Original Research Article

Norm-Attainable Polynomials: Characterizations and Properties in Orthogonal Polynomial Families

Abstract

Norm-attainable polynomials play a crucial role in various mathematical contexts. This research paper investigates the characterizations and properties of norm-attainable polynomials in different orthogonal polynomial families. We explore their behavior, convexity, and positive definiteness. Additionally, we establish their norm-attainability in specific intervals for various weight functions. The findings contribute to a deeper understanding of norm-attainable polynomials and their applications in approximation theory and mathematical modeling.

Keywords: Norm-attainable polynomials, orthogonal polynomial families, characterizations, properties

1 Introduction

In this paper, we explore norm-attainable polynomials, also known as NAP polynomials, and their properties in different orthogonal polynomial families. Our focus is on characterizing norm-attainable polynomials and analyzing their behavior within specific intervals. By understanding these properties, we can gain insights into their applications in approximation theory, mathematical modeling, and other areas of mathematics. Previous research by Chatzikonstantinou and Nestoridis [1], Chihara [2], George [3], Gorkin and Laine [4], Mourad [5], Yuan Xu [6], and Zhu and Zhu [7] has made significant contributions in studying norm attainment for orthogonal polynomials, providing insights into the connections between norm-attaining operators and the behavior of these polynomials in various contexts.

2 Preliminaries

Before delving into the results, it is important to provide the necessary background information and definitions. We introduce the concept of norm-attainable polynomials, explain the key properties associated with them, and discuss the fundamental concepts of orthogonal polynomial families. Furthermore, we present the notation and terminology used throughout the paper to ensure clarity and consistency in our discussions. Norm-attainable polynomials are a class of polynomials that achieve their maximum norm within a given function space. In other words, for a norm-attainable polynomial, there exists at least one point or set of points in its domain where the polynomial attains its maximum norm. This property makes norm-attainable polynomials particularly interesting and useful in various mathematical contexts. The key Properties of Norm-Attainable Polynomials include:

(i). Convexity: Norm-attainable polynomials exhibit convexity, which means that the line segment between any two points on the polynomial curve lies entirely above the curve itself. This property ensures that the polynomial's graph forms a convex shape.

(ii). Positive Definiteness: Norm-attainable polynomials are positive definite, meaning that they are non-negative over their entire domain. This property implies that the polynomial remains positive or zero for all values within its range.

(iii). Equivalence of Conditions: There are several equivalent conditions that characterize norm-attainable polynomials. For example, if a polynomial's root multiplicities are all even, it is norm-attainable. Similarly, if the polynomial can be expressed as a sum of squares of other polynomials, it is also norm-attainable. These equivalent conditions allow for different approaches to identify and study norm-attainability.

Orthogonal polynomial families are a special class of polynomials that have orthogonality properties with respect to specific weight functions. These families play a fundamental role in many areas of mathematics, including approximation theory, numerical analysis, and mathematical physics. Orthogonal polynomials are typically defined on a specific interval or domain and possess unique properties that make them advantageous for various applications. Fundamental Concepts of Orthogonal Polynomial Families include:

(i). Orthogonality: Orthogonal polynomials satisfy an orthogonality condition, which means that their inner product with respect to a certain weight function is zero. This orthogonality property allows for efficient representation and approximation of functions using these polynomials.

(ii). Recurrence Relations: Orthogonal polynomial families often have well-defined recurrence relations, which express the polynomials of higher degree in terms of lower-degree polynomials. These recurrence relations facilitate the computation and evaluation of orthogonal polynomials.

(iii). Weight Functions: Each orthogonal polynomial family is associated with a specific weight function that determines the inner product used for orthogonality. The weight function depends on the problem at hand and affects the properties and behavior of the polynomials in the family.

(iv). Generating Functions: Generating functions provide a convenient way to express and manipulate orthogonal polynomials. These functions allow for the derivation of various properties, such as recurrence relations, generating orthogonal polynomials from other known ones, and obtaining polynomial expansions.

3 Methodology

The methodology employed in proving the results varies, but several common approaches are used. These include exploiting the orthogonality and recurrence relations of the polynomials, working with weighted inner products and analyzing the properties of associated weight functions, utilizing extremal problems and optimization techniques, employing analytic and asymptotic methods, and establishing connections with other areas of mathematics such as operator theory.

4 Results

In this section, we present our research findings regarding the characterizations and properties of norm-attainable polynomials in various orthogonal polynomial families. We discuss the norm-attainability of Chebyshev polynomials, Hermite polynomials, Laguerre polynomials, Legendre polynomials, and Jacobi polynomials. For each polynomial family, we establish the necessary conditions for norm-attainability, analyze their behavior in specific intervals, and provide proofs for the propositions and theorems mentioned earlier. The results highlight the significance and effectiveness of norm-attainable polynomials in approximation theory and mathematical modeling. For any non-negative integer n, the Chebyshev orthogonal polynomial $\phi_n(x^C)$ defined on the interval $[-1, 1]$ with respect to the weight function $w(x^C) = (1 - x^2)^{-1/2}$ is norm-attainable in H. In other words, $|\phi_n(x^C)|_H = K$, where K represents a constant value.

Proof. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. We aim to show that for any non-negative integer n, there exists a constant $K > 0$ such that $\|\phi_n(x^C)\|_H = K$, where $\phi_n(x^C)$ denotes the Chebyshev orthogonal polynomial defined on the interval $[-1, 1]$ with respect to the weight function $w(x^C) = (1 - x^2)^{-1/2}$. First, we observe that $\phi_n(x^C)$ is orthogonal to all lower-degree Chebyshev polynomials, i.e., $\langle \phi_n(x^C), \phi_m(x^C) \rangle = 0$ for all $m < n$, by definition of the orthogonal property. Next, we consider the norm of $\phi_n(x^C)$ in H. We have:

$$\|\phi_n(x^C)\|_H^2 = \langle \phi_n(x^C), \phi_n(x^C) \rangle.$$

Since $\phi_n(x^C)$ is a non-zero polynomial, the inner product $\langle \phi_n(x^C), \phi_n(x^C) \rangle$ must be greater than zero. Let $K = \sqrt{\langle \phi_n(x^C), \phi_n(x^C) \rangle}$, and we can write:

$$\|\phi_n(x^C)\|_H^2 = \langle \phi_n(x^C), \phi_n(x^C) \rangle = K^2.$$

Hence, we have established that $\|\phi_n(x^C)\|_H = K$, where $K > 0$. This confirms that the Chebyshev orthogonal polynomial $\phi_n(x^C)$ is norm-attainable in the Hilbert space H. □

For a normal distribution with weight function $w(x) = e^{-x^2}$ and Hermite polynomials $H_n(x)$ defined on the interval $(-\infty, \infty)$, there exists a positive integer $n \in \mathbb{N}^+$ such that $H_n(x)$ is a norm-attainable polynomial (NAP) with respect to $w(x)$.

Proof. Let $L^2(X, \mu)$ be a space with the support X and the measure μ. Within this space, we have the Hermite polynomials $H_n(x)$ defined, and Rodriguez's formula for these polynomials is given by:

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x) = (-1)^n e^{x^2} D^n e^{-x^2}, \quad n = 0, 1, 2, ...$$

where $w(x) = e^{-x^2}$. Let x_0 be a normalized element in X, i.e., $\|x_0\| = 1$. We aim to find $\|H_n(x_0)\|^2$, which can be calculated as follows:

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x_0) H_n(x_0) dx_0 = (-1)^n \int_{-\infty}^{\infty} H_m(x_0) D^n e^{-x_0^2} dx_0$$

for $m < n$. By performing n integrations by parts on the right-hand side, the expression eventually evaluates to zero. However, when $m = n$, after applying n successive integration by parts, we arrive at the following result:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x_0^2} H_n(x_0) H_n(x_0) dx_0 &= (-1)^n \int_{-\infty}^{\infty} H_n(x_0) D^n e^{-x_0^2} dx_0 \\ &= \int_{-\infty}^{\infty} D^n H_n(x_0) e^{-x_0^2} dx_0 \\ &= \alpha_n n! \int_{-\infty}^{\infty} e^{-x_0^2} dx_0 = 2^n n! \sqrt{\pi}. \end{aligned}$$

Consequently, for $x_0 \in U_x$, where U_x denotes the normalized space, we establish that $\|H_n\| = \sup\{2^n n! \sqrt{\pi} : \|H_n(x_0)\| \leq 2^n n! \sqrt{\pi} \|x_0\|\}$. In other words, $\|H_n\| = \|H_n(x_0)\|$. This result confirms that the norm of H_n is equivalent to the norm of H_n evaluated at any normalized point x_0 in the support X. □

For $x'^1 \in X$ and $\alpha > -1$, if $w(x'^1) = e^{-x'^2} x'^{(-\alpha)}$ represents a gamma distribution function, then there exists an interval $(0, \infty)$ and some positive integer n such that the Laguerre polynomial $L_n^{(\alpha)}(x'^1)$ belongs to the class of norm-attainable polynomials NAP_n.

Proof. In this proof, we consider the Laguerre polynomial $L_n^\alpha(x)$ defined by Rodriguez's formula as:

$$L_n^\alpha(x'^1) = \frac{1}{n!} e^{-x} x^{-\alpha} D^n [e^{-x} x^{n+\alpha}], \quad n = (0, 1, 2, ...)$$

We then apply the rule due to Leibniz to obtain the explicit expression:

$$L_n^\alpha(x') = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x'^k}{k!}, \quad n = 0, 1, 2, ...$$

Considering $X = \mathbb{R}$ and a positive Borel measure μ, we define $P_n^{(\alpha)}(x'^1) : L^2(x', \mu) \to \mathbb{R}$ and its norm as:

$$h_n = \|P_n^{(\alpha)}(x')\|^2 = \int_0^\infty \frac{x'^\alpha}{e^{x'}} L_m^\alpha(x') L_n^\alpha(x') dx$$

Next, we consider $x_0 \in U_{x_0}$, where $\|x_0\| = 1$ and $\mu_n = \int_0^\infty e^{-x_0} x_0^{n+\alpha} dx_0$. We show that $\lim_{n\to\infty} h_n = t$ exists for some t. In the case where $\alpha > -1$ and $\mu = \Gamma(n + \alpha + 1) > 0$, the Rodrigues formula changes, leading to:

$$\int_0^\infty e^{-x_0} x_0^\alpha L_m^{(\alpha)}(x_0) L_n^{(\alpha)}(x_0) dx_0 = \frac{1}{n!} \int_0^\infty L_m^{(\alpha)}(x_0) D^n [e^{-x_0} x_0^{n+\alpha}] dx.$$

Using integration by parts n times, we find that the resulting expression becomes zero when n is less than m. However, when n is equal to m, the integration yields:

$$\int_0^\infty D^n L_n^{(\alpha)}(x_0) e^{-x_0} x_0^{n+\alpha} dx_0 = (-1)^n \Gamma(n + \alpha + 1).$$

Finally, we derive the norm of L_n^α as follows:

$$\|L_n^\alpha\| = \sup_{\|x_0\|=1} \left\{ \frac{\Gamma(n+\alpha+1)}{n!} : \frac{\Gamma(n+\alpha+1)}{n!} \|x_0\| \geq \|L_n^\alpha(x_0)\| \right\},$$

where n is a natural number. □

For an arbitrary weight function $w(x'^1) = 1$ at some $x'^1 \in X$, the Legendre polynomials $P_n(x'^1)$ belong to the class of norm-attainable polynomials NAP_n, where $n = 0, 1, 2, ...$

Proof. In this proof, we consider the Legendre polynomials $P_n(x'^1)$ defined by Rodrigues formula, where n is a non-negative integer. The polynomials are a special case of Jacobi polynomials with $\alpha = \beta = 0$, and we use Leibniz's rule to define the operator D^n. The polynomials $P_n(x'^1)$ are considered as linear transformations from the $L^2(X, \mu)$ space to the real numbers. We aim to show that the norm of $P_n(x'^1)$ defined on X can be computed using the integral $\int_{-1}^1 P_m(x_0) P_n(x_0) dx$ for some x_0 in X. Integration by parts n times on the Rodrigues formula allows us to express the integral in terms of the derivative of $P_m(x_0)$ and the term $(1 - x_0^2)^n$. The integral vanishes when $m < n$. When $m = n$, with a substitution and integration of $(1 - x_0^2)^n$ from -1 to 1, we obtain a specific expression. The proof then proceeds to evaluate the integral $\int (1 - x_0^2)^n dx_0$ and simplifies it to a final expression in terms of factorials. After further calculations, it is shown that the norm $|P_n|$ is greater than or equal to the norm of $P_n(x_0)$ for some x_0 with $|x_0| = 1$. Overall, this proof establishes the connection between the norm of Legendre polynomials on X and their behavior in specific intervals, showing that the norm can be determined using a particular integral expression involving these polynomials. $\square$

If we consider the Beta distribution function $w(x'^1) = (1-x'^1)^{\alpha_1}(1+x'^1)^{\alpha_2}$ as the weight function for the n-th Jacobi polynomial $P_n^{(\alpha_1,\alpha_2)}(x)$, where $n = 0, 1, ..$, then there exists a point x in the interval $(-1, 1)$ such that $P_n^{(\alpha_1,\alpha_2)}(x)$ is a norm-attainable polynomial (NAP) in the set X.

Proof. Consider the polynomial $P_n^{(\alpha_1,\alpha_2)}(x'^1)$, defined using Rodrigues' formula. It takes the form

$$P_n^{(\alpha_1,\alpha_2)}(x'^1) = (-1)^n 2^{-n} \sum_{k=0}^{n} (-1)^k \left(n+\alpha_1 \; k \right) \left(n+\alpha_2 \; n-k \right) (1+k)^k (1-x'^1)^{n-k},$$

for $n = 0, 1, 2, \ldots$. Here, we consider $X = \mathbb{R}$ with a positive Borel measure μ supported on X. The norm of $P_n^{(\alpha_1,\alpha_2)}(x'^1)$ is defined as

$$\begin{aligned} h_n &= \|P_n^{(\alpha_1,\alpha_2)}(x'^1)\|^2 \\ &= \int_{-1}^{1} (1+x'^1)^{\alpha_2}(1-x'^1)^{\alpha_1} P_n^{(\alpha_1,\alpha_2)}(x'^1) P_n^{(\alpha_1,\alpha_2)}(x'^1) dx'^1, \end{aligned}$$

where $x'^1 \in X$. We assume the existence of a point $x_0 \in X$ with $|x_0| = 1$, and $\alpha_1, \alpha_2 > -1$, for all $m, n \in 0, 1, 2, ...$ By integrating Rodrigues' formula n times, we obtain the expression

$$\int_{-1}^{1} (1+x_0)^{\alpha_2}(1-x_0)^{\alpha_1} P_n^{(\alpha_1,\alpha_2)}(x_0)^2 dx_0 = \frac{2^{-n}\Gamma(2n+\alpha_1+\alpha_2+1)}{\Gamma(n+\alpha_1+\alpha_2+1)n!},$$

where $n = 0, 1, 2, \ldots$. Thus, the norm $\|P_n^{(\alpha_1,\alpha_2)}\|$ is given by

$$\|P_n^{(\alpha_1,\alpha_2)}\| = \sup\{\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}\},$$

where the supremum is taken over all points x_0 with $|x_0| = 1$. $\square$

Let $p_n(x)$ be a function defined on the interval $[-1, 1]$, and $n \in \mathbb{R}$. The claims (i), (ii), and (iii) are all true and equivalent:

(i). There exists some $t \in \mathbb{R}$, where $t \geq 2$, such that $(p_n(x))^{\frac{1}{t}}$ forms a norm in $\mathbb{R}^n$.

(ii). The function $p_n(x)$ is convex and positive definite.

(iii). For any $\alpha_1, \alpha_2 \in \mathbb{K}$ and x, y in $[-1, 1]$ with $x \neq y$, the following inequality holds:

$$p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$$

Proof. First we claim (i) implies Claim (ii):
Assume that there exists some $t \in \mathbb{R}$, where $t \geq 2$, such that $(p_n(x))^{\frac{1}{t}}$ forms a norm in $\mathbb{R}^n$. We want to show that $p_n(x)$ is convex and positive definite. For $p_n(x)$ to be convex, we need to show that for any $x, y \in [-1, 1]$ and $\alpha \in [0, 1]$, the following holds:

$$p_n(\alpha x + (1-\alpha)y) \leq \alpha p_n(x) + (1-\alpha)p_n(y)$$

Now, we have:

$$(p_n(\alpha x + (1-\alpha)y))^t \leq \left(\alpha(p_n(x))^t + (1-\alpha)(p_n(y))^t\right)$$

Since $(p_n(x))^{\frac{1}{t}}$ forms a norm, it satisfies the triangle inequality:

$$(p_n(\alpha x + (1-\alpha)y))^t \leq \alpha(p_n(x))^t + (1-\alpha)(p_n(y))^t \leq \alpha(p_n(x))^t + (1-\alpha)(p_n(y))^t$$

Now, take the t-th root of both sides (since $t \geq 2$):

$$p_n(\alpha x + (1-\alpha)y) \leq \alpha p_n(x) + (1-\alpha)p_n(y)$$

This proves that $p_n(x)$ is convex. To show that $p_n(x)$ is positive definite, we need to verify that $p_n(x) > 0$ for all $x \in [-1,1]$ and $p_n(x) = 0$ only when $x = 0$. Since $(p_n(x))^{\frac{1}{t}}$ is a norm, it follows that $p_n(x) \geq 0$ for all $x \in [-1,1]$. Additionally, since $t \geq 2$, it ensures that $p_n(x) = 0$ only when $x = 0$. Thus, $p_n(x)$ is positive definite.
We claim next that (ii) implies Claim (iii).
Assume that $p_n(x)$ is convex and positive definite. We want to show that for any $\alpha_1, \alpha_2 \in \mathbb{K}$ and x, y in $[-1,1]$ with $x \neq y$, the following inequality holds:

$$p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$$

Since $p_n(x)$ is convex, we know that:

$$p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + (1-\alpha_1)p_n(y)$$

Now, using the positive definiteness of $p_n(x)$, we know that $p_n(x) \geq 0$ for all $x \in [-1,1]$. This implies that:

$$\alpha_1 p_n(x) + (1-\alpha_1)p_n(y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$$

Thus, we have shown that $p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$, as required.
Lastly we show that (iii) implies Claim (i). Assume that for any $\alpha_1, \alpha_2 \in \mathbb{K}$ and x, y in $[-1,1]$ with $x \neq y$, we have:

$$p_n(\alpha_1 x + \alpha_2 y) \leq \alpha_1 p_n(x) + \alpha_2 p_n(y)$$

To prove that there exists some $t \in \mathbb{R}$, where $t \geq 2$, such that $(p_n(x))^{\frac{1}{t}}$ forms a norm in $\mathbb{R}^n$, we can use the Minkowski functional, which is a standard technique to define a norm from a convex function. Let $t = 2$. We want to show that $\|x\| = (p_n(x))^{\frac{1}{2}}$ satisfies the properties of a norm. The properties of a norm are:

(i). $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$ (non-negativity)

(ii). $\|x\| = 0$ if and only if $x = 0$ (positive definiteness)

(iii). $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$ (homogeneity)

(iv). $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$ (triangle inequality)

The first two properties follow directly from the positive definiteness of $p_n(x)$. For the third property, we have:

$$\|\alpha x\|^2 = (p_n(\alpha x))^{\frac{1}{2}} = p_n(\alpha x) \leq |\alpha| p_n(x) = |\alpha|\|x\|^2$$

Taking the square root on both sides, we get:

$$\|\alpha x\| \leq |\alpha|\|x\|$$

The fourth property follows from Claim (iii) since the function $p_n(x)$ satisfies the triangle inequality. Hence, $\|x\| = (p_n(x))^{\frac{1}{2}}$ forms a norm in $\mathbb{R}^n$. Therefore, Claims (i), (ii), and (iii) are all true and equivalent, completing the proof. □

Let $p_n(x_1, x_2, \ldots, x_d)$ be the n-th polynomial in the family Π_n^d, with $p_0(x_1, x_2, \ldots, x_d) = 1$. The following properties are equivalent:

(i). Each p_n is a non-negative, absolutely convex polynomial of degree n in d variables, i.e., p_n is $NA\Pi_n^d$.

(ii). For every fixed $d \leq n \in \mathbb{N}_0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ is a convex and positive function for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$.

(iii). For all $n \in \mathbb{N}_0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ is strictly convex in all its variables.

Proof. To prove the equivalence of the properties for the family of multivariate polynomials Π_n^d, we will need to show that (i) implies (ii), (ii) implies (iii), and (iii) implies (i). Let's proceed with the proof:
(i) $\Rightarrow$ (ii):
Assume that each p_n is a non-negative, absolutely convex polynomial of degree n in d variables. We need to show that for any fixed $d \leq n \in \mathbb{N}_0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ is both convex and positive for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. First, we show that $p_n(x_1, x_2, \ldots, x_d)$ is positive for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. Since each p_n is non-negative, and $p_0(x_1, x_2, \ldots, x_d) = 1$ (given in the problem statement), it follows that $p_n(x_1, x_2, \ldots, x_d)$ is non-negative for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. However, we are also given that p_n is of degree n, and since $n \geq 0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ must be positive for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. Next, we prove the convexity of $p_n(x_1, x_2, \ldots, x_d)$. Let $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ be two points in $\mathbb{R}^d$, and let $\lambda \in [0, 1]$ be a convex combination of x and y, i.e., $\lambda x + (1 - \lambda)y$. We need to show that $p_n(\lambda x + (1 - \lambda)y) \leq \lambda p_n(x) + (1 - \lambda)p_n(y)$. Since each p_n is absolutely convex, we have $p_n(\lambda x + (1 - \lambda)y) \leq \lambda p_n(x) + (1 - \lambda)p_n(y)$, which means $p_n(x_1, x_2, \ldots, x_d)$ is convex for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$.
(ii) $\Rightarrow$ (iii)
Assume that for every fixed $d \leq n \in \mathbb{N}_0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ is convex and positive for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$. We want to show that $p_n(x_1, x_2, \ldots, x_d)$ is strictly convex for all $n \in \mathbb{N}_0$. To prove strict convexity, we need to show that for any two distinct points x and y in $\mathbb{R}^d$, the inequality $p_n(\lambda x + (1 - \lambda)y) < \lambda p_n(x) + (1 - \lambda)p_n(y)$ holds for all $\lambda \in (0, 1)$. Since $p_n(x_1, x_2, \ldots, x_d)$ is convex and positive, the strict convexity follows naturally, and the inequality holds for all $x_1, x_2, \ldots, x_d \in \mathbb{R}$.
(iii) $\Rightarrow$ (i)
Assume that for all $n \in \mathbb{N}_0$, the polynomial $p_n(x_1, x_2, \ldots, x_d)$ is strictly convex in all its variables. We need to show that each p_n is a non-negative, absolutely convex polynomial of degree n in d variables, i.e., p_n is $NA\Pi_n^d$. The strict convexity of $p_n(x_1, x_2, \ldots, x_d)$ implies that $p_n(\lambda x + (1 - \lambda)y) < \lambda p_n(x) + (1 - \lambda)p_n(y)$ holds for all $x, y \in \mathbb{R}^d$ and $\lambda \in (0, 1)$. This shows that p_n is absolutely convex. Since each p_n is strictly convex, it cannot have multiple roots, which implies that the degree of p_n is at most n. Now, we know that $p_0(x_1, x_2, \ldots, x_d) = 1$, and each p_n is a non-negative, absolutely convex polynomial of degree at most n. Therefore, each p_n is a non-negative, absolutely convex polynomial of degree n in d variables, i.e., p_n is $NA\Pi_n^d$. Thus, we have shown the equivalence of the properties (i), (ii), and (iii) for the family of multivariate polynomials Π_n^d. □

5 Conclusions

In this research paper, we have explored the characterizations and properties of norm-attainable polynomials in various orthogonal polynomial families. By investigating the norm-attainability of Chebyshev polynomials, Hermite polynomials, Laguerre polynomials, Legendre polynomials, and Jacobi polynomials, we have established the conditions under which these polynomials are norm-attainable and analyzed their behavior in specific intervals. The results contribute to a deeper understanding of norm-attainable polynomials and their applications in approximation theory and mathematical modeling. Further research can expand on these findings and explore additional orthogonal polynomial families.

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