Original Research Article

Wider classes of estimators in Adaptive cluster sampling

Abstract

Various efficient estimators using single and dual auxiliary variables with different functions including log and exponential have been developed in the SRSWOR design. Since the Adaptive cluster sampling (ACS) design is relatively new, estimators using functions like log and exponential with single and dual auxiliary variables have not been explored much. Therefore in this article, we propose two wider classes of estimators using single and dual auxiliary variables respectively so that the properties like bias and mean squared errors of various estimators using functions like log and exponential or any other function which belong to the proposed wider classes and have not been developed and studied yet would be known in advance. Formulae of the bias and mean squared error have been derived and presented. Further, since log type estimators have not been studied extensively in the ACS design we have developed new log type classes from each of the proposed wider classes and developed and studied some new log type member estimators. To examine the performance of these new developed log-type estimators over some competing estimators simulation studies have been conducted and all the estimators are further applied to a real data to estimate the average number of Mules in the Indian state of Assam. The studies show that the developed log-type estimators perform better.

Keywords: Wider class, Log-type estimator, Generalized estimator, Ratio type estimator, Adaptive sampling designs.

MSC Classification: 62D05

Statements and Declarations

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1 Introduction

In survey sampling, we might come across data that is highly clumped or patchy, in such a case the traditional sampling designs like the Simple random sampling without replacement (SRSWOR), Stratified random sampling, and Systematic sampling among other sampling designs do not provide a representative sample. In such cases, adaptive sampling designs are needed. Adaptive cluster sampling proposed by Thompson [1] is one such adaptive sampling design where an initial sample using SRSWOR is drawn and based on specified condition and defined neighborhood additional units are selected. Due to its wide applicability, it has been used in a variety of disciplines such as ecological science [2, 3], environmental science [4, 5] and social science[6].

Since Cochran [7] proposed the ratio estimator numerous ratio and product type estimators have been developed using various transformations in SRSWOR, Stratified random sampling, and Systematic sampling among other sampling designs. Drawing up a list of all such estimators will be of no use. Some noteworthy works where the authors introduced concepts like a class of estimators, constants which minimize MSE are Grover and Kaur [8], Khoshnevisan et al.[9], Haq and Shabbir [10]. The use of functions like log and exponential was also explored and it was found that their use increased the efficiency of the estimators and thus various estimators have been developed using log and exponential functions Bahal and Tuteja [11], Latpate et al. [12], Grover and Kaur [8], Singh and Khare [13] and Singh and Rai [14]. Upon observation of such estimators it can be said that it might be possible to develop a class of estimators such that various estimators existing and non-existing which use different functions like log and exponential and different transformations can be derived from only one such developed class. Developing such a class was first attempted by Srivastava [15] where he developed a general class of estimators. This work was followed by Srivastava [16] and Srivastava and Jhajj [17].

In ACS design since the work of Dryver and Chao [18] where they proposed the transformed population approach numerous ratio-type estimators using a single auxiliary variable have been developed and studied for estimating the finite population mean. Dryver and Chao [18] first developed their ratio type estimator for estimating the finite population mean. Chutiman [19] proposed their ratio type estimator to estimate the finite population mean of the survey variable using some known parameters of the auxiliary variable. Qureshi et al. [20] proposed their estimator for the population mean of the survey variable using some robust measures. Chaudhry and Hanif [21] proposed an estimator using two auxiliary variables for estimating the population mean of the survey variable.

ACS is relatively new and its theory is being developed [22, 23] therefore the use of multi-auxiliary variables and different functions like log and exponential under different transformations have not been extensively studied to develop efficient estimators. Thus in this article we have combined all these ideas and proposed wider classes of estimators based on single and dual auxiliary variables for estimating the finite population mean of the survey variable so that the properties like bias and mean squared errors of various member estimators using several functions like log and exponential under different transformations would be known in advance. Further since log type estimators have not been studied much in the ACS design we have proposed a log-type class of estimators from each of the proposed wider classes respectively.

Since the ACS design has been of a lot of interest its methodology is known to most readers. For new readers, we recommend reading Dryver and Chao [18]. In Section 2 all the notations and terminologies used for derivations are presented. Section 3 comprises of some related estimators in the ACS design. In Section 4 we present the proposed wider class based on single auxiliary variable along with its derivation of bias and MSE. In sub-section 4.1 we propose a log type class of estimators along with some new log type estimators from the proposed wider class based on single auxiliary variable. In sub-section 4.2 two simulation studies have been conducted to demonstrate the performance of the proposed log type estimators over competing estimators in ACS design presented in this paper. In Section 5 we present the proposed wider class based on dual auxiliary variables along with its derivation of bias and MSE. In sub-section 5.1 we propose a log type class of estimators based on dual auxiliary variables along with some new log type estimators from the proposed wider class based on dual auxiliary variables. In sub-section 5.2 we have conducted two simulation studies to demonstrate the performance of these proposed log type estimators over competing estimators in ACS design. Finally all the estimators presented in this paper are applied to a real population to estimate the average number of Mules in the Indian state of Assam to highlight the efficiency of the proposed log-type estimators. In Section 7 we present the concluding remarks on this article with fruitful future areas of research.

2 Notations and Terminologies

Consider a finite population of N units with label $i = 1, 2, ..., N$. The survey variable is denoted by Y and the auxiliary variables by X_1 and X_2 respectively. Using SRSWOR a sample of size n is drawn and network averages $w_y = w_{y_1}, w_{y_2} ..., w_{y_n}$, $w_{x_1} = w_{x_{1_1}}, w_{x_{1_2}} ..., w_{x_{1_n}}$ and $w_{x_2} = w_{x_{2_1}}, w_{x_{2_2}} ..., w_{x_{2_n}}$ are obtained (Dryver and Chao (2007)[18]), where the network averages $w_{y_i} = \frac{1}{m_i}\sum_{j\in\xi_i} w_{y_j}$, $w_{x_{1_i}} = \frac{1}{m_i}\sum_{j\in\xi_i} w_{x_{1_j}}$ and $w_{x_{2_i}} = \frac{1}{m_i}\sum_{j\in\xi_i} w_{x_{2_j}}$ are the average values of y, x_1 and x_2 in the network ξ_i containing the i^{th} unit where m_i is the total number of units in the network ξ_i respectively. We are interested in estimating μ_y. Now consider the following notations:

$\bar{w_y} = \frac{1}{n}\sum_{i=1}^{n} w_{y_i}$, $\bar{w_{x_1}} = \frac{1}{n}\sum_{i=1}^{n} w_{x_{1_i}}$, $\bar{w_{x_2}} = \frac{1}{n}\sum_{i=1}^{n} w_{x_{2_i}}$,

$\mu_y = \frac{1}{N}\sum_{i=1}^{N} w_{y_i}$, $\mu_{x_1} = \frac{1}{N}\sum_{i=1}^{N} w_{x_{1_i}}$, $\mu_{x_2} = \frac{1}{N}\sum_{i=1}^{N} w_{x_{2_i}}$,

$S^2_{w_y} = \frac{1}{N-1}\sum_{i=1}^{N}(w_{y_i} - \mu_y)^2$, $S^2_{w_{x_1}} = \frac{1}{N-1}\sum_{i=1}^{N}(w_{x_{1_i}} - \mu_{x_1})^2$,

$S^2_{w_{x_2}} = \frac{1}{N-1}\sum_{i=1}^{N}(w_{x_{2_i}} - \mu_{x_2})^2$, $S_{w_y w_{x_1}} = \frac{1}{N-1}\sum_{i=1}^{N}((w_y - \mu_y)(w_{x_{1_i}} - \mu_{x_1}))$, $S_{w_y w_{x_2}} = \frac{1}{N-1}\sum_{i=1}^{N}((w_y - \mu_y)(w_{x_{2_i}} - \mu_{x_2}))$, $S_{w_{x_1} w_{x_2}} = \frac{1}{N-1}\sum_{i=1}^{N}((w_{x_{1_i}} - \mu_{x_1})(w_{x_{2_i}} - \mu_{x_2}))$, $C^2_{w_y} = \frac{S^2_{w_y}}{\mu^2_y}$, $C^2_{w_{x_1}} = \frac{S^2_{w_{x_1}}}{\mu^2_{x_1}}$, $C^2_{w_{x_2}} = \frac{S^2_{w_{x_2}}}{\mu^2_{x_2}}$.
The error terms used are follows:
$e_0 = \frac{\bar{w_y}}{\mu_y} - 1$, $e_1 = \frac{\bar{w_{x_1}}}{\mu_{x_1}} - 1$, $e_2 = \frac{\bar{w_{x_2}}}{\mu_{x_2}} - 1$, such that
$E(e_0) = E(e_1) = E(e_2) = 0$ and $E(e_0^2) = fC^2_{w_y}$, $E(e_1^2) = fC^2_{w_{x_1}}$, $E(e_2^2) = fC^2_{w_{x_2}}$, $E(e_0 e_1) = f\rho_{w_y w_{x_1}} C_{w_y} C_{w_{x_1}}$, $E(e_0 e_2) = f\rho_{w_y w_{x_2}} C_{w_y} C_{w_{x_2}}$ $E(e_1 e_2) = f\rho_{w_{x_1} w_{x_2}} C_{w_{x_1}} C_{w_{x_2}}$, $f = \frac{1}{n} - \frac{1}{N}$.

3 Some related estimators

In this section, we present some related estimators in the ACS design using one and two auxiliary variables.

3.1 Related estimators using one auxiliary variable

In this section, all the estimators presented use one auxiliary variable which will be denoted by X. The Hansen-Hurwitz type estimator for population mean proposed by Thompson [1] is:

$$t_{HH} = \frac{1}{n}\sum_{i=1}^{n} w_{y_i}. \tag{1}$$

The expression of variance proposed by Thompson [1] is as follows:

$$Var = fS^2_{w_y}. \tag{2}$$

Dryver and Chao [18] developed their estimator as follows:

$$t_{DC} = \mu_x \frac{\bar{w_y}}{\bar{w_x}}. \tag{3}$$

The expressions of bias and MSE proposed by Dryver and Chao [18] are:

$$Bias(t_{DC}) = f\mu_y(C^2_{w_x} - \rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{4}$$

and

$$MSE(t_{DC}) = f\mu_y^2(C^2_{w_y} + C^2_{w_x} - 2\rho_{w_y w_x} C_{w_y} C_{w_x}) \tag{5}$$

respectively.
A transformed estimator developed by Chutiman using known coefficient of variation and kurtosis[19] is as follows:

$$t_{CH_1} = \bar{w_y}\left(\frac{\mu_x + C_{w_x}}{\bar{w_y} + C_{w_x}}\right), \tag{6}$$

$$t_{CH_2} = \bar{w_y}\left(\frac{\mu_x + \beta_2(w_x)}{\bar{w_y} + \beta_2(w_x)}\right). \tag{7}$$

The expression of MSE proposed by Chutiman [19] is:

$$MSE(t_{CH_1}) = f\mu_y^2(C_{w_y}^2 + \alpha_{CH_1}^2 C_{w_x}^2 - 2\alpha_{CH_1}\rho_{w_y w_x} C_{w_y} C_{w_x}) \tag{8}$$

and

$$MSE(t_{CH_2}) = f\mu_y^2(C_{w_y}^2 + \alpha_{CH_2}^2 C_{w_x}^2 - 2\alpha_{CH_2}\rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{9}$$

where $\alpha_{CH_1} = \frac{\mu_x}{\mu_x + C_{w_x}}$ and $\alpha_{CH_2} = \frac{\mu_x}{\mu_x + \beta_2(w_x)}$.
Qureshi et al. [20] proposed some ratio type estimators using robust measure as:

$$t_{KQ_1} = \bar{w_y}\left(\frac{\mu_x MR + \beta_1(w_x)}{\bar{w_x} MR + \beta_1(w_x)}\right) \tag{10}$$

and

$$t_{KQ_2} = \bar{w_y}\left(\frac{\mu_x MR + TM}{\bar{w_x} MR + TM}\right). \tag{11}$$

The expressions of bias and MSE give by Qureshi et al. [20] are:

$$Bias(t_{KQ_1}) = f\mu_y(\alpha_{KQ_1}^2 C_{w_x}^2 - \alpha_{KQ_1}\rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{12}$$

$$Bias(t_{KQ_2}) = f\mu_y(\alpha_{KQ_2}^2 C_{w_x}^2 - \alpha_{KQ_2}\rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{13}$$

$$MSE(t_{KQ_1}) = f\mu_y^2(C_{w_y}^2 + \alpha_{KQ_1}^2 C_{w_x}^2 - 2\alpha_{KQ_1}\rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{14}$$

and

$$MSE(t_{KQ_2}) = f\mu_y^2(C_{w_y}^2 + \alpha_{KQ_2}^2 C_{w_x}^2 - 2\alpha_{KQ_2}\rho_{w_y w_x} C_{w_y} C_{w_x}), \tag{15}$$

where $\alpha_{KQ_1} = \frac{\mu_x MR}{\mu_x MR + \beta_1(w_x)}$ and $\alpha_{KQ_2} = \frac{\mu_x MR}{\mu_x MR + TM}$.

3.2 Related estimators using dual auxiliary variables

Since ACS is relatively new, estimators using multi-auxiliary information are not much explored and thus in this sub-section, we present a related estimator based on two auxiliary variables proposed by Chaudhry and Hanif [21] as follows:

$$t_H = (\bar{w_y} + \beta(\mu_{x_2} - \bar{w_{x_2}}))exp\left(\frac{\mu_{x_1} - \bar{w_{x_1}}}{\mu_{x_1}}\right). \tag{16}$$

The expression of MSE proposed by Chaudhry and Hanif [21] is:

$$\begin{aligned} MSE(t_H) = \mu_y^2 f C_{w_y}^2 + \mu_y^2 f C_{w_{x_1}}^2 + \beta^2\mu_{x_2}^2 f C_{w_{x_2}}^2 - 2\beta\mu_y\mu_{x_2} f\rho_{w_y w_{x_2}} \\ C_{w_y} C_{w_{x_2}} - 2\mu_y^2 f\rho_{w_y w_{x_1}} C_{w_y} C_{w_{x_1}} + 2\beta\mu_y\mu_{x_2} f\rho_{w_{x_1} w_{x_2}} C_{w_{x_1}} C_{w_{x_2}}. \end{aligned} \tag{17}$$

4 Proposed wider class of estimators based one auxiliary variable

In this section we propose a wider class of estimators using one auxiliary variable as follows:

$$t_{w_I} = (k_1 \bar{w}_y + k_2(\mu_x - \bar{w}_x))\delta(u) \quad (18)$$

where $u = \frac{\alpha\mu_x+\gamma}{\alpha\bar{w}_x+\gamma}$, k_1, k_2 are constants such that they minimize the $MSE_{t_{w_I}}$ and $\delta(.)$ is a parametric function and satisfies the following condition:

1. $\delta(1) = 1$.
2. The first and the second order partial derivatives of δ with respect to u exists and are known constants at $\delta = 1$.

To derive the expression of bias and MSE of the proposed wider class t_{w_I}, we first expand $\delta(u)$ about the value 1 in second order Taylor's series as

$$\delta(u) = \delta(1) + (u-1)H_1 + (u-1)^2 H_2,$$

where $H_1 = \frac{\partial\delta}{\partial u}\Big|_{u=1}$ and $H_2 = \frac{1}{2}\frac{\partial^2\delta}{\partial^2 u}\Big|_{u=1}$.

Note that upon simplification we get $u - 1 = \frac{-e_1}{\theta} + \frac{e_1^2}{\theta^2}$ and $(u-1)^2 = \frac{e_1^2}{\theta^2}$ where $\theta = 1 + \frac{\gamma}{\alpha\mu_x}$. Using above values and the error terms defined in Section 2, we get

$$t_{w_I} = (k_1\mu_y + k_1\mu_y e_0 - \mu_x e_1 k_2)\left((1 + (\frac{-e_1}{\theta} + \frac{e_1^2}{\theta^2})H_1 + \frac{e_1^2}{\theta^2}H_2\right). \quad (19)$$

Simplifying and subtracting μ_y from both sides and taking expectation we get

$$\begin{aligned} Bias(t_{w_I}) = k_1\mu_y(1 + \frac{1}{\theta^2} f C_{w_x}^2 (H_1 + H_2) \\ -\frac{1}{\theta} H_1 f \rho_{w_y w_x} C_{w_y} C_{w_x}) + k_2\mu_x \frac{1}{\theta} H_1 f C_{w_x}^2 - \mu_y \end{aligned} \quad (20)$$

Similarly, the expression of MSE is obtained as

$$MSE(t_{w_I}) = \mu_y^2 + k_1^2 A + k_2^2 B + 2k_1k_2C - 2k_1D - 2k_2E \quad (21)$$

where
$A = \mu_y^2 + \frac{1}{\theta^2}\mu_y^2 f C_{w_x}^2 H_1^2 + \mu_y^2 f C_{w_y}^2 + 2\left(\frac{1}{\theta^2}\mu_y^2 f C_{w_x}^2 (H_1 + H_2) - \frac{2}{\theta}\mu_y^2 f \rho_{w_y w_x} C_{w_y} C_{w_x} H_1\right)$,
$B = \mu_x^2 f C_{w_x}^2$,
$C = \frac{2}{\theta}\mu_y\mu_x f C_{w_x}^2 H_1 - \mu_y\mu_x f \rho_{w_y w_x} C_{w_y} C_{w_x}$,
$D = \mu_y^2 + \frac{1}{\theta^2}\mu_y^2 f C_{w_x}^2 H_1 + \frac{1}{\theta^2}\mu_y^2 f C_{w_x}^2 H_2 - \frac{1}{\theta}\mu_y^2 f \rho_{w_y w_x} C_{w_y} C_{w_x} H_1$,
$E = \frac{1}{\theta}\mu_y\mu_x f C_{w_x}^2 H_1$.

Partially differentiating equation (21) with respect to k_1 and k_2 and equating it to zero, we get the optimum values as

$$k_{1_{opt}} = \frac{BD - CE}{AB - C^2} \quad (22)$$

$$k_{2_{opt}} = \frac{AE - CD}{AB - C^2} \tag{23}$$

Putting $k_{1_{opt}}$ and $k_{2_{opt}}$ in (21) the minimum MSE of t_{w_I} is

$$MSE(t_{w_{I_{min}}}) = \mu_y^2 + k_{1_{opt}}^2 A + k_{2_{opt}}^2 B + 2k_{1_{opt}} k_{2_{opt}} C - 2k_{1_{opt}} D - 2k_{2_{opt}} E \tag{24}$$

Is should be noted that all the estimators based on single auxiliary variable presented in this article are members of the proposed wider class t_{w_I}.

1. For $(k_1, k_2, \delta(u)) = (1, 0, 1)$, $t_{w_I} \rightarrow t_{HH}$[1].
2. For $(k_1, k_2, \delta(u)) = (1, 0, \frac{\mu_x}{\bar{w_x}})$, $t_{w_I} \rightarrow t_{DC}$[18].
3. For $(k_1, k_2, \delta(u)) = (1, 0, \frac{(\mu_x + C_{w_x})}{(\bar{w_x} + C_{w_x})})$, $t_{w_I} \rightarrow t_{CH_1}$[19].
4. For $(k_1, k_2, \delta(u)) = (1, 0, \frac{(\mu_x + \beta_2(w_x))}{(\bar{w_x} + \beta_2(w_x))})$, $t_{w_I} \rightarrow t_{CH_2}$[19].
5. For $(k_1, k_2, \delta(u)) = (1, 0, \frac{(MR\mu_x + \beta_1(w_x))}{(MR\bar{w_x} + \beta_1(w_x))})$, $t_{w_I} \rightarrow t_{KQ_1}$[20].
6. For $(k_1, k_2, \delta(u)) = (1, 0, \frac{(MR\mu_x + TM)}{(MR\bar{w_x} + TM)})$, $t_{w_I} \rightarrow t_{KQ_2}$[20].

4.1 Log type class derived from t_{w_I}

In this sub-section we have developed a log type class from the proposed wider class t_{w_I} using $\delta(u) = 1 + log(u)$ where $u = \frac{\alpha\mu_x + \gamma}{\alpha\bar{w_x} + \gamma}$. The developed class is as follows:

$$t_L = (k_1\bar{w_y} + k_2(\mu_x - \bar{w_x}))\left(1 + log\left(\frac{\alpha\mu_x + \gamma}{\alpha\bar{w_x} + \gamma}\right)\right). \tag{25}$$

The bias of this class can be obtained by putting $H_1 = \frac{\partial\delta}{\partial u}\Big|_{u=1} = 1$ and $H_2 = \frac{1}{2}\frac{\partial^2\delta}{\partial^2 u}\Big|_{u=1} = -\frac{1}{2}$ in (20). Similarly the MSE can be obtained by putting $H_1 = 1$ and $H_2 = -\frac{1}{2}$ in (22)-(23) and putting it in (24). The exact expressions of bias and MSE are

$$Bias(t_L) = k_1\mu_y(1 + \frac{1}{2}\frac{1}{\theta^2} fC_{w_x}^2 - \frac{1}{\theta} f\rho_{w_y w_x} C_{w_y} C_{w_x}) + k_2\mu_x \frac{1}{\theta} fC_{w_x}^2 - \mu_y, \tag{26}$$

and

$$MSE(t_{L_{min}}) = \mu_y^2 + k_{1_{L_{opt}}}^2 A_L + k_{2_{L_{opt}}}^2 B_L + 2k_{1_{L_{opt}}} k_{2_{L_{opt}}} C_L - 2k_{1_{L_{opt}}} D_L - 2k_{2_{L_{opt}}} E_L \tag{27}$$

respectively where $A_L = \mu_y^2 + \frac{2}{\theta^2}\mu_y^2 fC_{w_x}^2 + \mu_y^2 fC_{w_y}^2 - \frac{4}{\theta}\mu_y^2 f\rho_{w_y w_x} C_{w_y} C_{w_x}$,
$B_L = \mu_x^2 fC_{w_x}^2$,
$C_L = \frac{2}{\theta}\mu_y\mu_x fC_{w_x}^2 - \mu_y\mu_x f\rho_{w_y w_x} C_{w_y} C_{w_x}$,
$D_L = \mu_y^2 + \frac{1}{2}\frac{1}{\theta^2}\mu_y^2 fC_{w_x}^2 - \frac{1}{\theta}\mu_y^2 f\rho_{w_y w_x} C_{w_y} C_{w_x}$,
$E_L = \frac{1}{\theta}\mu_y\mu_x fC_{w_x}^2$ with $k_{1_{L_{opt}}} = \frac{B_L D_L - C_L E_L}{A_L B_L - C_L^2}$ and $k_{2_{L_{opt}}} = \frac{A_L E_L - C_L D_L}{A_L B_L - C_L^2}$.

From this developed class we propose the following estimators:

1. $t_{L_1} = (k_1\bar{w}_y + k_2(\mu_x - \bar{w}_x))\left(1 + log\left(\frac{TM(w_x)\mu_x + MR(w_x)}{TM(w_x)\bar{w}_x + MR(w_x)}\right)\right)$,
2. $t_{L_2} = (k_1\bar{w}_y + k_2(\mu_x - \bar{w}_x))\left(1 + log\left(\frac{TM(w_x)\mu_x + \beta_2(w_x)}{TM(w_x)\bar{w}_x + \beta_2(w_x)}\right)\right)$,
3. $t_{L_3} = (k_1\bar{w}_y + k_2(\mu_x - \bar{w}_x))\left(1 + log\left(\frac{MR(w_x)\mu_x + S^2_{w_x}}{MR(w_x)\bar{w}_x + S^2_{w_x}}\right)\right)$.

The bias and MSE of these estimators can be easily obtained from equation $(26)-(27)$ using $\theta = 1 + \frac{\gamma}{\alpha\mu_x}$.

4.2 Simulation study

To study the performance of the developed log type estimators ($t_{L_{1-3}}$) over competing estimators based on single auxiliary varaible presented in Section 2 we have conducted two simulation studies. The performance of the estimators are compared on the basis of Relative root mean square error (RRMSE). For the simulation studies, we use the Blue Winged Teal data used by Dryver and Chao (2007)[18] and Smith et al.[24]. The following algorithm is used for conducting the simulation studies:

1. Population-1 of size 50 is generated using the model $y = 4x + e$ where x takes the values of Blue-Winged Teal[18, 24] and $e \sim N(0, 4x)$.
2. Using sample sizes $n = 9, 11, 13, 15$ sampling procedure of ACS is repeated 20000 times and several values of estimates of population mean μ_y are obtained.
3. MSE for each sample size is calculated using the formula $MSE(t_i) = \frac{1}{20000}\sum_{r=1}^{20000}(t_i - \mu_y)^2$ where $t_i = t_{HH}, t_{DC}, t_{CH_1}, t_{CH_2}, t_{KQ_1}, t_{KQ_2}, t_{L_1}, t_{L_2}, t_{L_3}$.
4. For comparison RRMSE is calculated as $RRMSE(t_i) = \frac{1}{\mu_y}\sqrt{MSE(t_i)}$ where t_i is defined in step 3 and the values of RRMSEs are presented in Table-1.

For the second simulation study population-2 is generated using model $y = 3x + e$. Following steps 2-4, RRMSEs are obtained and presented in Table-2.

Table 1 Relative root mean square errors of all estimators in case of population-1

Estimators	$n = 9$	$n = 11$	$n = 13$	$n = 15$
t_{HH}	0.7472	0.6673	0.5943	0.5434
t_{DC}	0.3392	0.2884	0.2345	0.1942
t_{CH_1}	0.4399	0.3627	0.2921	0.2394
t_{CH_2}	0.4361	0.3594	0.2893	0.2371
t_{KQ_1}	0.3770	0.3045	0.2429	0.1976
t_{KQ_2}	0.3770	0.3045	0.2429	0.1976
t_{L_1}	0.1609	0.1408	0.1239	0.1122
t_{L_2}	3.8993	3.3459	1.2320	0.7745
t_{L_3}	0.1710	0.1492	0.1309	0.1184

Table 2 Relative root mean square errors of all estimators in case of population-2

Estimators	$n = 9$	$n = 11$	$n = 13$	$n = 15$
t_{HH}	0.7036	0.6297	0.5628	0.5106
t_{DC}	7.0590	5.5925	4.4035	3.3961
t_{CH_1}	1.7612	1.4310	1.1637	0.9297
t_{CH_2}	1.9853	1.6180	1.3200	1.0556
t_{KQ_1}	6.9331	5.5329	4.3742	3.3807
t_{KQ_2}	6.9351	5.5344	4.3754	3.3814
t_{L_1}	0.1639	0.1428	0.1255	0.1139
t_{L_2}	3.9165	3.3120	1.2339	0.7685
t_{L_3}	0.1735	0.1506	0.1327	0.1198

5 Proposed wider class of estimators based on two auxiliary variabales

In this section we propose the wider class of estimators based on dual auxiliary variables as follows

$$t_{w_{II}} = (m_1 \bar{w_y} + m_2(\mu_{x_1} - \bar{w_{x_1}}) + m_3(\mu_{x_2} - \bar{w_{x_2}}))\delta(s, v), \tag{28}$$

where $s = \frac{\alpha_1 \mu_{x_1} + \gamma_1}{\alpha_1 \bar{w_{x_1}} + \gamma_1}$, $v = \frac{\alpha_2 \mu_{x_2} + \gamma_2}{\alpha_2 \bar{w_{x_2}} + \gamma_2}$, m_1, m_2, m_3 are constants such that they minimize the $MSE_{t_{w_{II}}}$ and $\delta(s, v)$ is a function of s and v such that it satisfies the following condition:

1. The point (s, v) assumes the value in a closed convex subset of R_2 of two-dimensional real space containing the point $(1, 1)$.
2. $\delta(s, v)$ is bounded and continuous in R_2.
3. $\delta(1, 1) = 1$.
4. The partial derivatives of first and second order of $\delta(s, v)$ exists and are continuous and bounded in R_2.

In order to derive the expression of bias and MSE of the proposed wider class $t_{w_{II}}$, we first expand $\delta(s, v)$ about the value $(1, 1)$ in second-order Taylor's series as

$$\delta(s, v) = \delta(1, 1) + (s - 1)\Delta_1 + (v - 1)\Delta_2 + (s - 1)^2\Delta_3 + (v - 1)^2\Delta_4 + (s - 1)(v - 1)\Delta_5,$$

where $\Delta_1 = \left.\frac{\partial \delta}{\partial s}\right|_{(s=1,v=1)}$, $\Delta_2 = \left.\frac{\partial \delta}{\partial v}\right|_{(s=1,v=1)}$, $\Delta_3 = \left.\frac{1}{2}\frac{\partial^2 \delta}{\partial s^2}\right|_{(s=1,v=1)}$, $\Delta_4 = \left.\frac{1}{2}\frac{\partial^2 \delta}{\partial v^2}\right|_{(s=1,v=1)}$ and $\Delta_5 = \left.\frac{\partial^2 \delta}{\partial s \partial v}\right|_{(s=1,v=1)}$.

Note that upon simplification we get $s - 1 = \frac{-e_1}{\theta_1} + \frac{e_1^2}{\theta_1^2}$, $v - 1 = \frac{-e_2}{\theta_2} + \frac{e_2^2}{\theta_2^2}$, $(s - 1)^2 = \frac{e_1^2}{\theta_1^2}$, $(v - 1)^2 = \frac{e_2^2}{\theta_2^2}$, $(s - 1)(v - 1) = \frac{e_1 e_2}{\theta_1 \theta_2}$, $(s - 1)^2 = \frac{e_1^2}{\theta_1^2}$ where $\theta_1 = 1 + \frac{\gamma_1}{\alpha_1 \mu_{x_1}}$ and

$\theta_2 = 1 + \frac{\gamma_2}{\alpha_2 \mu_{x_2}}$. Using above values and the error terms defined in Section 2, we get

$$\begin{aligned} t_{w_{II}} &= (m_1\mu_y + m_1\mu_y e_0 - m_2\mu_{x_1}e_1 - m_3\mu_{x_2}e_2)(1 + (\frac{-e_1}{\theta_1} \\ &+ \frac{e_1^2}{\theta_1^2})h_1 + (\frac{-e_2}{\theta_2} + \frac{e_2^2}{\theta_2^2})h_2 + \frac{e_1^2}{\theta_1^2}h_3 + \frac{e_2^2}{\theta_2^2}h_4 + \frac{e_{12}}{\theta_1\theta_2}h_5). \end{aligned} \tag{29}$$

Simplifying and subtracting μ_y from both sides and taking expectation we get

$$\begin{aligned} Bias(t_{w_I}) &= m_1\mu_y(1 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_1 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_2 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_3 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_4 + \frac{1}{\theta_1\theta_2}f \\ &\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_5 - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_yC_{w_{x_1}}h_1 - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_yC_{w_{x_2}}h_2) + m_2\mu_{x_1}(\frac{1}{\theta_1}fC_{w_{x_1}}^2 h_1 \\ &+ \frac{1}{\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_2) + m_3\mu_{x_2}(\frac{1}{\theta_2}fC_{w_{x_2}}^2 h_2 + \frac{1}{\theta_1}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_1) - \mu_y. \end{aligned} \tag{30}$$

Similarly, the expression of MSE is obtained as

$$\begin{aligned} MSE(t_{w_{II}}) &= \mu_y^2 + m_1^2A_w + m_2^2B_w + m_3^2C_w + 2m_1m_2D_w \\ &+ 2m_1m_3E_w + 2m_2m_3F_w - 2m_1G_w - 2m_2H_w - 2m_3I_w, \end{aligned} \tag{31}$$

where
$A_w = \mu_y^2(1 + fC_{w_y}^2 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_1^2 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_2^2 + 2(\frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_1 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_2 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_3 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_4 + \frac{1}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_5 - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}}h1 - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}}h_2 + \frac{1}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_1h_2 - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}}h1 - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}}h_2)$,
$B_w = \mu_{x_1}^2 fC_{w_{x_1}}^2$,
$C_w = \mu_{x_2}^2 fC_{w_{x_2}}^2$,
$D_w = \mu_y\mu_{x_1}(\frac{2}{\theta_1}fC_{w_{x_1}}^2 h_1 + \frac{2}{\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_2 - f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}})$,
$E_w = \mu_y\mu_{x_2}(\frac{2}{\theta_2}fC_{w_{x_2}}^2 h_2 + \frac{2}{\theta_1}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_1 - f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}})$,
$F_w = \mu_{x_1}\mu_{x_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}$,
$G_w = \mu_y^2(1 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_1 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_2 + \frac{1}{\theta_1^2}fC_{w_{x_1}}^2 h_3 + \frac{1}{\theta_2^2}fC_{w_{x_2}}^2 h_4 + \frac{1}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_5 - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}}h_1 - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}}h_2)$,
$H_w = \mu_y\mu_{x_1}(\frac{1}{\theta_1}fC_{w_{x_1}}^2 h_1 + \frac{1}{\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_2)$,
$I_w = \mu_y\mu_{x_2}(\frac{1}{\theta_2}fC_{w_{x_2}}^2 h_2 + \frac{1}{\theta_1}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}h_1)$.

Partially differentiating (31) with respect to m_1, m_2 and m_3 and equating them to zero we get

$$m_1A_w + m_2D_w + m_3E_w = G_w, \tag{32}$$

$$m_1D_w + m_2B_w + m_3F_w = H_w, \tag{33}$$

$$m_1 E_w + m_2 F_w + m_3 C_w = I_w \tag{34}$$

This system of linear equation can be written as

$$\begin{bmatrix} A_w & D_w & E_w \\ D_w & B_w & F_w \\ E_w & F_w & C_w \end{bmatrix} \times \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} G_w \\ H_w \\ I_w \end{bmatrix} \tag{35}$$

Using Cramer rule we get

$$m_{1_{opt}} = \frac{\Delta_{m_1}}{\Delta}, m_{2_{opt}} = \frac{\Delta_{m_2}}{\Delta}, m_{3_{opt}} = \frac{\Delta_{m_3}}{\Delta}, \tag{36}$$

where

$$\Delta_{m_1} = \begin{vmatrix} G_w & D_w & E_w \\ H_w & B_w & F_w \\ I_w & F_w & C_w \end{vmatrix} \tag{37}$$

$$\Delta_{m_2} = \begin{vmatrix} A_w & G_w & E_w \\ D_w & H_w & F_w \\ E_w & I_w & C_w \end{vmatrix} \tag{38}$$

$$\Delta_{m_3} = \begin{vmatrix} A_w & D_w & G_w \\ D_w & B_w & H_w \\ E_w & F_w & I_w \end{vmatrix} \tag{39}$$

and

$$\Delta = \begin{vmatrix} A_w & D_w & E_w \\ D_w & B_w & F_w \\ E_w & F_w & C_w \end{vmatrix} \tag{40}$$

Solving the determinants in equations (37) – (40) and using them in (36) we get

$$m_{1_{opt}} = \frac{G_w(B_w C_w - F_w^2) - D_w(H_w C_w - I_w F_w) + E_w(H_w F_w - I_w B_w)}{A_w B_w C_w - A_w F_w^2 - D_w^2 C_w + 2D_w E_w F_w - B_w E_w^2}, \tag{41}$$

$$m_{2_{opt}} = \frac{A_w(H_w C_w - I_w F_w) - G_w(D_w C_w - E_w F_w) + E_w(D_w I_w - E_w H_w)}{A_w B_w C_w - A_w F_w^2 - D_w^2 C_w + 2D_w E_w F_w - B_w E_w^2}, \tag{42}$$

$$m_{3_{opt}} = \frac{A_w(I_w B_w - H_w F_w) - D_w(I_w D_w - H_w E_w) + G_w(D_w F_w - E_w B_w)}{A_w B_w C_w - A_w F_w^2 - D_w^2 C_w + 2D_w E_w F_w - B_w E_w^2}. \tag{43}$$

Using (41) – (43) in (31) we get

$$\begin{aligned} MSE(t_{w_{II_{min}}}) &= \mu_y^2 + m_{1_{opt}}^2 A_w + m_{2_{opt}}^2 B_w + m_{3_{opt}}^2 C_w + 2m_{1_{opt}} m_{2_{opt}} D_w \\ &+ 2m_{1_{opt}} m_{3_{opt}} E_w + 2m_{2_{opt}} m_{3_{opt}} F_w - 2m_{1_{opt}} G_w - 2m_{2_{opt}} H_w - 2m_{3_{opt}} I_w. \end{aligned} \tag{44}$$

5.1 Log type class derived from $t_{w_{II}}$

In this sub-section we have developed a log type class from the proposed wider class $t_{w_{II}}$ using $\delta(s,v) = (1+log(s))(1+log(v))$ where $s = \frac{\alpha_1\mu_{x_1}+\gamma_1}{\alpha_1\bar{w_{x_1}}+\gamma_1}$, $v = \frac{\alpha_2\mu_{x_2}+\gamma_2}{\alpha_2\bar{w_{x_2}}+\gamma_2}$. The developed class is:

$$t_{L_{II}} = (m_1\bar{w_y} + m_2(\mu_{x_1} - \bar{w_{x_1}}) + m_3((\mu_{x_2} - \bar{w_{x_2}}))) \left(1 + log\left(\frac{\alpha_1\mu_{x_1}+\gamma_1}{\alpha_1\bar{w_{x_1}}+\gamma_1}\right)\right)\left(1 + log\left(\frac{\alpha_2\mu_{x_2}+\gamma_2}{\alpha_2\bar{w_{x_2}}+\gamma_2}\right)\right) \quad (45)$$

The bias and MSE_{min} of this class can be obtained using $\Delta_1 = \frac{\partial\delta}{\partial s}\Big|_{(s=1,v=1)} = 1$, $\Delta_2 = \frac{\partial\delta}{\partial v}\Big|_{(s=1,v=1)} = 1$, $\Delta_3 = \frac{1}{2}\frac{\partial^2\delta}{\partial s^2}\Big|_{(s=1,v=1)} = -\frac{1}{2}$, $\Delta_4 = \frac{1}{2}\frac{\partial^2\delta}{\partial v^2}\Big|_{(s=1,v=1)} = -\frac{1}{2}$ and $\Delta_5 = \frac{\partial^2\delta}{\partial s\partial v}\Big|_{(s=1,v=1)} = 1$ in (30) and (41) $-$ (44) respectively.The expressions of bias and MSE_{min} of the developed class $t_{L_{II}}$ are

$$\begin{aligned} Bias(t_{L_{II}}) &= m_1\mu_y(1 + \frac{1}{2\theta_1^2}fC^2_{w_{x_1}} + \frac{1}{2\theta_2^2}fC^2_{w_{x_2}} + \frac{1}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}} \\ & C_{w_{x_2}} - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_yC_{w_{x_1}} - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_yC_{w_{x_2}}) + m_2\mu_{x_1}(\frac{1}{\theta_1}fC^2_{w_{x_1}} \\ &+ \frac{1}{\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}) + m_3\mu_{x_2}(\frac{1}{\theta_2}fC^2_{w_{x_2}} + \frac{1}{\theta_1}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}) - \mu_y, \end{aligned} \quad (46)$$

$$\begin{aligned} MSE(t_{L_{II_{min}}}) &= \mu_y^2 + m^2_{1_{opt}}A_{w_{L_{II}}} + m^2_{2_{opt}}B_{w_{L_{II}}} + m^2_{3_{opt}}C_{w_{L_{II}}} + 2m_{1_{opt}}m_{2_{opt}}D_{w_{L_{II}}} \\ &+ 2m_{1_{opt}}m_{3_{opt}}E_{w_{L_{II}}} + 2m_{2_{opt}}m_{3_{opt}}F_{w_{L_{II}}} - 2m_{1_{opt}}G_{w_{L_{II}}} - 2m_{2_{opt}}H_{w_{L_{II}}} - 2m_{3_{opt}}I_{w_{L_{II}}}, \end{aligned} \quad (47)$$

where $A_{w_{L_{II}}} = \mu_y^2(1 + fC^2_{w_y} + \frac{2}{\theta_1^2}fC^2_{w_{x_1}} + \frac{2}{\theta_2^2}fC^2_{w_{x_2}} + \frac{4}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}} - \frac{4}{\theta_2}f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}} - \frac{4}{\theta_1}f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}})$,
$B_{w_{L_{II}}} = \mu^2_{x_1}fC^2_{w_{x_1}}$,
$C_{w_{L_{II}}} = \mu^2_{x_2}fC^2_{w_{x_2}}$,
$D_{w_{L_{II}}} = \mu_y\mu_{x_1}(\frac{2}{\theta_1}fC^2_{w_{x_1}} + \frac{2}{\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}} - f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}})$,
$E_{w_{L_{II}}} = \mu_y\mu_{x_2}(\frac{2}{\theta_2}fC^2_{w_{x_2}} + \frac{2}{\theta_1}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}} - f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}})$,
$F_{w_{L_{II}}} = \mu_{x_1}\mu_{x_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}$,
$G_{w_{L_{II}}} = \mu_y^2(1 + \frac{1}{2\theta_1^2}fC^2_{w_{x_1}} + \frac{1}{2\theta_2^2}fC^2_{w_{x_2}} + \frac{1}{\theta_1\theta_2}f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}} - \frac{1}{\theta_2}f\rho_{w_yw_{x_2}}C_{w_y}C_{w_{x_2}} - \frac{1}{\theta_1}f\rho_{w_yw_{x_1}}C_{w_y}C_{w_{x_1}})$,
$H_{w_{L_{II}}} = \mu_y\mu_{x_1}(\frac{fC^2_{w_{x_1}}}{\theta_1} + \frac{f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}}{\theta_2})$,
$I_{w_{L_{II}}} = \mu_y\mu_{x_2}(\frac{fC^2_{w_{x_2}}}{\theta_2} + \frac{f\rho_{w_{x_1}w_{x_2}}C_{w_{x_1}}C_{w_{x_2}}}{\theta_1})$,

$$m_{1_{opt}} = \frac{G_{w_{L_{II}}}(B_{w_{L_{II}}}C_{w_{L_{II}}} - F^2_{w_{L_{II}}}) - D_{w_{L_{II}}}(H_{w_{L_{II}}}C_{w_{L_{II}}} - I_{w_{L_{II}}}F_{w_{L_{II}}}) + E_{w_{L_{II}}}(H_{w_{L_{II}}}F_{w_{L_{II}}} - I_{w_{L_{II}}}B_{w_{L_{II}}})}{A_{w_{L_{II}}}B_{w_{L_{II}}}C_{w_{L_{II}}} - A_{w_{L_{II}}}F^2_{w_{L_{II}}} - D^2_{w_{L_{II}}}C_{w_{L_{II}}} + 2D_{w_{L_{II}}}E_{w_{L_{II}}}F_{w_{L_{II}}} - B_{w_{L_{II}}}E^2_{w_{L_{II}}}},$$
$$m_{2_{opt}} = \frac{A_{w_{L_{II}}}(H_{w_{L_{II}}}C_{w_{L_{II}}} - I_{w_{L_{II}}}F_{w_{L_{II}}}) - G_{w_{L_{II}}}(D_{w_{L_{II}}}C_{w_{L_{II}}} - E_{w_{L_{II}}}F_{w_{L_{II}}}) + E_{w_{L_{II}}}(D_{w_{L_{II}}}I_{w_{L_{II}}} - E_{w_{L_{II}}}H_{w_{L_{II}}})}{A_{w_{L_{II}}}B_{w_{L_{II}}}C_{w_{L_{II}}} - A_{w_{L_{II}}}F^2_{w_{L_{II}}} - D^2_{w_{L_{II}}}C_{w_{L_{II}}} + 2D_{w_{L_{II}}}E_{w_{L_{II}}}F_{w_{L_{II}}} - B_{w_{L_{II}}}E^2_{w_{L_{II}}}},$$
$$m_{3_{opt}} = \frac{A_{w_{L_{II}}}(I_{w_{L_{II}}}B_{w_{L_{II}}} - H_{w_{L_{II}}}F_{w_{L_{II}}}) - D_{w_{L_{II}}}(I_{w_{L_{II}}}D_{w_{L_{II}}} - H_{w_{L_{II}}}E_{w_{L_{II}}}) + G_{w_{L_{II}}}(D_{w_{L_{II}}}F_{w_{L_{II}}} - E_{w_{L_{II}}}B_{w_{L_{II}}})}{A_{w_{L_{II}}}B_{w_{L_{II}}}C_{w_{L_{II}}} - A_{w_{L_{II}}}F^2_{w_{L_{II}}} - D^2_{w_{L_{II}}}C_{w_{L_{II}}} + 2D_{w_{L_{II}}}E_{w_{L_{II}}}F_{w_{L_{II}}} - B_{w_{L_{II}}}E^2_{w_{L_{II}}}}.$$

From this developed class$t_{L_{II}}$ we propose the following estimators:

1. $t_{L_{II_1}} = (m_1\bar{w_y} + m_2(\mu_{x_1} - \bar{w_{x_1}}) + m_3((\mu_{x_2} - \bar{w_{x_2}})))$
   $\left(1 + log\left(\frac{TM(w_{x_1})\mu_{x_1} + MR(w_{x_1})}{TM(w_{x_1})\bar{w_{x_1}} + MR(w_{x_1})}\right)\right)\left(1 + log\left(\frac{TM(w_{x_2})\mu_{x_2} + MR(w_{x_2})}{TM(w_{x_2})\bar{w_{x_2}} + MR(w_{x_2})}\right)\right)$,

2. $t_{L_{II_2}} = (m_1\bar{w_y} + m_2(\mu_{x_1} - \bar{w_{x_1}}) + m_3((\mu_{x_2} - \bar{w_{x_2}})))$
   $\left(1 + log\left(\frac{TM(w_{x_1})\mu_{x_1} + \beta_2(w_{x_1})}{TM(w_{x_1})\bar{w_{x_1}} + \beta_2(w_{x_1})}\right)\right)\left(1 + log\left(\frac{TM(w_{x_2})\mu_{x_2} + \beta_2(w_{x_2})}{TM(w_{x_2})\bar{w_{x_2}} + \beta_2(w_{x_2})}\right)\right)$,

3. $t_{L_{II_3}} = (m_1\bar{w_y} + m_2(\mu_{x_1} - \bar{w_{x_1}}) + m_3((\mu_{x_2} - \bar{w_{x_2}})))$
   $\left(1 + log\left(\frac{MR(w_{x_1})\mu_{x_1} + S^2_{w_{x_1}}}{MR(w_{x_1})\bar{w_{x_1}} + S^2_{w_{x_1}}}\right)\right)\left(1 + log\left(\frac{MR(w_{x_2})\mu_{x_2} + S^2_{w_{x_2}}}{MR(w_{x_2})\bar{w_{x_2}} + S^2_{w_{x_2}}}\right)\right)$.

The bias and MSE of these estimators can be easily obtained from equation $(46) - (47)$ using $\theta_1 = 1 + \frac{\gamma_1}{\alpha_1\mu_{x_1}}$ and $\theta_2 = 1 + \frac{\gamma_2}{\alpha_2\mu_{x_2}}$.

5.2 Simulation study

In this section we conduct two simulation studies to assess the performance of the developed log type estimators over competing estimators that are based on two auxiliary variables presented in Section 2 on the basis of Relative root mean square error (RRMSE). For the simulation studies, we again use the Blue Winged Teal data from Dryver and Chao (2007)[18] and Smith et al.[24].
The following algorithm is used for conducting the simulation studies:

1. Population-3 of size 50 is generated using the model $y = \frac{1}{3}x_1 + \frac{1}{3}x_2 + e$ where x_1 takes the values of Blue-Winged Teal [18] and x_2 is generated using the model $x_2 = 3x_1 + e$.
2. Using sample sizes $n = 9, 11, 13, 15$ sampling procedure of ACS is repeated 20000 times and several values of estimates of population mean μ_y are obtained.
3. MSE for each sample size is calculated using the formula $MSE(t_i) = \frac{1}{20000}\sum_{r=1}^{20000}(t_i - \mu_y)^2$ where $t_i = t_H, t_{L_{II_1}}, t_{L_{II_2}}, t_{L_{II_3}}$.
4. For comparison RRMSE is calculated as $RRMSE(t_i) = \frac{1}{\mu_y}\sqrt{MSE(t_i)}$ where t_i is defined in step 3 and the values of RRMSEs are presented in Table-3.

For the second simulation study population-4 is generated using model $y = \frac{1}{3}x_1 + \frac{1}{3}x_2 + e$ where x_1 takes the values of Blue-Winged Teal [18] and x_2 is generated using the model $x_2 = 3.5x_1 + e$. Following steps 2-4, RRMSEs are obtained and presented in Table-4.

Table 3 Relative root mean square errors of all estimators in case of population-3

Estimators	$n = 9$	$n = 11$	$n = 13$	$n = 15$
t_{HH}	0.748470	0.668437	0.595328	0.544375
t_H	0.887735	0.778505	0.683279	0.620539
t_{LII_1}	0.430774	0.336801	0.277607	0.240991
t_{LII_2}	0.741258	0.372371	0.501833	0.887541
t_{LII_3}	0.685537	0.475919	0.370864	0.311403

Table 4 Relative root mean square errors of all estimators in case of population-4

Estimators	$n = 9$	$n = 11$	$n = 13$	$n = 15$
t_{HH}	0.748383	0.668359	0.595258	0.544311
t_H	0.887737	0.778506	0.683280	0.620538
t_{LII_1}	0.742780	0.502529	0.387352	0.323269
t_{LII_2}	0.744425	0.371564	0.507063	0.896693
t_{LII_3}	0.685473	0.475888	0.370845	0.311389

14	1	0	0	3	1
4	0	2	10	13	0
3	4	0	0	6	0
0	0	0	7	0	0
0	0	0	24	0	0

chart **1** Network of Mules formed using the condition $C_y : y_i > 0$.

6 Application on real data

In this section we use all the estimators presented in this article to estimate the average number of Mules in the Indian state of Assam using number of Mules over three years and number of male Donkeys in the same district as the auxiliary variables. In this section we use the 19^{th} Livestock Census data [25] of the state of Assam. In order to use the ACS design, each of the twenty seven districts of Assam are treated as a quadrat and the entire population of Mules is divided into twnety seven quadrats. In order to have $5X6$ quadrats (for easily applying the ACS design) we add three quadrats having zero Mules and finally a population of Mules of Assam divided amongst thrity quadrats is used for this study. For comparison of estimators the formula of MSE of each estimator is used and the values obtained are presented in Table-5.

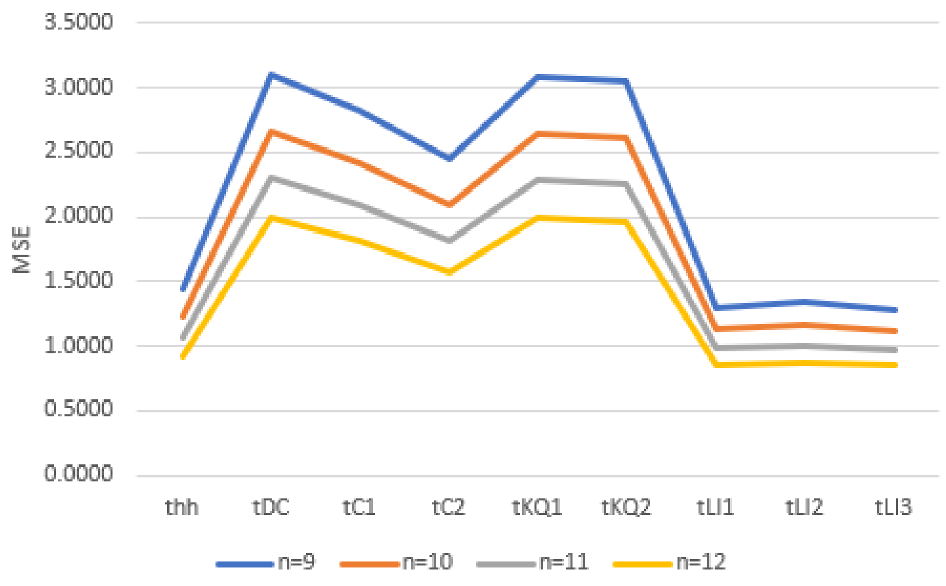

Fig. 1MSEs of all estimators based on one auxiliary variable in estimating average number of Mules.

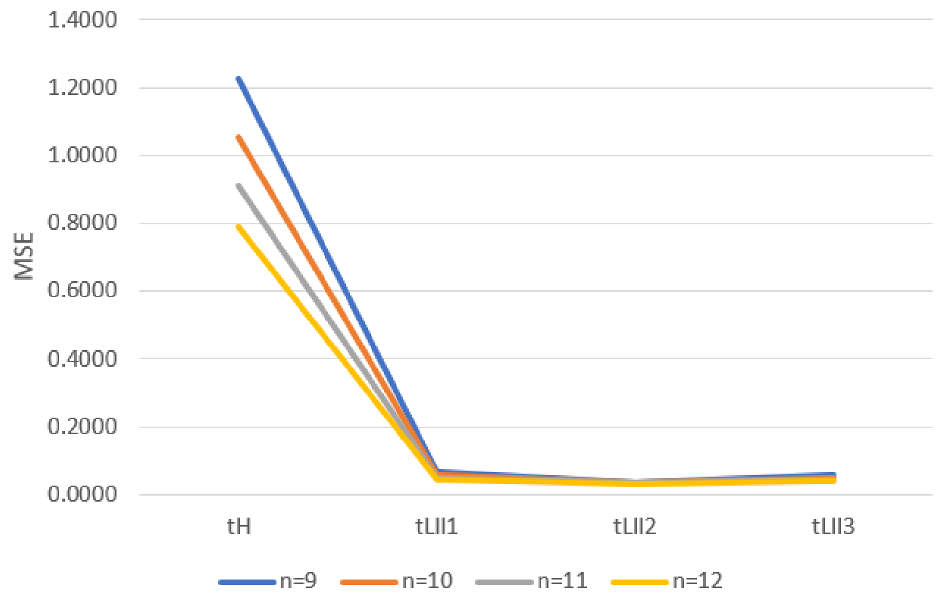

Fig. 2 MSEs of all estimators based on two auxiliary variable in estimating average number of Mules.

Table 5 Mean squared errors of all estimators in estimating average number of Mules in Assam

Estimators	$n = 9$	$n = 10$	$n = 11$	$n = 12$
t_{HH}	1.4421	1.2361	1.0676	0.9271
t_{DC}	3.1050	2.6621	2.2991	1.9966
t_{CH_1}	2.8298	2.4255	2.0948	1.8191
t_{CH_2}	2.4508	2.1007	1.8142	1.5755
t_{KQ_1}	3.0923	2.6505	2.2891	1.9879
t_{KQ_2}	3.0600	2.6200	2.2600	1.9700
t_H	1.2277	1.0523	0.9088	0.7892
$t_{L_{I_1}}$	1.3000	1.1300	0.9830	0.8610
$t_{L_{I_2}}$	1.3400	1.1600	1.0100	0.8810
$t_{L_{I_3}}$	1.2795	1.1100	0.9731	0.8530
$t_{L_{II_1}}$	0.0642	0.0562	0.0493	0.0433
$t_{L_{II_2}}$	0.0361	0.0365	0.0351	0.0330
$t_{L_{II_3}}$	0.0561	0.0506	0.0452	0.0403

7 Conclusion

The ACS design is relatively new and has not been explored much. Various efficient estimators based on single and dual auxiliary variables studied in the SRSWOR design have not been explored in the ACS design. Thus the aim of this research paper was to develop wider classes of estimators based on single and dual auxiliary variables in the ACS design so that the properties like bias and MSE of numerous member estimators of these proposed wider classes will be known in advance. The proposed wider classes using single and dual auxiliary variables have been presented in section 4 and 5 with their formulae of bias and MSE. In sections 4.1 and 5.1 we developed new log type classes of estimators using single and dual auxiliary variables $t_{w_{L_I}}$ and $t_{w_{L_{II}}}$ from the proposed wider classes t_{w_I} and $t_{w_{II}}$ and further proposed some log type estimators from each developed class $t_{w_{L_I}}$ and $t_{w_{L_{II}}}$ respectively.

The proposed log type estimators $t_{w_{L_{I_1}}} - t_{w_{L_{I_3}}}$ and $t_{w_{L_{II_1}}} - t_{w_{L_{II_3}}}$ have been developed using known parameters of auxiliary variables namely Tri-mean, Mid-range, coefficient of kutrosis and population variance. The performance of these estimators are then compared with several competing estimators presented in this article using various simulation studies presented in sections 4.2 and 5.2 respectively. The performance is compared using the Relative root mean square errors or RRMSEs. The results of the simulation studies have been tabulated in Tables 1-4. From the results we can see that the developed log type estimators $t_{w_{L_{I_1}}}$, $t_{w_{L_{I_3}}}$ and $t_{w_{L_{II_1}}}$, $t_{w_{L_{II_3}}}$ result in lower RRMSEs than the competing estimators.

Further we studied the performance of all the estimators on real data in section 6 using the 19^{th} Livestock Census data[25] to estimate the average number of Mules in the Indian state of Assam using all the estimators that have been presented in this article. The performance of all of these estimators have been compared on the criteria of MSE which is calculated using the formulae of MSE of all the estimators.

The result of this study is tabulated in Table-5. Note that amongst all the estimators based on single auxiliary variable namely $t_{HH}, t_{DC}, t_{CH_1}, t_{CH_2}, t_{KQ_1}, t_{KQ_2}$ our proposed estimators $t_{w_{L_{I_1}}}$-$t_{w_{L_{I_3}}}$ result in much lower MSE. Further we can see that the competing estimator based on two auxiliary variables namely t_H results in much higher MSE than the proposed log type estimators based on two auxiliary variables namely $t_{w_{L_{II_1}}} - t_{w_{L_{II_3}}}$.

Our aim in this study was to develop wider classes of estimators based on single and dual auxiliary variables so that the properties like bias and MSE of numerous member estimators based on any function which have not been developed yet will be known in advance without much effort. Since the ACS design is relatively new, estimators using functions like log and multiauxiliary variables have not been explored much therefore from the proposed wider classes t_{w_I} and $t_{w_{II}}$ we develop log type classes $t_{w_{L_I}}$ and $t_{w_{L_{II}}}$ and further propose some log type estimators $t_{w_{L_{I_1}}} - t_{w_{L_{I_3}}}$ and $t_{w_{L_{II_1}}} - t_{w_{L_{II_3}}}$ from the proposed log type classes $t_{w_{L_I}}$ and $t_{w_{L_{II}}}$ respectively. From the results of the simulation studies and application on real data, we recommend using the proposed log type estimators $t_{w_{L_{I_1}}}$, $t_{w_{L_{I_3}}}$ and $t_{w_{L_{II_1}}}$, $t_{w_{L_{II_3}}}$ when the population under study is rare or hidden clustered and ACS design is to be used. For future research studies we recommend studying different functions like exponential and different definition of neighborhood.

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