ON HETERODOX NON-KL GENERALIZED DIVERGENCE METRIC WITH CHARACTERISTICS IN FUZZY ENVIRONMENT

Abstract

In this paper, we suggest a novel divergence metric on a fuzzy set. Some scholars have used the fuzzy set extension and one that integrated with other theories. Axioms are proven in order to demonstrate the viability of measure. We create a way about decision-making criteria using the suggested measure and provide a workable method. We discuss the divergence metric metric for fuzzy sets in this post. The discussed properties of the proposed proposal. Multicriteria decision making is a very useful technique with a wide range of applications in the real world.

Keywords: Fuzzy Set, Divergence metric, Decision Making, etc.

Introduction

The fuzzy set theory introduced by Zadeh has achieved great success in a variety of domains. The actual world is full of uncertainty. When dealing with uncertainty and fuzziness, entropy is a crucial tool. Information theory and fuzzy theory are used to tackle problems in the study of information distribution, storage retrieval, and decision-making. Entropy is the term used to describe the measure of information theory for the first time, according to Shannon [13]. The measure of information related to the two probability distributions of discrete random variables is then evaluated by Kullback and Liebler [8], and is given as

$$D(p,q) = \sum_{i=1}^{n} p_i \; log \frac{p_i}{q_i}$$

referred to as guided divergence. The fuzzy set theory was developed by L. Zadeh [15] and is utilised in many branches of research and industry, including image processing, pattern identification, and decision-making.

Renyi [12] introduced a new divergence metric ,

$$D_\alpha\,(p,q) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i\, q_i^{1-\alpha}\,, \qquad \alpha \neq 1$$

Havrda–Charvat [6] also gave a new measure of divergence metric,

$$D_\alpha\,(p,q) = \frac{1}{\alpha - 1} \left(\sum_{i=1}^{n} p_i^{\alpha}\, q_i^{1-\alpha} - 1\right), \quad \alpha \neq 1$$

Bhandari and Pal [2] used the idea of fuzzy measure conditioning, which corresponds to Kullbck and Leibler [8] probabilistic divergence metric, to develop a fuzzy distance measure between two fuzzy sets. A measure was introduced by Bhandari and Pal [2].

$$I(A,B)=\sum_{i=1}^{n}\mu_A(x_i)log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}$$

Later, according to the exponential fuzzy entropy provided by Pal and Pal [6], Fan and Xie [5] offered discriminating of fuzzy information of fuzzy set against. A generalised divergence metric measure similar to Havrda and Charvat [6] was introduced by Kapur [9]. Along with R-norm divergence metric, Hooda and Bajaj [7] proposed a divergence metric measure. A measure of the divergence metric of two sets was provided by Bhatia and Singh [4]. Some form of fuzzy divergence metric was proposed by Tomar, Ohlan, Priya, and Tomar [1, 14]. In their article "Decision-making with Parameterized Hesitant Fuzzy Soft Set Theory," Zahari Md Rodzi and Abd Ghafur Ahmad [16] established this concept. We suggest divergence metric while keeping in mind the aforementioned literature, and certain significant properties are also investigated.

It has been demonstrated that the suggested measure is widely applicable. A brief study on the fuzzy set, measure, and divergence metric is provided in section II. A novel divergence metric measure is discussed in section III. Properties are provided together with their proof in section IV. Section V discusses how the proposed measure would be applied. The sixth segment concludes the work discussed previously.

Preliminaries

In this part, we define a few terms and notations related to divergence metric measure and fuzzy sets. We shall outline the features of the fuzzy set and its measure that will be relevant to our upcoming discussion.

Definition 1. Let $P=(p_1,p_2,\ldots\ldots p_n),p_i\geq 0$ is the set of all complete finite discrete probability distribution then measure of information was defined firstly by Shannon as .

$$H(P)=\sum_{i=1}^{n}p_i\ \log p_i\,,$$

Definition 2. Let $X=\{x_1,x_2,\ldots\ldots x_n\}$be universe of discourse then $A=\{<x,\mu_A(x_i)>/x\in X\}$ is called fuzzy set where $\mu_A(x_i)\ :X\rightarrow[0,1]$is a membership function defined as follows

$$\mu_A(x_i)=\ 0\ if\ x\ \notin A$$

$$\mu_A(x_i)=\ 0\ if\ x\in A$$

$$\mu_A(x_i)=\ 0.5\ if\ x\ \notin A\text{ or }x\in A$$

Some notation for two fuzzy sets

1. $A\cup B=\{<x,\max(\mu_A(x),\mu_B(x))>/x\in X\}$
2. $A\cap B=\{<x,\min(\mu_A(x),\mu_B(x))>/x\in X\}$
3. $A=B=\{<x,\mu_A(x)=\mu_B(x)>/x\in X\}$
4. $A.B=\{<x,\mu_A(x).\mu_B(x)>/x\in X\}$

5. $A^c = \{< x, \mu_A(x) = 1 - \mu_A(x) > / x \in X$

Definition 3. Let $X = \{x_1, x_2, \ldots\ldots\ldots, x_n\}$ be universe of discourse and F(X) be the set of all family subset. A mapping $I: F(X) \times F(X) \rightarrow R$ is called divergence measure between fuzzy sets if

i. $I(A:B) \geq 0$
ii. $I(A:B) = I(B:A)$
iii. $I(A:B) = 0 \;\; iff\; A = B$
iv. $Max\{I(A \cup C, B \cup C), I(A \cap C, B \cap C)\} \leq I(A:B)$

Bajaj et.al. [3] define the measure of fuzzy divergence metric as

$$I_\alpha (A:B) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \log[\mu_A^\alpha (x_i) \mu_B^{1-\alpha}(x_i) + (1 - \mu_A(x_i))^\alpha (1 - \mu_B(x_i))^{1-\alpha}]$$

$$I_{\alpha,\beta} (A:B) = \frac{1}{1 - 2^{\beta - 1}} \sum_{i=1}^{n} \left\{ \left[\mu_A^\alpha (x_i) \mu_B^{1-\alpha} (x_i) + \left(1 - \mu_A(x_i)\right)^\alpha \left(1 - \mu_B(x_i)\right)^{1-\alpha} \right]^{\frac{\beta - 1}{\alpha - 1}} - 1 \right\}$$

Entropy measure for fuzzy sets was introduced by Prakash et al. [10] as

$$H_\alpha^\beta (A) = \frac{1}{(1-\alpha)\beta} \sum_{i=1}^{n} \left\{ \left[\mu_A^\alpha (x_i) + \left(1 - \mu_A(x_i)\right)^\alpha \right]^\beta - 1 \right\} \quad ; \alpha > 0, \alpha \neq 1, \beta \neq 0$$

2. Our Results

New Divergence metric Measure

In accordance with Prakash et al. [10], we suggest the following fuzzy divergence metric measure:

$$H_{\alpha,\beta} (A, B) = \frac{1}{(\alpha - 1)\beta} \sum_{i=1}^{n} \log\{ \left[\mu_A^\alpha (x_i) \mu_B^{1-\alpha} (x_i) + \left(1 - \mu_A(x_i)\right)^\alpha \left(1 - \mu_B(x_i)\right)^{1-\alpha} \right]^\beta - 1\} \quad ; \alpha > 0, \alpha \neq 1, \beta \neq 0 \qquad (1)$$

Theorem 1. Show that $H_{\alpha,\beta} (A, B)$ is valid measure of fuzzy divergence metric.

Proof. To show that proposed measure in (1) is valid we have to prove following axioms:

1. We can clearly check in figure that $H_{\alpha,\beta}\ (A,B)$ is non – negative.

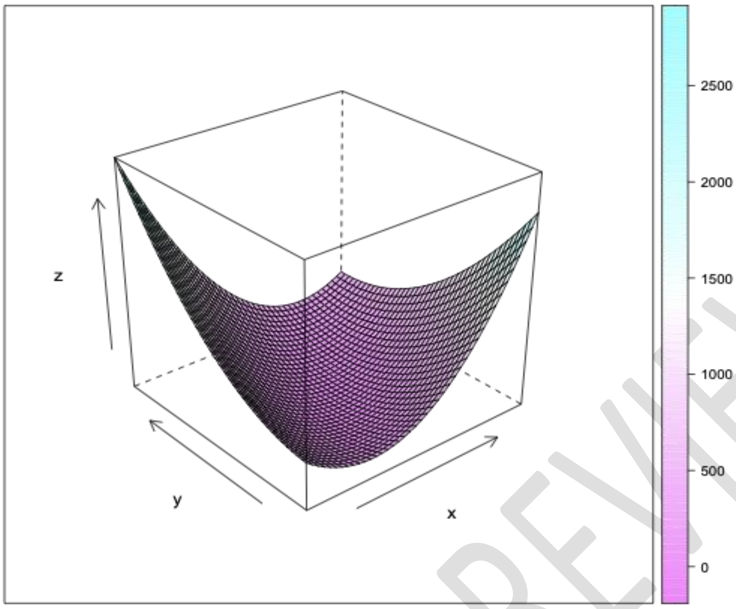

Figure 1. $H_{\alpha,\beta}\ (A,B)$

2. $H_{\alpha,\beta}\ (A,B) \neq H_{\alpha,\beta}\ (B,A)$

3. $H_{\alpha,\beta}\ (A,B)\ = 0$, if A = B

4. We have to check the convexity of $H_{\alpha,\beta}\ (A,B)$

So now,

$$\frac{\partial H_{\alpha,\beta}}{\partial \mu_A\ (x_i\ )} = \Big\{\alpha\beta\Big[\mu_A^{\alpha-1}\ (x_i\ )\mu_\beta^{1-\alpha}\,(x_i\ )$$

$$+\left(1-\mu_A\ (x_i\ )\right)^{\alpha-1}\ \left(1-\mu_\beta\ (x_i\ )\right)^{1-\alpha}\Big]\Big[\mu_A^{\alpha}\ (x_i\ )\mu_B^{1-\alpha}\,(x_i\ )$$

$$+\left(1-\mu_A\ (x_i\ )\right)^{\alpha}\ \left(1-\mu_B\ (x_i\ )\right)^{1-\alpha}\Big]^{\beta-2}\Big\}$$

$$\frac{\partial^2\ H_{\alpha,\beta}}{\partial \mu_A^2\ (x_i\,)} = \Big\{\alpha(\alpha-1)\beta\Big[\mu_A^{\alpha-2}\ (x_i\ )\mu_B^{1-\alpha}\ (x_i\ )$$

$$+\left(1-\mu_A\ (x_i\ )\right)^{\alpha-2}\ \left(1-\mu_B\ (x_i)\right)^{1-\alpha}\Big]\Big[\mu_A^{\alpha}\ (x_i\ )\mu_B^{1-\alpha}\ (x_i\ )$$

$$+\left(1-\mu_A\ \ (x_i\ )\right)^{\alpha}\ \ (1-\mu_B\ \ (x_i\ )^{1-\alpha}\ \Big]^{\beta-2}$$

$$+\,\alpha(\beta-1)\beta\Big[\mu_A^{\alpha-1}\,(x_i\ )\mu_B^{1-\alpha}\ \ (x_i\,)$$

$$+\left(1-\mu_A\ \ (x_i)\right)^{\alpha-1}\left(1-\mu_B\ (x_i\ )\right)^{1-\alpha}\Big]\Big[\mu_A^{\alpha}\ \ (x_i\ )\mu_B^{1-\alpha}\ (x_i\ )$$

$$+\left(1-\mu_A\,(x_i\ )\right)^{\alpha}\ \ \left(1-\mu_B\ (x_i)\right)^{1-\alpha}\Big]^{\beta-3}\ \Big\}$$

$$\frac{\partial^2 H_{\alpha,\beta}}{\partial \mu_A^2 (x_i)} > 0 \qquad for\ \alpha > 0, \quad \beta > 0, \quad \alpha \neq 1, \quad \beta \neq 1,2$$

Similarly we can show that

$$\Rightarrow \frac{\partial^2 H_{\alpha,\beta}}{\partial \mu_B^2 (x_i)} > 0 \quad for\ \alpha > 0, \quad \beta > 0, \quad \alpha \neq 1,2 \quad \beta \neq 1$$

Therefore, it follows that the proposed measures are sound axiomatically.

Some Important Properties

Assume that the family of all fuzzy set of universe X, is denoted by FS(X) and A, B, C $\in$ FS(X) is given

$$A = [< x, \mu_A (x) >/x \in X]$$

$$B = [< x, \mu_B (x) >/x \in X]$$

$$C = [< x, \mu_C (x) >/x \in X]$$

and we have

$$\Delta_1 = [x_i / x_i \in X, \mu_A (x_i) \geq \mu_B (x_i)]$$

$$\Delta_2 = [x_i / x_i \in X, \mu_A (x_i) < \mu_B (x_i)]$$

Theorem 2. Prove that proposed measure in (1) satisfies the following properties:

1. $H_{\alpha,\beta} (A \cup B, A) + H_{\alpha,\beta} (A \cap B, A) = H_{\alpha,\beta} (B, A)$
2. $H_{\alpha,\beta} (A, A \cap B) = H_{\alpha,\beta} (A \cup B, B)$
3. $H_{\alpha,\beta} (A, A \cup B) = H_{\alpha,\beta} (A \cap B, B)$
4. $H_{\alpha,\beta} (A \cup B, C) + H_{\alpha,\beta} (A \cap B, C) = H_{\alpha,\beta} (A, C) + H_{\alpha,\beta} (B, C)$
5. $H_{\alpha,\beta} (A \cup B, A \cap B) = H_{\alpha,\beta} (A \cup B, B) + H_{\alpha,\beta} (B, A \cap B)$
6. $H_{\alpha,\beta} (A, A^C) = H_{\alpha,\beta} (A^C, A)$
7. $H_{\alpha,\beta} (A^C, B^C) = H_{\alpha,\beta} (A, B)$
8. $H_{\alpha,\beta} (A, B^C) = H_{\alpha,\beta} (A^C, B)$
9. $H_{\alpha,\beta} (A, B) + H_{\alpha,\beta} (A^C, B) = H_{\alpha,\beta} (A^C, B^C) + H_{\alpha,\beta} (A, B^C)$

Proof:

1. $\left[H_{\alpha,\beta}(A\cup B,A)+H_{\alpha,\beta}(A\cap B,A)\right]=$
$\frac{1}{(\alpha-1)\beta}\sum_{i=1}^{n}\log\left\{\left[\mu_{A\cup B}^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_{A\cup B}(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}+\left[\mu_{A\cap B}^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_{A\cap B}(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-2\right\}$

$=\frac{1}{(\alpha-1)\beta}\left\{\sum_{\Delta_1}log\left\{\left[\mu_A^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}+\sum_{\Delta_2}log\left\{\left[\mu_B^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_B(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}\right\}+\{\sum_{\Delta_1}log\{\left[\mu_A^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}+\sum_{\Delta_2}log\left\{\left[\mu_B^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_B(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}\}$

$$=\frac{1}{(\alpha-1)\beta}\sum_{i=1}^{n}\log\{[\mu_B^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_\beta(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}]^{\beta}-1\}$$

Hence we can say that

$$H_{\alpha,\beta}(A\cup B,A)+H_{\alpha,\beta}(A\cap B,A)=H_{\alpha,\beta}(B,A)$$

2. $H_{\alpha,\beta}(A,A\cap B)=\frac{1}{(\alpha-1)\beta}\sum_{i=1}^{n}log\left\{\left[\mu_A^{\alpha}(x_i)\mu_{A\cap B}^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_{A\cap B}(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}$ $=$
$\frac{1}{(\alpha-1)\beta}\left\{\sum_{\Delta_1}\log\{\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}+\sum_{\Delta_2}log\left\{\left[\mu_A^{\alpha}(x_i)\mu_A^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_A(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}\}$

3. $=\frac{1}{(\alpha-1)\beta}\{\sum_{\Delta_1}log\left\{\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta}-1\right\}\}$

$$H_{\alpha,\beta}(A \cup B, B) = \frac{1}{(\alpha-1)\beta} \sum_{i=1}^{n} log\left\{\left[\mu_{A\cup B}^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta} - 1\right\}$$

$$= \frac{1}{(\alpha-1)\beta}\{\sum_{\Delta_1} log\left\{\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta} - 1\right\} + \sum_{\Delta_2} log\left\{\left[\mu_B^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + \left(1-\mu_B(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta} - 1\right\}$$

$$= \frac{1}{(\alpha-1)\beta}\{\sum_{\Delta_1} log\left\{\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]^{\beta} - 1\right\}\}$$

Hence we can say that

$$H_{\alpha,\beta}(A, A \cap B) = H_{\alpha,\beta}(A \cup B, B)$$

All other properties can be proved as above

Conclusions

We describe the divergence metric measure for fuzzy sets in this study. The discussed properties of the proposed proposal. We tested the proposed function in this research and found that it satisfies all the crucial criteria. We see that the proposed function has more application flexibility due to the presence of the argument in it. Therefore, whenever alterations are made or limiting constraints are put in place, we have produced some significant and intriguing findings that may be helpful for the generalised fuzzy divergence metric. Finally, several other significant findings are made that are helpful in statistics and information science.

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