Original Research Article

SOME NEW OPTIMAL BOUNDS FOR WALLIS RATIO

Abstract. Wallis ratio can be expressed as an asymptotic expansion using Stirling series and Bernoulli numbers. We prove the general inequalities for Wallis ratio for arbitray number of terms in the asymptotic expansion. We show that the coefficients in the asymptotic expansion are the best possible.

1. Introduction

The Wallis ratio is defined as

$$w_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

Throughout its long history, w_n has had many important applications in mathematics such as in combinatorics and statistics. The estimates of the Wallis ratio have interested many mathematicians and as a result, we have recently seen many remarkable results in this direction. For more on this subject, the reader is referred to [2, 5, 8, 10, 16, 15] as well as the comprehensive surveys [11, 12, 13].

The motivation of this paper is from the famous Stirling series. It is well known that $\log \Gamma$ has an asymptotic expansion for any fixed t as

$$\log \Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right) \log x - x + \frac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(t)}{n(n+1)} x^{-n},$$

where $B_{n+1}(t)$ is Bernoulli polynomial. If we use this asymptotic expansion for both $t=1$ and $t=1/2$, we can get an asymptotic expansion for $\log w_n$, that is,

$$\log w_n \sim -\frac{1}{2}\log(n\pi) + \sum_{k=1}^{\infty} \frac{4^{-k}-1}{k(2k-1)} \frac{B_{2k}}{n^{2k-1}},$$

or

$$\log(w_n\sqrt{n\pi}) \sim \sum_{k=1}^{\infty} \frac{4^{-k}-1}{k(2k-1)} \frac{B_{2k}}{n^{2k-1}}, \tag{1.1}$$

where B_{2k} are Bernoulli numbers. A related asymptotic expansion can be found in [14].

Thus for any fixed $N \geq 1$, the remainder

$$\log(w_n\sqrt{n\pi}) - \sum_{k=1}^{N} \frac{4^{-k}-1}{k(2k-1)} \frac{B_{2k}}{n^{2k-1}} = o\left(\frac{1}{n^{2N-1}}\right).$$

In Theorem 1.1, we will show that for all $n \geq 1$, this remainder is always positive if N is odd and always negative if N is even.

2000 *Mathematics Subject Classification.* Primary 26D10, 05A10; Secondary 11B68.
Key words and phrases. Wallis ratio, Bernoulli numbers, Stirling series.

Now consider the case when N is odd. $\log(w_n\sqrt{n\pi})$ is between Nth partial sum and $(N+1)$th partial sum. Since the $(N+1)$th term in the series has a positive coefficient $\frac{4^{-k}-1}{k(2k-1)}B_{2N+2}$, we naturally want to know if this positive number can be smaller, that is, we want to find the smallest constant A_{2N+1} such that

$$\sum_{k=1}^{N}\frac{4^{-k}-1}{k(2k-1)}\frac{B_{2k}}{n^{2k-1}} < \log(w_n\sqrt{n\pi}) < \sum_{k=1}^{N}\frac{4^{-k}-1}{k(2k-1)}\frac{B_{2k}}{n^{2k-1}} + \frac{A_{2N+1}}{n^{2N+1}}.$$

By Theorem 1.1, the above the inequality holds if $A_{2N+1} = \frac{4^{-k}-1}{k(2k-1)}B_{2N+2}$. In fact, this coefficient is the smallest constant for the inequality to hold. So these coefficients are the best possible constants. The same is true when N is even.

Theorem 1.1. *For any odd $N_1 \geq 1$ and even $N_2 \geq 1$, we have*

$$\frac{1}{\sqrt{n\pi}}\exp\left(\sum_{k=1}^{N_1}\frac{4^{-k}-1}{k(2k-1)}\frac{B_{2k}}{n^{2k-1}}\right) < w_n < \frac{1}{\sqrt{n\pi}}\exp\left(\sum_{k=1}^{N_2}\frac{4^{-k}-1}{k(2k-1)}\frac{B_{2k}}{n^{2k-1}}\right) \tag{1.2}$$

for all $n \geq 1$. The coefficients in the series are the best possible in the sense as discussed.

We can just list a first few special cases as

$$\frac{1}{\sqrt{n\pi}}e^{-\frac{1}{8n}} < w_n < \frac{1}{\sqrt{n\pi}}e^{-\frac{1}{8n}+\frac{1}{192n^3}},$$

$$\frac{1}{\sqrt{n\pi}}e^{-\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}} < w_n < \frac{1}{\sqrt{n\pi}}e^{-\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}}.$$

2. Some identities for Bernoulli numbers

We start by introducing the some basic properties for Bernoulli numbers. Bernoulli numbers are defined as the coefficients of the expansion of

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}B_n\frac{x^n}{n!}, \qquad |x| < 2\pi.$$

Thus we see that $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, ...$ and $B_{2k+1} = 0$ for $k \geq 1$. For any $k \geq 1$, we know $B_{4k} < 0$ and $B_{4k-2} > 0$. For $n \geq 2$, we have

$$\sum_{k=0}^{n}C(n+1,k)B_k = 0. \tag{2.1}$$

For more about Bernoulli numbers, please refer to [6].

We first introduce a new identity for Bernoulli numbers.

Theorem 2.1. *For any integer $n \geq 1$, we have*

$$\sum_{k=0}^{n-1}(2\cdot 4^{-k}-1)B_{2k}C(2n,2k) = n\cdot 4^{-n+1}.$$

Proof. By ([6], p. 260), we know that

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^k(2-2^{2k})\frac{B_{2k}}{(2k)!}x^{2k-1}, 0 < |x| < \pi.$$

On the other side, we know

$$\sin x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)!}x^{2n-1}$$

and

$$1-\cos(2x) = 1-\sum_{j=0}^{\infty}(-1)^j\frac{2^{2j}x^{2j}}{(2j)!} = x\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j}}{(2j)!}x^{2j-1}.$$

for all x. From the trigonometric identity

$$\frac{1-\cos(2x)}{\sin x} = 2\sin x,$$

we know that

$$\begin{aligned}
&\left(\frac{1}{x}+\sum_{k=1}^{\infty}(-1)^k(2-2^{2k})\frac{B_{2k}}{(2k)!}x^{2k-1}\right)\left(x\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j}}{(2j)!}x^{2j-1}\right)\\
= &\left(\sum_{k=0}^{\infty}(-1)^k(2-2^{2k})\frac{B_{2k}}{(2k)!}x^{2k}\right)\left(\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j}}{(2j)!}x^{2j-1}\right)\\
= &\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sum_{k=0}^{n-1}(2-2^{2k})\frac{B_{2k}}{(2k)!}\frac{2^{2n-2k}}{(2n-2k)!}\right)x^{2n-1}\\
= &\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sum_{k=0}^{n-1}(2-2^{2k})B_{2k}2^{2n-2k}C(2n,2k)\right)\frac{x^{2n-1}}{(2n)!}.
\end{aligned}$$

Comparing both sides, we know

$$\frac{1}{2n}\sum_{k=0}^{n-1}(2-2^{2k})B_{2k}2^{2n-2k}C(2n,2k) = 2.$$

Moving the factors $\frac{1}{2n}$ and 2^{2n} to the right side, we have the desired identity. □

Lemma 2.2. *For $v \geq 1$,*

$$\sum_{k=0}^{v-1}C(2v,2k)B_{2k} = v$$

and

$$\sum_{k=0}^{v}C(2v+1,2k)B_{2k} = \frac{2v+1}{2}.$$

Proof. Since $B_1 = -\frac{1}{2}$, $B_{2k+1} = 0$ for $k \geq 1$, we know by (2.1) for $n \geq 2$ that

$$\sum_{j=\text{even},j\leq n-1}C(n,j)B_j = -C(n,1)B_1 = \frac{n}{2}.$$

Lemma 2.2 follows directly from the above identity. □

Theorem 2.3. *For any $u \geq 0$, we have*

$$\sum_{k=0}^{\lceil \frac{u}{2} \rceil} (4^{-k} - 1) C(u+2, 2k) B_{2k} = \frac{u+2}{4}(2^{-u} - 1), \tag{2.2}$$

where $\lceil \cdot \rceil$ is the ceiling function.

Proof. Clearly if $u = 0$, the identity holds, as both sides of (2.2) are 0.

If $u = 2v$ for $v \geq 1$, then by Theorem 2.1 and then Lemma 2.2, we see that

$$\begin{aligned} & \sum_{k=0}^{v} (4^{-k} - 1) C(2v+2, 2k) B_{2k} \\ = \; & \frac{1}{2} \sum_{k=0}^{v} (2 \cdot 4^{-k} - 1) C(2v+2, 2k) B_{2k} - \frac{1}{2} \sum_{k=0}^{v} C(2v+2, 2k) B_{2k} \\ = \; & \frac{1}{2}(v+1) 4^{-v} - \frac{1}{2}(v+1) = \frac{2v+2}{4}(2^{-2v} - 1). \end{aligned}$$

If $u = 2v - 1$ for some $v \geq 1$, then by Corollary 1(b) of Liu and Guo [9], we know that

$$\sum_{k=0}^{v} (2 \cdot 4^{-k} - 1) C(2v+1, 2k) B_{2k} = (2v+1) 4^{-v},$$

hence

$$\begin{aligned} & \sum_{k=0}^{v} (4^{-k} - 1) C(2v+1, 2k) B_{2k} \\ = \; & \frac{1}{2} \sum_{k=0}^{v} (2 \cdot 4^{-k} - 1) C(2v+1, 2k) B_{2k} - \frac{1}{2} \sum_{k=0}^{v} C(2v+1, 2k) B_{2k} \\ = \; & \frac{2v+1}{2} 4^{-v} - \frac{1}{4}(2v+1) = \frac{u+2}{4}(2^{-u} - 1). \end{aligned}$$

Therefore (2.2) holds for all $u \geq 0$. □

Next we will need some estimates for B_{2n}. The following (and better) inequalities can be found in [1], [3], and [4].

Lemma 2.4. *For any even $k \geq 1$, we have*

$$\frac{2(2k)!}{(2\pi)^{2k}} \frac{1}{1 - 4^{-k}} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}} \frac{1}{1 - 2 \cdot 4^{-k}}.$$

Using Lemma 2.4, we can prove the following inequalities which will be used in next section.

Lemma 2.5. *For $k \geq 1$, $1 \leq u - 2k \leq 4$, we have*

$$(1 - 4^{-k}) C(u+2, 2k) |B_{2k}| - (1 - 4^{-(k+1)}) C(u+2, 2k+2) |B_{2k+2}| > 0.$$

Proof.

$$
\begin{aligned}
& (1-4^{-k})C(u+2,2k)|B_{2k}| - (1-4^{-(k+1)})C(u+2,2k+2)|B_{2k+2}| \\
> \ & (1-4^{-k})C(u+2,2k)\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-4^{-k}} \\
& -(1-4^{-k-1})C(u+2,2k+2)\frac{2(2k+2)!}{(2\pi)^{2k+2}}\frac{1}{1-2\cdot 4^{-k-1}} \\
= \ & \frac{2(u+2)!}{(u-2k+2)!(2\pi)^{2k}}\left[1-\frac{1-4^{-(k+1)}}{1-2\cdot 4^{-(k+1)}}\frac{(u-2k+2)(u-2k+1)}{4\pi^2}\right].
\end{aligned}
$$

When $k \geq 1$, $1 \leq u-2k \leq 4$, it is easy to see that

$$
1-\frac{1-4^{-(k+1)}}{1-2\cdot 4^{-(k+1)}}\frac{(u-2k+2)(u-2k+1)}{4\pi^2} \geq 1-\frac{15}{14}\cdot\frac{30}{4\pi^2} > 0,
$$

which proves the desired inequality. □

The next lemma shows when $u-2k$ is a little larger, the inequality in Lemma 2.5 reverses.

Lemma 2.6. *For $k \geq 1$,$u \geq 2k+7$, we have*

$$
(1-4^{-(k+1)})C(u+2,2k+2)|B_{2k+2}| - (1-4^{-k})C(u+2,2k)|B_{2k}| > 0.
$$

Proof.

$$
\begin{aligned}
& (1-4^{-(k+1)})C(u+2,2k+2)|B_{2k+2}| - (1-4^{-k})C(u+2,2k)|B_{2k}| \\
> \ & (1-4^{-k-1})C(u+2,2k+2)\frac{2(2k+2)!}{(2\pi)^{2k+2}}\frac{1}{1-\cdot 4^{-k-1}} \\
& -(1-4^{-k})C(u+2,2k)\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2\cdot 4^{-k}} \\
= \ & \frac{2(u+2)!}{(u-2k+2)!(2\pi)^{2k}}\frac{1-4^{-k}}{1-2\cdot 4^{-k}}\left[\frac{(u-2k+2)(u-2k+1)}{4\pi^2}\cdot\frac{1-2\cdot 4^{-k}}{1-4^{-k}}-1\right].
\end{aligned}
$$

When $k \geq 1$, $u \geq 2k+7$, it is easy to see that

$$
\frac{(u-2k+2)(u-2k+1)}{4\pi^2}\frac{1-2\cdot 4^{-k}}{1-4^{-k}}-1 > \frac{72}{4\pi^2}\cdot\frac{2}{3}-1 = \frac{12}{\pi^2}-1 > 0,
$$

which proves the desired inequality. □

3. Proof of main theorem

To prove Theorem 1.1, we first introduce an expansion related to w_n. Let $a = \frac{1}{n}$, $b = \frac{1}{n+1}$. Then $a = \frac{b}{1-b}$ and

$$
\frac{1}{n^{2k-1}} - \frac{1}{(n+1)^{2k-1}} = \left(\frac{b}{1-b}\right)^{2k-1} - b^{2k-1}.
$$

For fixed $N \geq 1$, set

$$
v_n^{(N)} = \log w_n + \log\sqrt{n\pi} - \sum_{k=1}^{N}\frac{4^{-k}-1}{k(2k-1)}\frac{B_{2k}}{n^{2k-1}},
$$

then

$$v_n^{(N)} - v_{n+1}^{(N)} = \log\left(\frac{2n+2}{2n+1}\right) + \log\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right) + \sum_{k=1}^{N} \frac{1-4^{-k}}{k(2k-1)} B_{2k}\left(\frac{1}{n^{2k-1}} - \frac{1}{(n+1)^{2k-1}}\right).$$

Now

$$\log\left(\frac{2n+2}{2n+1}\right) + \log\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right) = -\log\left(1-\frac{b}{2}\right) + \frac{1}{2}\log(1-b).$$

Thus $v_n^{(N)} - v_{n+1}^{(N)}$ can be expressed in terms of b. Set

$$f_N(y) = -\log\left(1-\frac{y}{2}\right) + \frac{1}{2}\log(1-y) + \sum_{k=1}^{N} \frac{1-4^{-k}}{k(2k-1)} B_{2k}\left(\left(\frac{y}{1-y}\right)^{2k-1} - y^{2k-1}\right).$$

Then $f_N(b) = v_n^{(N)} - v_{n+1}^{(N)}$.

$$\begin{aligned} f_N{}'(y) &= \frac{1}{2-y} - \frac{1}{2(1-y)} + \sum_{k=1}^{N} \frac{1-4^{-k}}{k(2k-1)} B_{2k}(2k-1)\left(\frac{y^{2k-2}}{(1-y)^{2k}} - y^{2k-2}\right) \\ &= \frac{1}{2-y} - \frac{1}{2(1-y)} + \sum_{k=1}^{N} \frac{1-4^{-k}}{k} B_{2k} y^{2k-2}((1-y)^{-2k} - 1). \end{aligned}$$

Expanding all terms in power series, we have

$$\begin{aligned} \frac{1}{2-y} - \frac{1}{2(1-y)} &= \sum_{u=0}^{\infty} \frac{1}{2}\left(\frac{y}{2}\right)^n - \sum_{u=0}^{\infty} \frac{1}{2} y^n = \sum_{u=1}^{\infty} \left(\frac{1}{2^{u+1}} - \frac{1}{2}\right) y^u, \\ (1-y)^{-2k} &= \sum_{j=0}^{\infty} C(2k+j-1, j) y^j. \end{aligned}$$

We have

$$\begin{aligned} & \sum_{k=1}^{N} \frac{1-4^{-k}}{k} B_{2k} y^{2k-2}((1-y)^{-2k} - 1) \\ = & \sum_{k=1}^{N} \frac{1-4^{-k}}{k} B_{2k} y^{2k-2} \sum_{j=1}^{\infty} C(2k+j-1, j) y^j \\ = & \sum_{k=1}^{N} \sum_{j=1}^{\infty} \frac{1-4^{-k}}{k} B_{2k} C(2k+j-1, j) y^{2k-2+j} \\ = & \sum_{u=1}^{\infty} \left(\sum_{k=1}^{\min(N, \lceil u/2 \rceil)} \frac{1-4^{-k}}{k} B_{2k} C(u+1, 2k-1) \right) y^u. \end{aligned}$$

Since $\frac{2}{u+2} C(u+2, 2k) = \frac{C(u+1, 2k-1)}{k}$, we can define

$$M_u^{(N)} = \frac{1}{2^{u+1}} - \frac{1}{2} + \frac{2}{u+2} \sum_{k=1}^{\min(N, \lceil u/2 \rceil)} (1-4^{-k}) B_{2k} C(u+2, 2k).$$

Then

$$f_N{}'(y) = \sum_{u=1}^{\infty} M_u^{(N)} y^u.$$

Now we can complete the proof of our main theorems.

Proof of Theorem 1.1

Let $N \geq 1$ be fixed. By Theorem 2.3, if $1 \leq u \leq 2N$,

$$M_u^{(N)} = \frac{1}{2^{u+1}} - \frac{1}{2} + \frac{2}{u+2}\frac{u+2}{4}(1 - 2^{-u}) = 0. \tag{3.1}$$

If $u = 2N + 1$ or $u = 2N + 2$, then by (3.1), we $M_u^{(N+1)} = 0$. But

$$M_u^{(N)} = M_u^{(N+1)} - \frac{2}{u+2}(1 - 4^{-N-1})B_{2N+2}C(u + 2, 2N + 2).$$

This implies that if $N \geq 1$ is odd, then $B_{2N+2} < 0$ and hence $M_{2N+1}^{(N)}$ and $M_{2N+2}^{(N)}$ are both positive, and if $N \geq 1$ is even, then $M_{2N+1}^{(N)}$ and $M_{2N+2}^{(N)}$ are both negative.

If $u = 2N+3$ or $u = 2N+4$, we know by (3.1) $M_u^{(N+2)} = 0$. If follows that that

$$\begin{aligned} M_u^{(N)} &= 0 - \frac{2}{u+2}(1 - 4^{-N-1})B_{2N+2}C(u + 2, 2N + 2) \\ &\quad - \frac{2}{u+2}(1 - 4^{-N-2})B_{2N+4}C(u + 2, 2N + 4). \end{aligned}$$

If N is odd, then $B_{2N+2} < 0$ and hence by Lemma 2.5, $M_u^{(N)} > 0$ and if N is even, then then $B_{2N+2} > 0$ and hence $M_u^{(N)} < 0$.

Suppose now $u \geq 2N + 5$. If N is odd, then we regroup the terms of $M_u^{(N)}$ to make

$$\begin{aligned} & M_u^{(N)} \\ = \; & \frac{2}{u+2}\left[(1 - 4^{-N})B_{2N}C(u + 2, 2N) + (1 - 4^{-N+1})B_{2N-2}C(u + 2, 2N - 2)\right] \\ & + \cdots \\ & + \frac{2}{u+2}\left[(1 - 4^{-3})B_6C(u + 2, 6) + (1 - 4^{-2})B_4C(u + 2, 4)\right] \\ & + \left[\frac{2}{u+2}(1 - 4^{-1})B_2C(u + 2, 2) + \frac{1}{2^{u+1}} - \frac{1}{2}\right]. \end{aligned}$$

We see that

$$\frac{2}{u+2}(1 - 4^{-1})B_2C(u + 2, 2) + \frac{1}{2^{u+1}} - \frac{1}{2} = \frac{1}{8}(u + 1) + \frac{1}{2^{u+1}} - \frac{1}{2} > 0$$

and all the other sums are positive by Lemma 2.6, since $u \geq (2N - 2) + 7$. Hence if N is odd, then $M_u^{(N)} > 0$ for all $u \geq 2N + 5$.

Similarly, if N is even, and $u \geq 2N + 5$, we regroup the terms in $M_u^{(N)}$ as

$$\begin{aligned} & M_u^{(N)} \\ = \; & \frac{2}{u+2}(1 - 4^{-N})B_{2N}C(u + 2, 2N) \\ & + \frac{2}{u+2}\left[(1 - 4^{-N+1})B_{2N-2}C(u + 2, 2N - 2) + (1 - 4^{-N+2})B_{2N-4}C(u + 2, 2N - 4)\right] \\ & + \cdots \\ & + \frac{2}{u+2}\left[(1 - 4^{-2})B_4C(u + 2, 4) + (1 - 4^{-2})B_2C(u + 2, 2)\right] \\ & + \frac{1}{2^{u+1}} - \frac{1}{2}. \end{aligned}$$

It is easy to see the first term is negative as $B_{2N} < 0$, the last sum is negative and all the other sums are negative by Lemma 2.6.

Therefore we have showed that $M_u^{(N)} = 0$ for $u = 1, 2, \cdots, 2N$ and when $u \geq 2N+1$, if N is odd, then $M_u^{(N)} > 0$ and if N is even, then $M_u^{(N)} < 0$. It follows that $f_N'(y) > 0$ when N is odd, so f_N is increasing. But $f_N(0) = 0$, so $f_N(y) > 0$ for $y > 0$, which implies $v_1^{(N)} > v_2^{(N)} > ... > v_n^{(N)} > \ldots$. As $v_n^{(N)} \to 0$, we know that $v_n^{(N)} > 0$ for all n. This proves the left inequality of (1.1). Similarly if N is even, then $f_N'(y) < 0$, so f_N is decreasing, and $f_N(y) < 0$ for $y > 0$, which implies $v_1^{(N)} < v_2^{(N)} < \ldots < v_n^{(N)} < \ldots < 0$. This proves the right inequality of (1.1).

Now let us show that the coefficients in (1.1) are the best possible. Let N be odd. We know from (1.1)

$$w_n > \frac{1}{\sqrt{n\pi}} \exp\left(\sum_{k=1}^{N} \frac{4^{-k}-1}{k(2k-1)} \frac{B_{2k}}{n^{2k-1}}\right)$$

for all $n \geq 1$. Now we want to find the smallest C such that

$$w_n < \frac{1}{\sqrt{n\pi}} \exp\left(\sum_{k=1}^{N} \frac{4^{-k}-1}{k(2k-1)} \frac{B_{2k}}{n^{2k-1}} + \frac{C}{n^{2N+1}}\right) \tag{3.2}$$

for all $n \geq 1$.

By (1.1), we know (3.2) holds for $C = C_{N+1} = \frac{4^{-N-1}-1}{(N+1)(2N+1)} B_{2N+2}$ for all $n \geq 1$. Now let us show this C_{N+1} is the smallest constant of C for (3.2) to hold. Suppose now $C < C_{N+1}$. Let

$$\tilde{v}_n^{(N+1)} = v_n^{(N+1)} + \frac{C_{N+1} - C}{n^{2N+1}},$$

$$\tilde{f}_{N+1}(y) = f_{N+1}(y) + (C_{N+1} - C)\left(\left(\frac{y}{1-y}\right)^{2N+1} - y^{2N+1}\right).$$

$$\tilde{f}_{N+1}{}'(y) = f_{N+1}{}'(y) + (C_{N+1} - C)(2N+1)y^{2N}\left[(1-y)^{-2N-2} - 1\right].$$

We already showed that for $N+1$, $M_u^{(N+1)} = 0$ for $u = 1, \ldots, 2N+2$. Thus $f_{N+1}{}'(y) = o\left(y^{2N+2}\right)$. It follows that for sufficiently small y, we have $\tilde{f}_{N+1}{}'(y) > 0$. This implies $\tilde{v}_n^{(N+1)} > 0$, and hence the inequality in (3.3) reverses. Thus C_{N+1} is the smallest constant of C for (3.2) to hold. The same way can show the coefficients are the best possible when N is even.

$\square$

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