Method Article
Derived Reduced Balanced Incomplete Block Design

ABSTRACT

Background: Construction of Balanced Incomplete Block Designs (BIBD) is considered a combination problem that involves the arrangement of v treatments into b blocks each of size k such that each treatment is replicated exactly r times in the design and a pair of treatments occur together in λ blocks.
Statement of the Problem: Several methods of constructing BIBDs have been developed, however, the methods still cannot be used to construct all the BIBDs. This has left the existence of a number of BIBDs still unknown because a definite construction method for all has not been established.
Aim: The study aimed at developing a new construction method that could aid the construction of more BIBDs.
Methodology: The new class of BIBD derived in the study used un-reduced BIBD in the construction of the new class of BIBD known as the Derived Reduced Balanced Incomplete Block Design (DRBIBD).
Results: Consider an un-reduced BIBD with parameters v and k such that $k \geq 3$ the derived new method of constructing BIBDs resulted in a design with the following parameters $v^* = v-1, b^* = \binom{v-1}{k-1}, k^* = k-1, r^* = \binom{v-2}{k-2}, \lambda = \binom{v-3}{k-3}$.
Conclusion: The new method of construction was determined to be simple and could be used to construct a number of BIBDs which could assist in solving problem about BIBD whose existence are still unknown.

Comment [JCL1]: Present the abstract in paragraph form. Only essentials elements in the background, statement of the problem, methodology, results and conclusion must be included in the paragraph.

Keywords: Un-reduced BIBD; Block; Treatment; Reduced; Derived; Incomplete Block.

1. INTRODUCTION

A Balanced Incomplete Block Design (BIBD) was first developed by Yates (1936) as a way of finding solutions to treatment arrangements in the agricultural field with constrained block sizes. However, with time, the construction of BIBD has been determined to be a combinatory problem that entails the arrangements of v treatments into b blocks of size k each such that each treatment occurs r times in the entire design and each pair of treatments occurs together λ number of times [2, 4, 10, 13].

Over the years, a lot of research in combinatorial mathematics has emphasized on the construction of BIBDs as a way of finding solutions to a given set of BIBDs. This has led to the development of different techniques for constructing BIBD. Most of these methods were first introduced by Bose [4], but later other researchers also develop further other methods of constructing the designs [5, 7, 15, 16, 19, 21]. This has led to the existing known methods of constructing BIBD which include; Projection Geometry, Euclidean Geometry, Cyclic Difference Sets Method, Symmetric Repeated Difference Method, Latin Square Method, Linear Integer Programming, Use of the IBD package in R software, and Using Existing BIBD Designs [1, 9, 11, 13,17, 18, 20].

Different classes of BIBD have been discovered through the construction of BIBD using other existing BIBDs. Some of these classes of BIBD include Derived, Residual, Dual, and Complementary BIBD [3, 12]. A Derived Balanced Incomplete Block Design with parameters $v^* = k, b^* = b - 1, r^* = r - 1, k^* = \lambda, \lambda^* = \lambda - 1$ can be constructed from a BIBD with parameters (v, b, r, k, λ) if a block from the BIBD is deleted and then in the remaining $b - 1$ blocks all the other treatments are deleted except the ones that were in the deleted block [3, 12].

Among all classes of BIBDs, a universal class of BIBD known as the Un-reduced BIBD exist. This class of BIBD is known to be created from a set of all possible combinations$\binom{v}{k}$. The design is believed to contain all other BIBDs with parameters v and k. This BIBD has parameters $v^* = v, b^* = \binom{v}{k}, r^* = \binom{v-1}{k-1}, k^* = k, \lambda^* = \binom{v-2}{k-2}$[3, 12].

The general, overview of the literature shows extensive work has been done over the years by various scholars on the construction of BIBDs. However, a serious gap still exists because all these techniques have been found to be able to construct all the BIBDs. According to Jane [14] a number of BIBD designs still haven't been determined to exist or not exist because despite them satisfying all the design parameters conditions, the existing construction methods cannot be used to construct them. This has left the existence of a number of BIBD designs to still be unknown [6, 8, 13, 14, 17].

The present study intended to bridge this gap by introducing a new class of BIBD known as Derived Reduced Balanced Incomplete Block Design (DRBIBD) which uses the un-reduced BIBD to construct a new class of BIBD. The idea behind the technique was motivated by the fact that un-reduced BIBD was a universal set of several classes of BIBD, therefore, all the unknown classes of BIBDs still can be derived from the design using appropriate technique.

2. MATERIAL AND METHODS

2.1 CONSTRUCTION OF DERIVED REDUCED BIBD

Theorem 1. *Consider an un-reduced BIBD with parameters v and $k \geq 3$. When blocks from the design containing treatment $i$are extracted from the design and in the extracted blocks treatment i is deleted from all the blocks leaving the rest of the treatments. Then a BIBD is formed with parameters $v^* = v - 1, b^* = \binom{v-1}{k-1}, k^* = k - 1, r^* = \binom{v-2}{k-2}, \lambda = \binom{v-3}{k-3}$.*

Proof

We need to prove that the Derived Reduced BIBD satisfies the necessary conditions for the existence of a BIBD. First, we begin with the first condition.

$$
\begin{aligned}
b^*k^* &= \left(\frac{(v-1)!}{(k-1)!\,(v-k)!}\right)k-1 \\
&= \left(\frac{(v-1)!\,(k-1)}{(k-1)(k-2)!\,(v-k)!}\right) \\
&= \frac{(v-1)!}{(k-2)!\,(v-k)!} \\
&= \left(\frac{(v-1)(v-2)!}{(k-2)!\,(v-k)!}\right) \\
&= \left(\frac{(v-2)!}{(k-2)!\,(v-k)!}\right)v-1 \\
&= r^*v^*
\end{aligned}
$$

Hence the first condition of BIBD is satisfied. Next, we check the second condition for the existence of BIBD.

$$\begin{aligned}
\lambda^*(v^*-1) &= \left(\frac{(v-3)!}{(k-3)!\,(v-k)!}\right)(v-2) \\
&= \left(\frac{(v-2)(v-3)!}{(k-3)!\,(v-k)!}\right) \\
&= \frac{(v-2)!}{(k-3)!\,(v-k)!} \\
&= \left(\frac{(k-2)(v-2)!}{(k-2)(k-3)!\,(v-k)!}\right) \\
&= \left(\frac{(v-2)!}{(k-2)!\,(v-k)!}\right)(k-2) \\
&= r^*(k^*-1)
\end{aligned}$$

The second condition of BIBD is also satisfied therefore, the Derived Reduced BIBD is indeed a BIBD.

3. RESULTS AND DISCUSSION

The study illustrated how the DRBIBD could be constructed from an un-reduced BIBD with parameters v and k. The illustration was as discussed below;

3.1 Illustration 1

Let's consider an un-reduced BIBD generated $v = 8$ treatments and $k = 7$ plots as shown in Table 1

Table 1: Un-reduced BIBD with $v = \mathbf{8}$ and $k = \mathbf{7}$

Un-reduced BIBD Blocks
Block 1 = $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7}\}$
Block 2 = $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{8}\}$
Block 3 = $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{7},\mathbf{8}\}$
Block 4 = $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{7},\mathbf{8}\}$
Block 5 = $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$
Block 6 = $\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$
Block 7 = $\{\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$
Block 8 = $\{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}\}$

From the BIBD in Table 1 if we select only the blocks that contain treatment 1 from the design and from the selected blocks delete treatment 1 then the BIBD illustrated in Table 2 will result.

Table 2: Derived Reduced BIBD with $v = 7, k = 6, b = 7, r = 6, \lambda = 5$

Resulting Derived Reduced BIBD Blocks
Block 1 = $\{2, 3, 4, 5, 6, 7\}$
Block 2 = $\{2, 3, 4, 5, 6, 8\}$
Block 3 = $\{2, 3, 4, 5, 7, 8\}$
Block 4 = $\{2, 3, 4, 6, 7, 8\}$
Block 5 = $\{2, 3, 5, 6, 7, 8\}$
Block 6 = $\{2, 4, 5, 6, 7, 8\}$
Block 7 = $\{3, 4, 5, 6, 7, 8\}$

3.2 Illustration 2

Let's consider an un-reduced BIBD generated $v = 6$ treatments and $k = 3$ plots as shown in Table 3

Table 3: Un-reduced BIBD with $v = 6$ and $k = 3$

Un-reduced BIBD Blocks	
Block 1 = $\{1, 2, 3\}$	Block 11 = $\{2, 3, 4\}$
Block 2 = $\{1, 2, 4\}$	Block 12 = $\{2, 3, 5\}$
Block 3 = $\{1, 2, 5\}$	Block 13 = $\{2, 3, 6\}$
Block 4 = $\{1, 2, 6\}$	Block 14 = $\{2, 4, 5\}$
Block 5 = $\{1, 3, 4\}$	Block 15 = $\{2, 4, 6\}$
Block 6 = $\{1, 3, 5\}$	Block 16 = $\{2, 5, 6\}$
Block 7 = $\{1, 3, 6\}$	Block 17 = $\{3, 4, 5\}$
Block 8 = $\{1, 4, 5\}$	Block 18 = $\{3, 4, 6\}$
Block 9 = $\{1, 4, 6\}$	Block 19 = $\{3, 5, 6\}$
Block 10 = $\{1, 5, 6\}$	Block 20 = $\{4, 5, 6\}$

From the BIBD in Table 3 if we select only the blocks that contain treatment 6 from the design and from the selected blocks delete treatments 6 then the BIBD illustrated in Table 4 will result.

Table 4: Derived Reduced BIBD with $v = 5, k = 2, b = 10, r = 4, \lambda = 1$

Resulting Derived Reduced BIBD Blocks	
Block 1 = $\{1, 2\}$	Block 6 = $\{2, 3\}$
Block 2 = $\{1, 3\}$	Block 7 = $\{2, 4\}$
Block 3 = $\{1, 4\}$	Block 8 = $\{2, 5\}$
Block 4 = $\{1, 5\}$	Block 9 = $\{3, 4\}$
Block 5 = $\{3, 5\}$	Block 10 = $\{4, 5\}$

3.3 List of Some Potential Derived Reduced BIBDs

In general, the Derived Reduced BIBD technique can be used to construct a number of BIBDs as illustrated in Table 5.

Table 5: List of some Derived Reduced BIBD

No.	Un-reduced BIBD	Derived Reduced BIBD
1	$v = 4, k = 3$	$v = 3, b = 3, r = 2, k = 2, \lambda = 1$
2	$v = 5, k = 3$	$v = 4, b = 6, r = 3, k = 2, \lambda = 1$
3	$v = 5, k = 4$	$v = 4, b = 4, r = 3, k = 3, \lambda = 2$
4	$v = 6, k = 3$	$v = 5, b = 10, r = 4, k = 2, \lambda = 1$
4	$v = 6, k = 4$	$v = 5, b = 10, r = 6, k = 3, \lambda = 3$
5	$v = 6, k = 5$	$v = 5, b = 5, r = 4, k = 4, \lambda = 3$
6	$v = 7, k = 3$	$v = 6, b = 15, r = 5, k = 2, \lambda = 1$
7	$v = 7, k = 4$	$v = 6, b = 20, r = 10, k = 3, \lambda = 4$
8	$v = 7, k = 5$	$v = 6, b = 15, r = 10, k = 4, \lambda = 6$
9	$v = 7, k = 6$	$v = 6, b = 6, r = 5, k = 5, \lambda = 4$
10	$v = 8, k = 3$	$v = 7, b = 21, r = 6, k = 2, \lambda = 1$
11	$v = 8, k = 4$	$v = 7, b = 35, r = 15, k = 3, \lambda = 5$
12	$v = 8, k = 5$	$v = 7, b = 35, r = 20, k = 4, \lambda = 10$
13	$v = 8, k = 6$	$v = 7, b = 21, r = 15, k = 5, \lambda = 10$
14	$v = 8, k = 7$	$v = 7, b = 7, r = 6, k = 6, \lambda = 5$
15	$v = 9, k = 3$	$v = 8, b = 28, r = 7, k = 2, \lambda = 1$
16	$v = 9, k = 4$	$v = 8, b = 56, r = 21, k = 3, \lambda = 6$
17	$v = 9, k = 5$	$v = 8, b = 70, r = 35, k = 4, \lambda = 15$
18	$v = 9, k = 6$	$v = 8, b = 56, r = 35, k = 5, \lambda = 20$
19	$v = 9, k = 7$	$v = 8, b = 28, r = 21, k = 6, \lambda = 15$
20	$v = 9, k = 8$	$v = 8, b = 8, r = 7, k = 7, \lambda = 6$
21	$v = 10, k = 3$	$v = 9, b = 36, r = 8, k = 2, \lambda = 1$
22	$v = 10, k = 4$	$v = 9, b = 84, r = 28, k = 3, \lambda = 7$
23	$v = 10, k = 5$	$v = 9, b = 126, r = 56, k = 4, \lambda = 21$
24	$v = 10, k = 6$	$v = 9, b = 126, r = 70, k = 5, \lambda = 35$
25	$v = 10, k = 7$	$v = 9, b = 84, r = 56, k = 6, \lambda = 35$
26	$v = 10, k = 8$	$v = 9, b = 36, r = 28, k = 7, \lambda = 21$
27	$v = 10, k = 9$	$v = 9, b = 9, r = 8, k = 8, \lambda = 7$
28	$v = 11, k = 6$	$v = 10, b = 252, r = 18, k = 5, \lambda = 56$
29	$v = 11, k = 7$	$v = 10, b = 210, r = 18, k = 6, \lambda = 70$
30	$v = 11, k = 8$	$v = 10, b = 120, r = 18, k = 7, \lambda = 56$

4. CONCLUSION

In conclusion, the study was able to establish that for un-reduced BIBD with parameters v and $k \geq 3$. If blocks from the design containing treatment i are extracted from the design and in the extracted blocks treatment i is deleted from all the blocks leaving the rest of the treatments, then a BIBD is formed with parameters $v^* = v - 1, b^* = \binom{v-1}{k-1}, k^* = k - 1, r^* = \binom{v-2}{k-2}, \lambda = \binom{v-3}{k-3}$. This class of BIBD is known as Derived Reduced Balanced Incomplete Block Design. The study was, therefore, able to derive a new technique of constructing BIBDs from existing BIBDs which adds to the list of techniques that could be used in constructing BIBDs.

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