\mathcal{T}u - \mathcal{T}v| + |\mathcal{T}u - \mathcal{T}v|)^2 \\ &= (2|\mathcal{T}u - \mathcal{T}v|)^2 \\ &= 4 \left| \frac{1}{36}u^2e^{-u^2} - \frac{1}{36}v^2e^{-v^2} \right|^2 = \left| \frac{1}{18}u^2e^{-u^2} - \frac{1}{18}v^2e^{-v^2} \right|^2 \\ &\leq \frac{1}{9} |u^2e^{-u^2} - v^2e^{-v^2}|^2 \\ &\leq \frac{4}{9} |u - v|^2 = \frac{1}{9} |2(u - v)|^2 \\ &\leq \frac{1}{3} \mathcal{S}(u, v, w) \\ &\leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(u, u, \mathcal{T}v)}{\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)}. \end{aligned}$$

Clearly by taking  $a_2 = \frac{1}{2}$ , we have  $a_1 + a_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1$ . Then, from Theorem 2.1 we conclude that,  $\mathcal{T}$  has a unique fixed point. Also, 0 is the only fixed point of  $\mathcal{T}$ .

**Theorem 2.3:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space and  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping such that

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &\leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)} \\ &\quad + a_3 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(u, u, \mathcal{T}v)}{\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)}, \end{aligned}$$

(2.6)

where  $a_1, a_2, a_3$  are non-negative constant with  $a_1 + a_2 + a_3 < 1$ . Then,  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Choose  $u_0 \in X$  and construct a Picard iterative sequence  $\{u_n\}$  as  $\mathcal{T}u_n = u_{n+1}$ . If there exists  $n_0 \in \mathbb{N}$  such that  $u_{n_0} = u_{n_0+1}$ , then  $u_{n_0} = u_{n_0+1} = \mathcal{T}u_{n_0}$ , i.e.,  $u_{n_0}$  is a fixed point of  $\mathcal{T}$ . Next, without loss of generality, let  $u_n \neq u_{n+1}$  for all  $n \in \mathbb{N}$ , Using (2.6), we get

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&+ a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})} \\
&+ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})} \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&+ a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)} \\
&+ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)} \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_3 \mathcal{S}(u_n, u_n, u_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
(1 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) &\leq (a_1 + a_2)\mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
(2.7)
\end{aligned}$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1 + a_2}{1 - a_3}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \frac{a_1 + a_2}{1 - a_3}$ . In view of  $a_1 + a_2 + a_3 < 1$ , we have  $0 \leq \lambda < 1$ . Thus, from Lemma 1.13  $\{u_n\}$  is Cauchy sequence in  $\mathcal{X}$ . Since,  $(\mathcal{X}, \mathcal{S})$  is a complete S-metric space, so there exists some point  $u^* \in \mathcal{X}$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Again from (2.6) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.8)$$

$$\begin{aligned} &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) \\ &\quad + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)} \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) \\ &\quad + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, u_{n+1})}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u_n, u_n, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})}. \end{aligned} \quad (2.9)$$

Taking the limit as  $n \rightarrow \infty$  on both side of (2.9), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$ .

Hence,  $\mathcal{T}u^* = u^*$  it follows that  $u^*$  is a fixed point of  $\mathcal{T}$ .

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point  $v^*$ , then by (2.6), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)} \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(u^*, u^*, v^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)} \end{aligned}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq a_1 \mathcal{S}(u^*, u^*, v^*).$$

$$(2.10)$$

Since  $a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$ , we obtain that  $\mathcal{S}(u^*, u^*, v^*) = 0$ , i.e.,  $u^* = v^*$ .

Hence the fixed point is unique.

This completes the proof.  $\blacksquare$

**Theorem 2.4:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space. Let  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping satisfying

$$(2.11) \quad \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + a_3 \frac{\mathcal{S}(v, v, \mathcal{T}v)[1 + \mathcal{S}(u, u, \mathcal{T}u)]}{1 + \mathcal{S}(u, u, v)},$$

for all  $u, v \in \mathcal{X}$  and  $a_1, a_2, a_3$  are non-negative constant with  $a_1 + a_2 + a_3 < 1$ . Then  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Choose  $u_0 \in \mathcal{X}$ . Construct a sequence  $\{u_n\}$  in  $\mathcal{X}$  by  $\mathcal{T}u_n = u_{n+1}$ . For all  $n \in \mathbb{N}$ , from condition (2.11) with  $u = u_n$  and  $v = u_{n-1}$ , we have

$$\begin{aligned} \mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)[1 + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})]}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, u_{n+1})[1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)]}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1}) + a_3 \mathcal{S}(u_n, u_n, u_{n+1}). \end{aligned}$$

It follows that

$$(2.12) \quad (1 - a_2 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left( \frac{a_1}{1 - a_2 - a_3} \right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \frac{a_1}{1-a_2-a_3}$ . In view of  $a_1 + a_2 + a_3 < 1$ , we have  $0 \leq \lambda < 1$ . Thus, from Lemma 1.13  $\{u_n\}$  is Cauchy sequence in  $\mathcal{X}$ . Since,  $(\mathcal{X}, \mathcal{S})$  is a complete S-metric space, so there exists some point  $u^* \in \mathcal{X}$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Again from (2.11) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.13)$$

$$\begin{aligned} &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n) \mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*) [1 + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)]}{1 + \mathcal{S}(u_n, u_n, u^*)} \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1}) \mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*) [1 + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n+1})]}{1 + \mathcal{S}(u_n, u_n, u^*)} \end{aligned}$$

(2.14) Taking the limit as  $n \rightarrow \infty$  on both side of (2.14), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$ .

Hence,  $\mathcal{T}u^* = u^*$  it follows that  $u^*$  is a fixed point of  $\mathcal{T}$ .

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point  $v^*$ , then by (2.11), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_1 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_1 \frac{\mathcal{S}(v^*, v^*, \mathcal{T}v^*) [1 + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)} \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*) \mathcal{S}(v^*, v^*, v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(v^*, v^*, v^*) [1 + \mathcal{S}(u^*, u^*, u^*)]}{1 + \mathcal{S}(u^*, u^*, v^*)} \end{aligned}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq a_1 \mathcal{S}(u^*, u^*, v^*).$$

Since  $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$ , thus, we obtain  $\mathcal{S}(u^*, u^*, v^*) = 0$ , i.e.,  $u^* = v^*$ .

Hence, we proved that  $\mathcal{T}$  have a unique fixed point in  $\mathcal{X}$ .

Here completes the proof.  $\blacksquare$

**Example 2.5:** Let  $\mathcal{X} = [0,1]$  and  $(\mathcal{X}, \mathcal{S})$  be a usual S-metric space which is complete, define by

$$\mathcal{S}(u, v, w) = |u - w| + |v - w|.$$

Consider a mapping  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be define as  $(u) = \frac{u}{8}$ , for all  $u, v, w \in \mathcal{X}$ .

Obviously,

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) = 2 |\mathcal{T}u - \mathcal{T}v| = 2 \left| \frac{u}{8} - \frac{v}{8} \right| = \frac{1}{4} |u - v|,$$

$$\mathcal{S}(u, u, v) = 2 |u - v|.$$

Also,

$$\mathcal{S}(u, u, \mathcal{T}u) = 2 |u - \mathcal{T}u| = 2 \left| u - \frac{u}{8} \right| = \frac{7u}{4},$$

$$\mathcal{S}(v, v, \mathcal{T}v) = 2 |v - \mathcal{T}v| = 2 \left| v - \frac{v}{8} \right| = \frac{7v}{4},$$

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &= \frac{1}{4} |u - v| \\ &= \frac{1}{8} 2 |u - v| \\ &\leq \frac{1}{8} \mathcal{S}(u, u, v) + \frac{1}{4} \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + \frac{1}{7} \frac{\mathcal{S}(v, v, \mathcal{T}v)[1 + \mathcal{S}(u, u, \mathcal{T}u)]}{1 + \mathcal{S}(u, u, v)}. \end{aligned}$$

It is clear that,  $a_1 + a_2 + a_3 = \frac{1}{8} + \frac{1}{4} + \frac{1}{7} = \frac{29}{56} < 1$ . Thus, we conclude that inequality (2.11) of Theorem 2.4 remains valid. Hence,  $\mathcal{T}$  has a unique fixed point and the fixed point is 0.

**Theorem 2.6:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space and  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping satisfying the following condition

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &\leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + a_3 \frac{\mathcal{S}(u, u, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, v)} \\ &\quad + a_4 [\mathcal{S}(u, u, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)] + a_5 [\mathcal{S}(v, v, \mathcal{T}u) + \mathcal{S}(u, u, \mathcal{T}v)], \end{aligned} \tag{2.15}$$

for all  $u, v \in \mathcal{X}$  and  $a_1, a_2, a_3, a_4, a_5$  are non-negative constant  $a_1 + a_2 + a_3 + 2 a_4 + 3 a_5 < 1$ . Then,  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Choose  $u_0 \in X$ . Construct a sequence  $\{u_n\}$  in  $X$  by  $\mathcal{T}u_n = u_{n+1}$ .

For all  $n \in \mathbb{N}$ , from condition (2.15) with  $u = u_n$  and  $v = u_{n-1}$ , we have

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\
&\quad + a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\
&\quad + a_4 [\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)] \\
&\quad + a_5 [\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)] \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\
&\quad + a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} + a_4 [\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})] \\
&\quad + a_5 [\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})] \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1}) \\
&\quad + a_4 [\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})] \\
&\quad + a_5 [2 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})].
\end{aligned}$$

It follows that

$$(1 - a_2 - a_4 - a_5)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (a_1 + a_4 + 2a_5) \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \quad (2.16)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left( \frac{a_1 + a_4 + 2a_5}{1 - a_2 - a_4 - a_5} \right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \frac{a_1 + a_4 + 2a_5}{1 - a_2 - a_4 - a_5}$ . In view of  $a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1$ , we have  $0 \leq \lambda < 1$ . Thus, from

Lemma 1.13,  $\{u_n\}$  is Cauchy sequence in  $X$ . Since,  $(X, \mathcal{S})$  is a complete S-metric space, so there exists some point  $u^* \in X$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Again from (2.15) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.17)$$

$$\begin{aligned} &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n) \mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, u^*)} + a_4 [\mathcal{S}(u_n, u_n, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)] \\ &\quad \quad \quad + a_5 [\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*)] \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1}) \mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &\quad + a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) \mathcal{S}(u^*, u^*, u_{n+1})}{1 + \mathcal{S}(u_n, u_n, u^*)} + a_4 [\mathcal{S}(u_n, u_n, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)] \\ &\quad \quad \quad + a_5 [\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*)] \end{aligned} \quad (2.18)$$

Taking the limit as  $n \rightarrow \infty$  on both side of (2.18), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$ .

Hence,  $\mathcal{T}u^* = u^*$  it follows that  $u^*$  is a fixed point of  $\mathcal{T}$ .

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point  $v^*$ , then by (2.15), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, v^*)} \\ &\quad + a_4 [\mathcal{S}(u^*, u^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)] + a_5 [\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}v^*)] \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*) \mathcal{S}(v^*, v^*, v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(u^*, u^*, v^*) \mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*)} \\ &\quad \quad \quad + a_4 [\mathcal{S}(u^*, u^*, u^*) + \mathcal{S}(v^*, v^*, v^*)] + a_5 [\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(u^*, u^*, v^*)] \end{aligned}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_3 + 2a_5) \mathcal{S}(u^*, u^*, v^*).$$

$$(2.19)$$

Since  $0 < a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1 \Rightarrow a_1 + a_3 + 2a_5 < 1$ , thus, we obtain  $\mathcal{S}(u^*, u^*, v^*) = 0$ , which further implies  $u^* = v^*$ .

Therefore,  $\mathcal{T}$  have a unique fixed point in  $\mathcal{X}$ .

Here completes the proof.  $\blacksquare$

**Theorem 2.7:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space and  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a self map that satisfies the following inequality

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &\leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + a_3 \frac{\mathcal{S}(u, u, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, v)} \\ &+ a_4 \frac{\mathcal{S}(v, v, \mathcal{T}v)[1 + \mathcal{S}(u, u, \mathcal{T}u)]}{1 + \mathcal{S}(u, u, v)}, \end{aligned} \quad (2.20)$$

for all  $u, v \in \mathcal{X}$  and  $a_1, a_2, a_3, a_4$  are non-negative constant  $a_1 + a_2 + a_3 + a_4 < 1$ . Then,  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Choose  $u_0 \in X$  and construct a Picard iterative sequence  $\{u_n\}$  as  $\mathcal{T}u_n = u_{n+1}$ .

For all  $n \in \mathbb{N}$ , from condition (2.20) with  $u = u_n$  and  $v = u_{n-1}$ , we have

$$\begin{aligned} \mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &+ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &+ a_4 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)[1 + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})]}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &+ a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} + a_4 \frac{\mathcal{S}(u_n, u_n, u_{n+1})[1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)]}{1 + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)} \\ &\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_n, u_n, u_{n+1}) + a_4 \mathcal{S}(u_n, u_n, u_{n+1}), \end{aligned}$$

which further implies,

$$(1 - a_2 - a_4)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \quad (2.21)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1}{1-a_2-a_4}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \frac{a_1}{1-a_2-a_4}$ . In view of  $a_1 + a_2 + a_3 + a_4 < 1$ , we have  $0 \leq \lambda < 1$ . Thus, from Lemma 1.13,  $\{u_n\}$  is Cauchy sequence in  $\mathcal{X}$ . Since,  $(\mathcal{X}, \mathcal{S})$  is a complete S-metric space, so there exists some point  $u^* \in \mathcal{X}$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Again from (2.20) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.22)$$

$$\begin{aligned} &= 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &+ a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, u^*)} + a_4 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)[1+\mathcal{S}(u_n, u_n, \mathcal{T}u_n)]}{1+\mathcal{S}(u_n, u_n, u^*)} \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)} \\ &+ a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, u_{n+1})}{1+\mathcal{S}(u_n, u_n, u^*)} + a_4 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)[1+\mathcal{S}(u_n, u_n, u_{n+1})]}{1+\mathcal{S}(u_n, u_n, u^*)}. \end{aligned}$$

$$(2.23)$$

Taking the limit as  $n \rightarrow \infty$  on both side of (2.23), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$ .

Hence,  $\mathcal{T}u^* = u^*$  it follows that  $u^*$  is a fixed point of  $\mathcal{T}$ .

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point  $v^*$ , then by (2.20), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_1 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_1 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}v^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, v^*)} \end{aligned}$$

$$\begin{aligned}
& +a_4 \frac{\mathcal{S}(v^*, v^*, \mathcal{T}v^*)[1+\mathcal{S}(u^*, u^*, \mathcal{T}u^*)]}{1+\mathcal{S}(u^*, u^*, v^*)} \\
& \leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, v^*)}{\mathcal{S}(u^*, u^*, v^*)} + a_3 \frac{\mathcal{S}(u^*, u^*, v^*)\mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*)} \\
& +a_4 \frac{\mathcal{S}(v^*, v^*, v^*)[1+\mathcal{S}(u^*, u^*, u^*)]}{1+\mathcal{S}(u^*, u^*, v^*)}.
\end{aligned}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_3) \mathcal{S}(u^*, u^*, v^*)$$

$$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_2 + a_3 + a_4) \mathcal{S}(u^*, u^*, v^*),$$

(2.24)

a contradiction.

Thus, we obtain  $\mathcal{S}(u^*, u^*, v^*) = 0$ , which further implies  $u^* = v^*$ .

Therefore,  $\mathcal{T}$  have a unique fixed point in  $\mathcal{X}$ .

Here completes the proof.  $\blacksquare$

**Example 2.8:** Let  $\mathcal{X} = [0,1]$  be equipped with complete S-metric space define by

$$\mathcal{S}(u, v, w) = (|u - v| + |u - w| + |v - w|)^2.$$

Let the mapping  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be defined by

$$\mathcal{T}(u) = \frac{u}{5}.$$

Then, for all  $u, v, w \in \mathcal{X}$ , we have,

$$\mathcal{S}(u, u, v) = 4|u - v|^2,$$

$$\mathcal{S}(u, u, \mathcal{T}u) = 4|u - \mathcal{T}u|^2 = 4 \left| u - \frac{u}{5} \right|^2 = \frac{16}{25} u^2$$

$$\mathcal{S}(v, v, \mathcal{T}v) = 4|v - \mathcal{T}v|^2 = 4 \left| v - \frac{v}{5} \right|^2 = \frac{16}{25} v^2$$

Also,

$$\begin{aligned}
\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &= (|\mathcal{T}u - \mathcal{T}u| + |\mathcal{T}u - \mathcal{T}v| + |\mathcal{T}u - \mathcal{T}v|)^2 \\
&= (2 |\mathcal{T}u - \mathcal{T}v|)^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{4}{25} |u - v|^2 \\
&\leq \frac{1}{25} \mathcal{S}(u, v, w) + \frac{4}{25} \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(v, v, \mathcal{T}v)}{\mathcal{S}(u, u, v)} + \frac{2}{5} \frac{\mathcal{S}(v, v, \mathcal{T}v)[1 + \mathcal{S}(u, u, \mathcal{T}u)]}{1 + \mathcal{S}(u, u, v)}
\end{aligned}$$

Clearly, we have  $a_1 + a_2 + a_3 = \frac{1}{25} + \frac{4}{25} + \frac{2}{5} = \frac{3}{5} < 1$ . Then, from Theorem 2.7 we conclude that,  $\mathcal{T}$  has a unique fixed point. Also,  $0$  is the only fixed point of  $\mathcal{T}$ .

**Theorem 2.9:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space. Let  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping satisfying

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1 \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}v)\mathcal{S}(v, v, \mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)} + a_3 \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v), \quad (2.25)$$

for all  $u, v \in \mathcal{X}$  and  $a_1, a_2, a_3 \geq 0$ .  $\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u) \neq 0$  with  $a_1 + a_2 + a_3 < 1$ . Then  $\mathcal{T}$  has a unique fixed point  $\mathcal{X}$ .

**Proof:** Choose  $u_0$  as an arbitrary point in  $X$ . We define a sequence  $\{u_n\}$  in  $\mathcal{X}$  by  $\mathcal{T}u_n = u_{n+1}$ . Then for all  $n \in \mathbb{N}$ , from condition (2.25) with  $u = u_n$  and  $v = u_{n-1}$ , we have

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&\quad + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})} \\
&\quad + a_3 \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&\quad + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)} \\
&\quad + a_3 \mathcal{S}(u_n, u_n, u_{n+1}) \\
&\leq a_1 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_3 \mathcal{S}(u_n, u_n, u_{n+1}).
\end{aligned}$$

Which further implies

$$\begin{aligned}
(1 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) &\leq (a_1 + a_2)\mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
(2.26)
\end{aligned}$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left( \frac{a_1 + a_2}{1 - a_3} \right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put  $\lambda = \frac{a_1 + a_2}{1 - a_3}$ . In view of  $a_1 + a_2 + a_3 < 1$ , we have  $0 \leq \lambda < 1$ . Thus, from Lemma 1.13  $\{u_n\}$  is Cauchy sequence in  $\mathcal{X}$ . Since,  $(\mathcal{X}, \mathcal{S})$  is a complete S-metric space, so there exists some point  $u^* \in \mathcal{X}$  such that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Again from (2.25) it is easy to see that

$$\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.27)$$

$$\begin{aligned} &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) \\ &\quad + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n) \mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*) \mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u_n)} \\ &\quad + a_3 \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\ &\leq 2 \mathcal{S}(u^*, u^*, u_{n+1}) + a_1 \mathcal{S}(u_n, u_n, u^*) \\ &\quad + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1}) \mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*) \mathcal{S}(u^*, u^*, u_{n+1})}{\mathcal{S}(u_n, u_n, \mathcal{T}u^*) + \mathcal{S}(u^*, u^*, u_{n+1})} \\ &\quad + a_3 \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \end{aligned}$$

(2.28) Taking the limit as  $n \rightarrow \infty$  on both side of (2.28), we have  $\lim_{n \rightarrow \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$ .

Hence,  $\mathcal{T}u^* = u^*$  it follows that  $u^*$  is a fixed point of  $\mathcal{T}$ .

Finally, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point  $v^*$ , then by (2.25), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*) \mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*) \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)} \\ &\quad + a_3 \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2 \frac{\mathcal{S}(u^*, u^*, u^*) \mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, v^*) \mathcal{S}(v^*, v^*, u^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)} \end{aligned}$$

$$+a_3 \mathcal{S}(u^*, u^*, v^*)$$

$$\mathcal{S}(u^*, u^*, v^*) \leq (a_1 + a_3) \mathcal{S}(u^*, u^*, v^*).$$

(2.29)

Since  $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 + a_3 < 1$ , thus, we obtain  $\mathcal{S}(u^*, u^*, v^*) = 0$ , i.e.,  $u^* = v^*$ .

Hence, we proved that  $\mathcal{T}$  have a unique fixed point in  $\mathcal{X}$ .

Here completes the proof. ■

**Remark 2.10.**

1. If we put  $a_2 = a_3 = a_4 = a_5 = 0$  in Theorem 2.6, we get the Banach Theorem [2].
2. If we put  $a_1 = a_2 = a_3 = a_5 = 0$  in Theorem 2.6, we get the Kanan Theorem [11].
3. If we put  $a_2 = a_3 = a_5 = 0$  in Theorem 2.6, we get the Fisher Theorem [9].
4. If we put  $a_1 = a_2 = a_3 = a_4 = 0$  in Theorem 2.6, we get the result of Chaterjee Theorem [5].
5. If we put  $a_2 = a_3 = 0$  in Theorem 2.9, we get the result of Dass and Gupta Theorem [8].

**Theorem 2.11:** Let  $(\mathcal{X}, \mathcal{S})$  be a complete S-metric space and  $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$  be a mapping such that  $F(\mathcal{T}) \neq \emptyset$  and that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2u) \leq \lambda \mathcal{S}(u, u, \mathcal{T}u),$$

(2.30)

for all  $u \in \mathcal{X}$ , where  $0 \leq \lambda < 1$  is a constant. Then  $\mathcal{T}$  has the P property.

**Proof:** We always assume that  $n > 1$ . Since the statement for  $n = 1$  is trivial. Let  $w \in F(\mathcal{T}^n)$ .

By the hypotheses, we get

$$\begin{aligned} \mathcal{S}(w, w, \mathcal{T}w) &= \mathcal{S}(\mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}^2\mathcal{T}^{n-1}w) \\ &\leq \lambda \mathcal{S}(\mathcal{T}^{n-1}w, \mathcal{T}^{n-1}w, \mathcal{T}^n w) \\ &\leq \lambda \mathcal{S}(\mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}^2\mathcal{T}^{n-2}w) \\ &\leq \lambda^2 \mathcal{S}(\mathcal{T}^{n-2}w, \mathcal{T}^{n-2}w, \mathcal{T}^{n-1}w) \\ &\leq \dots \leq \lambda^n \mathcal{S}(w, w, \mathcal{T}w) \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $\mathcal{S}(w, w, \mathcal{T}w) = 0$ , that is  $\mathcal{T}w = w$ .

**Theorem 2.12:** Under the condition of Theorem 2.3,  $\mathcal{T}$  has the P property.

**Proof:** We have to prove that the mapping  $\mathcal{T}$  satisfies (2.30). In fact, for any  $u \in \mathcal{X}$ , we have

$$\begin{aligned} \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2u) &= \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u) \\ &\leq a_1 \mathcal{S}(u, u, \mathcal{T}u) + a_2 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)} \\ &+ a_3 \frac{\mathcal{S}(u, u, \mathcal{T}u)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u)\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u)}{\mathcal{S}(u, u, \mathcal{T}\mathcal{T}u) + \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}u)} \\ &\leq a_1 \mathcal{S}(u, u, \mathcal{T}u) + a_2 \mathcal{S}(u, u, \mathcal{T}u) + a_3 \mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}\mathcal{T}u) \end{aligned}$$

$$(1 - a_3)\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2u) \leq (a_1 + a_2)\mathcal{S}(u, u, \mathcal{T}u)$$

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2u) \leq \frac{a_1 + a_2}{1 - a_3} \mathcal{S}(u, u, \mathcal{T}u)$$

Deduce that  $\lambda = \frac{a_1 + a_2}{1 - a_3}$ . Note that  $a_1 + a_2 + a_3 < 1$ , then  $\lambda < 1$ . Accordingly, (2.30) is satisfied.

Consequently, by Theorem 2.3,  $\mathcal{T}$  has the P property.

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