Method Article

Derived Reduced Balanced Incomplete Block Design

.

ABSTRACT

Construction of Balanced Incomplete Block Designs (BIBD) is a combination problem that involves the arrangement of v treatments into b blocks each of size k such that each treatment is replicated exactly r times in the design and a pair of treatments occur together in λ blocks. Several methods of constructing BIBDs exist. However, these methods still cannot be used to design all BIBDs. Therefore, several BIBDs are still unknown because a definite construction method for all BIBDs is still unknown. The study aimed to develop a new construction method that could aid in constructing more BIBDs. The study derived a new class of BIBD from un-reduced BIBD with parameters v and k such that $k \geq 3$ through selection of all blocks within the un-reduced BIBD that contains a particular treatment i then in the selected blocks treatment delete treatment i and retain all the other treatments. The resulting BIBD was Derived Reduced BIBD with parameters$v^* = v - 1, b^* = \binom{v-1}{k-1}, k^* = k - 1, r^* = \binom{v-2}{k-2}, \lambda = \binom{v-3}{k-3}$. In conclusion, the construction method was simple and could be used to construct several BIBDs, which could assist in solving the problem of BIBD, whose existence is still unknown.

Keywords: Un-reduced BIBD; Block; Treatment; Reduced; Derived; Incomplete Block.

1. INTRODUCTION

Yate (1936) developed the Balanced Incomplete Block Design (BIBD) to find solutions to treatment arrangements in the agricultural field with constrained block sizes. However, with time, the construction of BIBD has been determined to be a combinatory problem that entails the arrangements of v treatments into b blocks of size k each such that each treatment occurs r times in the entire design and each pair of treatments occurs together λ number of times [2, 4, 10, 13, 23-27, 29].

Over the years, much research in combinatorial mathematics has emphasized the construction of BIBDs to find solutions to a given set of BIBDs. The investigations have led to the development of different techniques for constructing BIBD. Most of these methods were first introduced by Bose [4], but later, other researchers also developed further methods of designing BIBDs [5, 7, 15, 16, 19, 21, 28]. The developments have led to the existing known methods of constructing BIBD, which include Projection Geometry, Euclidean Geometry, Cyclic Difference Sets Method, Symmetric Repeated Difference Method, Latin Square Method, Linear Integer Programming, Use of the IBD package in R software, and Using Existing BIBD Designs [1, 9, 11, 13,17, 18, 20, 25, 30].

Different classes of BIBD have been discovered through the construction of BIBD using other existing BIBDs. Some of these classes of BIBD include Derived, Residual, Dual, and Complementary BIBD [3, 12]. For a BIBD with parameters$(v, b, r, k, \lambda)$when a block from the BIBD is deleted and then in the remaining $b - 1$ blocks all the other treatments are deleted except the ones that were in the deleted block then a Derived Balanced Incomplete Block Design with parameters$v^* = k, b^* = b-1, r^* = r-1, k^* = \lambda, \lambda^* = \lambda - 1$ is constructed from [3, 12].

The un-reduced BIBD is considered a universal set of BIBDs with parameters v and k. The design comprises a set of all possible combinations $\binom{v}{k}$, each possible combination being a block. The design contains all other BIBDs with parameters v and k. This BIBD has parameters $v^* = v, b^* = \binom{v}{k}, r^* = \binom{v-1}{k-1}, k^* = k, \lambda^* = \binom{v-2}{k-2}$[3, 12].

The general overview of the literature shows that various scholars have done extensive work over the years on the construction of BIBDs. However, a severe gap exists because all these techniques can only construct some BIBDs. According to Jane [14], several BIBD designs still need to be determined whether to exist because despite them satisfying all the design parameters conditions, the construction methods cannot be used in their construction. The fact has left several BIBD designs still being discovered [6, 8, 13, 14, 17, 27].

The present study intended to bridge this gap by introducing a new class of BIBD known as Derived Reduced Balanced Incomplete Block Design (DRBIBD), which uses the un-reduced BIBD to construct a new type of BIBD. The idea behind the technique was motivated by the fact that un-reduced BIBD was a universal set of several classes of BIBD. Therefore, all the unknown types of BIBDs can still be derived from the design using the appropriate techniques.

2. MATERIAL AND METHODS

2.1 CONSTRUCTION OF DERIVED REDUCED BIBD

Theorem 1. *Consider an un-reduced BIBD with parameters v and $k \geq 3$. When blocks from the design containing treatment i are extracted from the design and in the extracted blocks, treatment i is deleted from all the blocks, leaving the rest of the treatments. Then a BIBD is formed with parameters $v^* = v-1, b^* = \binom{v-1}{k-1}, k^* = k-1, r^* = \binom{v-2}{k-2}, \lambda = \binom{v-3}{k-3}$.*

Proof

We need to prove that the Derived Reduced BIBD satisfies the necessary conditions for the existence of a BIBD. First, we begin with the first condition.

$$\begin{aligned} b^*k^* &= \left(\frac{(v-1)!}{(k-1)!\,(v-k)!}\right)k-1 \\ &= \left(\frac{(v-1)!\,(k-1)}{(k-1)(k-2)!\,(v-k)!}\right) \\ &= \frac{(v-1)!}{(k-2)!\,(v-k)!} \\ &= \left(\frac{(v-1)(v-2)!}{(k-2)!\,(v-k)!}\right) \\ &= \left(\frac{(v-2)!}{(k-2)!\,(v-k)!}\right)v-1 \\ &= r^*v^* \end{aligned}$$

Hence, the first condition of BIBD is satisfied. Next, we check the second condition for the existence of BIBD.

$$\begin{aligned}\lambda^*(v^*-1) &= \left(\frac{(v-3)!}{(k-3)!\,(v-k)!}\right)(v-2)\\ &= \left(\frac{(v-2)(v-3)!}{(k-3)!\,(v-k)!}\right)\\ &= \frac{(v-2)!}{(k-3)!\,(v-k)!}\\ &= \left(\frac{(k-2)(v-2)!}{(k-2)(k-3)!\,(v-k)!}\right)\\ &= \left(\frac{(v-2)!}{(k-2)!\,(v-k)!}\right)(k-2)\\ &= r^*(k^*-1)\end{aligned}$$

The second condition of BIBD is also satisfied; therefore, the Derived Reduced BIBD is indeed a BIBD.

3. RESULTS AND DISCUSSION

The study illustrated how the DRBIBD is from an un-reduced BIBD with parameters v and k. The illustration was as discussed below;

3.1 Illustration 1

Let's consider an un-reduced BIBD generated $v = 8$ treatments and $k = 7$ plots as shown in Table 1

Table 1: Un-reduced BIBD with $v = \mathbf{8}$ and $k = \mathbf{7}$

Un-reduced BIBD Blocks
Block 1 = $\{\mathbf{1,2,3,4,5,6,7}\}$
Block 2 = $\{\mathbf{1,2,3,4,5,6,8}\}$
Block 3 = $\{\mathbf{1,2,3,4,5,7,8}\}$
Block 4 = $\{\mathbf{1,2,3,4,6,7,8}\}$
Block 5 = $\{\mathbf{1,2,3,5,6,7,8}\}$
Block 6 = $\{\mathbf{1,2,4,5,6,7,8}\}$
Block 7 = $\{\mathbf{1,3,4,5,6,7,8}\}$
Block 8 = $\{\mathbf{2,3,4,5,6,7,8}\}$

From the BIBD in Table 1, if we select only the blocks that contain treatment 1 from the design and from the chosen blocks delete treatment 1, then the BIBD illustrated in Table 2 will result.

Table 2: Derived Reduced BIBD with $v = \mathbf{7}, \boldsymbol{k} = \mathbf{6}, \boldsymbol{b} = \mathbf{7}, \boldsymbol{r} = \mathbf{6}, \boldsymbol{\lambda} = \mathbf{5}$

Resulting Derived Reduced BIBD Blocks
Block 1 = $\{\mathbf{2,3,4,5,6,7}\}$
Block 2 = $\{\mathbf{2,3,4,5,6,8}\}$
Block 3 = $\{\mathbf{2,3,4,5,7,8}\}$
Block 4 = $\{\mathbf{2,3,4,6,7,8}\}$

Resulting Derived Reduced BIBD Blocks
Block 1 = $\{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$
Block 5 = $\{\mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$
Block 6 = $\{\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$
Block 7 = $\{\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$

3.2 Illustration 2

Let's consider an un-reduced BIBD generated $v = 6$ treatments and $k = 3$ plots as shown in Table 3

Table 3: Un-reduced BIBD with $\boldsymbol{v = 6}$ and $\boldsymbol{k = 3}$

Un-reduced BIBD Blocks	
Block 1 = $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$	Block 11 = $\{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$
Block 2 = $\{\mathbf{1}, \mathbf{2}, \mathbf{4}\}$	Block 12 = $\{\mathbf{2}, \mathbf{3}, \mathbf{5}\}$
Block 3 = $\{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$	Block 13 = $\{\mathbf{2}, \mathbf{3}, \mathbf{6}\}$
Block 4 = $\{\mathbf{1}, \mathbf{2}, \mathbf{6}\}$	Block 14 = $\{\mathbf{2}, \mathbf{4}, \mathbf{5}\}$
Block 5 = $\{\mathbf{1}, \mathbf{3}, \mathbf{4}\}$	Block 15 = $\{\mathbf{2}, \mathbf{4}, \mathbf{6}\}$
Block 6 = $\{\mathbf{1}, \mathbf{3}, \mathbf{5}\}$	Block 16 = $\{\mathbf{2}, \mathbf{5}, \mathbf{6}\}$
Block 7 = $\{\mathbf{1}, \mathbf{3}, \mathbf{6}\}$	Block 17 = $\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$
Block 8 = $\{\mathbf{1}, \mathbf{4}, \mathbf{5}\}$	Block 18 = $\{\mathbf{3}, \mathbf{4}, \mathbf{6}\}$
Block 9 = $\{\mathbf{1}, \mathbf{4}, \mathbf{6}\}$	Block 19 = $\{\mathbf{3}, \mathbf{5}, \mathbf{6}\}$
Block 10 = $\{\mathbf{1}, \mathbf{5}, \mathbf{6}\}$	Block 20 = $\{\mathbf{4}, \mathbf{5}, \mathbf{6}\}$

From the BIBD in Table 3, if we select only the blocks that contain treatment 6 from the design and from the chosen blocks delete treatments 6, then the BIBD illustrated in Table 4 will result.

Table 4: Derived Reduced BIBD with $\boldsymbol{v = 5, k = 2, b = 10, r = 4, \lambda = 1}$

Resulting Derived Reduced BIBD Blocks	
Block 1 = $\{\mathbf{1}, \mathbf{2}\}$	Block 6 = $\{\mathbf{2}, \mathbf{3}\}$
Block 2 = $\{\mathbf{1}, \mathbf{3}\}$	Block 7 = $\{\mathbf{2}, \mathbf{4}\}$
Block 3 = $\{\mathbf{1}, \mathbf{4}\}$	Block 8 = $\{\mathbf{2}, \mathbf{5}\}$
Block 4 = $\{\mathbf{1}, \mathbf{5}\}$	Block 9 = $\{\mathbf{3}, \mathbf{4}\}$
Block 5 = $\{\mathbf{3}, \mathbf{5}\}$	Block 10 = $\{\mathbf{4}, \mathbf{5}\}$

3.3 List of Some Potential Derived Reduced BIBDs

The Derived Reduced BIBD technique can construct several BIBDs, as illustrated in Table 5.

Table 5: List of some Derived Reduced BIBD

No.	Un-reduced BIBD	Derived Reduced BIBD
1	$v = 4, k = 3$	$v = 3, b = 3, r = 2, k = 2, \lambda = 1$
2	$v = 5, k = 3$	$v = 4, b = 6, r = 3, k = 2, \lambda = 1$
3	$v = 5, k = 4$	$v = 4, b = 4, r = 3, k = 3, \lambda = 2$
4	$v = 6, k = 3$	$v = 5, b = 10, r = 4, k = 2, \lambda = 1$
4	$v = 6, k = 4$	$v = 5, b = 10, r = 6, k = 3, \lambda = 3$
5	$v = 6, k = 5$	$v = 5, b = 5, r = 4, k = 4, \lambda = 3$
6	$v = 7, k = 3$	$v = 6, b = 15, r = 5, k = 2, \lambda = 1$
7	$v = 7, k = 4$	$v = 6, b = 20, r = 10, k = 3, \lambda = 4$
8	$v = 7, k = 5$	$v = 6, b = 15, r = 10, k = 4, \lambda = 6$
9	$v = 7, k = 6$	$v = 6, b = 6, r = 5, k = 5, \lambda = 4$
10	$v = 8, k = 3$	$v = 7, b = 21, r = 6, k = 2, \lambda = 1$
11	$v = 8, k = 4$	$v = 7, b = 35, r = 15, k = 3, \lambda = 5$
12	$v = 8, k = 5$	$v = 7, b = 35, r = 20, k = 4, \lambda = 10$
13	$v = 8, k = 6$	$v = 7, b = 21, r = 15, k = 5, \lambda = 10$
14	$v = 8, k = 7$	$v = 7, b = 7, r = 6, k = 6, \lambda = 5$
15	$v = 9, k = 3$	$v = 8, b = 28, r = 7, k = 2, \lambda = 1$
16	$v = 9, k = 4$	$v = 8, b = 56, r = 21, k = 3, \lambda = 6$
17	$v = 9, k = 5$	$v = 8, b = 70, r = 35, k = 4, \lambda = 15$
18	$v = 9, k = 6$	$v = 8, b = 56, r = 35, k = 5, \lambda = 20$
19	$v = 9, k = 7$	$v = 8, b = 28, r = 21, k = 6, \lambda = 15$
20	$v = 9, k = 8$	$v = 8, b = 8, r = 7, k = 7, \lambda = 6$
21	$v = 10, k = 3$	$v = 9, b = 36, r = 8, k = 2, \lambda = 1$
22	$v = 10, k = 4$	$v = 9, b = 84, r = 28, k = 3, \lambda = 7$
23	$v = 10, k = 5$	$v = 9, b = 126, r = 56, k = 4, \lambda = 21$
24	$v = 10, k = 6$	$v = 9, b = 126, r = 70, k = 5, \lambda = 35$
25	$v = 10, k = 7$	$v = 9, b = 84, r = 56, k = 6, \lambda = 35$
26	$v = 10, k = 8$	$v = 9, b = 36, r = 28, k = 7, \lambda = 21$
27	$v = 10, k = 9$	$v = 9, b = 9, r = 8, k = 8, \lambda = 7$
28	$v = 11, k = 6$	$v = 10, b = 252, r = 18, k = 5, \lambda = 56$
29	$v = 11, k = 7$	$v = 10, b = 210, r = 18, k = 6, \lambda = 70$
30	$v = 11, k = 8$	$v = 10, b = 120, r = 18, k = 7, \lambda = 56$

4. CONCLUSION

In conclusion, the study established that for un-reduced BIBD with parameters v and $k \geq 3$. If blocks from the design containing treatment i are extracted from the design and in the extracted blocks, treatment i is deleted from all the blocks, leaving the rest of the treatments, then a BIBD is formed with parameters $v^* = v - 1, b^* = \binom{v-1}{k-1}, k^* = k - 1, r^* = \binom{v-2}{k-2}, \lambda =$

$\binom{v-3}{k-3}$. This class of BIBD is known as Derived Reduced Balanced Incomplete Block Design. The study was, therefore, able to derive a new technique for constructing BIBDs from existing BIBDs, which adds to the list of methods used in designing BIBDs.

REFERENCES

[1.] Akra1, U. P., Akpan, S. S., Ugbe, T. A. and Ntekim, O. E. (2021). Finite Euclidean Geometry Approach for Constructing Balanced Incomplete Block Design (BIBD). Asian Journal of Probability and Statistics. 11(4): 47-59.

[2.] Alabi, MA (2018). Construction of balanced incomplete block design of lattice series I and II. International Journal of Innovative Scientific and Engineering Technologies Research.2018;6(4):10-22.

[3.] Alam, N. M. (2014). On Some Methods of Construction of Block Designs. IASRI, Library Avenue, New Delhi-110012.

[4.] Bose, R. C. (1939), On the construction of balanced incomplete block designs. Annals of Eugenics, Vol. 9, pp. 353–399.

[5.] Bose, R. C., Shrikhande, S. S., and Parker, E. T. (1960). Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture. Canadian Journal of Mathematics, 12, 189-203.

[6.] Bruck, R. H. and Ryser, H. J. (1949), The non-existence of certain finite projective planes. Canadian Journal of Mathematics, Vol. 1, pp. 88–93.

[7.] Fisher, R. A. (1940). An examination of the different possible solutions of a problem in incomplete blocks. Ann. Eugenics, 10, 52–75.

[8.] Greig, M., and Rees, D. H. (2003). Existence of balanced incomplete block designs for many sets of treatments. Discrete Mathematics, 261(1-3), 299-324.

[9.] Goud T. S. and Bhatra, C.N.Ch. (2016). Construction of Balanced Incomplete Block Designs. International Journal of Mathematics and Statistics Invention. 4(1) 2321-4767.

[10.] Hanani, H. (1961). The existence and construction of balanced incomplete block designs. The Annals of Mathematical Statistics, 32(2), 361-386.

[11.] Hinkelmann, K. and Kempthorne, O. (2005). Design and Analysis of Experiments. John Wiley and Sons, Inc., Hoboken, New Jersey.

[12.] Hsiao-Lih, J., Tai-Chang, H. and Babul, M. H. (2007). A study of methods for construction of balanced incomplete block design. Journal of Discrete Mathematical Sciences and Cryptography Vol. 10 (2007), No. 2, pp. 227–243.

[13.] Janardan, M. (2018). Construction of balanced incomplete block design: an application of Galois field. Open Science Journal of Statistics and Application.2013;5(3):32-39.

[14.] Jane di Paola, Jennifer Seberry Wallis and W. D. Wallis, A list of balanced incomplete block designs for r < 30, Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing, CongressusNumerantium, 8, (1973), 249-258.

[15.] Khare, M. and W. T. Federer (1981). A simple construction procedure for resolvable incomplete block designs for any number of treatments. Biom. J., 23, 121–132.

[16.] Mandal, B. N. (2015). Linear Integer Programming Approach to Construction of Balanced Incomplete Block Designs. Communications in Statistics - Simulation and Computation, 44:6, 1405-1411, DOI: 10.1080/03610918.2013.821482.

[17.] Montgomery, D. C. (2019). Design and analysis of experiment. John Wiley and Sons, New York.

[18.] Neil, J.S. (2010). Construction of balanced incomplete block design. Journal of Statistics and Probability.12(5);231–343.

[19.] Pachamuthu, A. R. M. (2018). On the construction of balanced incomplete block designs using MOLS of order six - a special case. International Journal of Creative Research Thoughts. 6(1) 2320-2882.

[20.] Wan, Z. X. (2009). Design theory. World Scientific Publishing Company.

[21.] Yasmin, F., Ahmed, R. and Akhtar, M. (2015). Construction of Balanced Incomplete Block Designs Using Cyclic Shifts. Communications in Statistics—Simulation and Computation 44: 525–532.DOI: 10.1080/03610918.2013.784984.

[22.] Yates, F. (1936). A new method of arranging variety trials involving a large number of varieties. J. Agric. Sci., 26, 424-445.

[23.] Gary, W. O. (2010). A First Course in Design and Analysis of Experiments. The University of Minnesota. The USA.

[24.] Jianxing, Y., &Busheng, G. (2020). Existence of G-designs with| V (G)|= 6. In Combinatorial Designs and Applications (pp. 201-218). CRC Press.

[25.] Sindu, M., &Rajarathinam, A. (2019). A New Method for Construction of Un-equally Replicated Pairwise Balanced DESIGN. International Journal of Agricultural & Statistical Sciences, 15(1).

[26.] Bailey, R. A. (2023). Resolved designs viewed as sets of partitions. In Combinatorial Designs and their Applications (pp. 17-47). Routledge.

[27.] Kathuria, A., &Batra, S. (2023). Balanced Incomplete Block Design (BIBD) as Traceable Codes. *International Journal of Science and Research Archive*, *8*(1), 734-739.

[28.] Khattree, R. (2019). A note on the non-existence of the constant block-sum balanced incomplete block designs. Communications in Statistics-Theory and Methods, 48(20), 5165-5168.

[29.] Huang, E. P., & Shih, J. H. (2021). Assigning readers to cases in imaging studies using balanced incomplete block designs. Statistical Methods in Medical Research, 30(10), 2288-2312.

[30.] Bassalygo, L. A., & Zinoviev, V. A. (2017). Remark on balanced incomplete block designs, near-resolvable block designs, and q-ary constant-weight codes. Problems of Information Transmission, 53, 51-54.