The number of spanning trees of general fan graphs

As we all know, the number of the spanning trees can be calculated by virtue of the famous Kirchhoff's matrix-tree theorem. However, it doesn't work when coming across complex graphs with thousands of edges and vertices or more. Hence, how to obtain accurate solutions of the number of spanning trees of general fan graphs becomes a subject to several studies of many places like computer science, physics and mathematics. In this paper, we focus on calculating different types of graphs generated by adding vertices and edges to the fan graph. Particularly, we define a new graph called "C-graph", which brings a unique angle of view for us to recognize the construction of original graphs. Moreover, we introduce a new recursive relation about the general fan graph, simplifying the calculation. Therefore, we can obtain functions of the number of spanning trees obtained from a given fan graphs, which are also suitable for larger and more complex conditions. Finally, we discuss the effect of vertices and edges on the number of spanning trees, finding out that edges have greater impact. Additionally, by using Kirchhoff's matrix-tree theorem, we verified the rationality of our results.

Keywords: fan graphs; spanning trees; iterate relations; self-similar graphs; Kirchhoff's matrix-tree theorem

1. Introduction

The knowledge of spanning trees has widely distributed in many fields, such as mathematics, engineering, computer science and image processing. Currently, researches on spanning trees are mainly focusing on two parts: applications and algorithms. For example, the stock market [4]can be studied by the maximum spanning trees, and the minimum spanning tree perfectly guarantees the minimum sum of all paths in the entire topology. Moreover, theory-based agglomerative hierarchical clustering technique and neural network for emotion classification are also both associated with the minimum spanning tree. [6]

In order to find the number of spanning trees of a graph, Kirchhoff had given his famous matrix tree theorem [2] which achieved the calculation by finding any cofactor of the graph's Laplacian matrix. However, this method fails when facing large graphs. Consequently, we want to find a formula by which the number of spanning trees can be obtained easily, just like what lots of scientists have already done [3] [7] [10] [12].

In this article, we concentrate on the number of spanning trees of general fan graphs. Our method stems from another one introduced by Kirchhoff [5]. He divides the spanning trees of a graph into two parts, which are spanning trees that contain edge"e" and spanning trees that do not contain edge"e". By doing this, an elimination of edge is finished ,and thus we only need to discuss spanning trees of a simpler graph. [1] [9]

Since the structure of a general fan graph can be considered as a combination of some similar subgraphs, we want to simplify the graph by discussing the relation between its subgraphs and spanning trees. Our goal is to obtain an iterative formula for the number of spanning trees from a given general fan graph. [8]

2. Definitions

We use the following notations and terminology:

Definition 1. A spanning tree can be defined as the subgraph of an undirected and connected graph, which contains all vertices of the original graph and the number of its edges equals to the number of all vertices substract 1. We will use $\tau(G)$ to represent the number of spanning trees of graph G.

Definition 2. Suppose that there is an undirected and connected graph $C_{d,e}$ which is composed of two vertices u, v linked by e identical paths with $d-1$ vertices. That is, $C_{d,e} = G(V,E), V = \{u, v, s_{i,j} | i = 1, 2, \ldots, e; j = 1, 2, \ldots, d-1\}, E = \{\{u, s_{i,1}\}, \{v, s_{i,d-1}\}, \{s_{i,j}, s_{i,j+1}\} | i = 1, 2, \ldots, e; j = 1, 2, \ldots, d-2\}$. We name it "**C-graph**".(see Figure.1) If the point of both ends of "C-graph" is u, v, the "C-graph" which have e identical paths with $d-1$ vertices is denoted by $C_{d,e}(u, v)$.

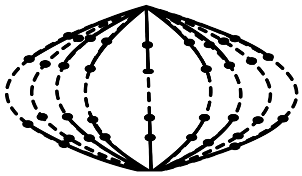

Fig. 1. C-graph

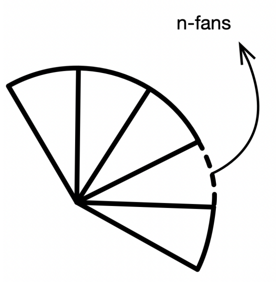

Fig. 2. $G_{1,1,1,1}(n)$

Definition 3. We now consider an undirected and connected graph $G_{d_1,d_2,e_1,e_2}(n)$ (see Figure 3), where $d_1, d_2, e_1, e_2 \in N$ and the simplest case is $G_{1,1,1,1}(n)$ (see Figure 2). We start from a "C-Graph" $C_{d_1,e_1}(O, s_1)$.

By introducing vertex s_2, other $[e_1(d_1 - 1) + e_2(d_2 - 1)]$ ones and related edges, we obtain a graph in which s_1, s_2 and $e_2(d_2 - 1)$ vertices form a "C-Graph" $C_{d_2,e_2}(s_1, s_2)$, and, similarly, s_2, O and other $e_1(d_1 - 1)$ vertices form another "C-Graph" $C_{d_1,e_1}(s, s_2)$. This is $G_{d_1,d_2,e_1,e_2}(1)$. Then $G_{d_1,d_2,e_1,e_2}(n)$ is obtained by continuously doing this method (start from $C_{d_1,e_1}(O, s_2), C_{d_1,e_1}(O, s_3), \ldots, C_{d_1,e_1}(O, s_n)$) for $n - 1$ times.

To clarify it, we introduce a more precise definition, that is, $G_{d_1,d_2,e_1,e_2}(n) = (V, E)$ where
$V = \{O, s_i, r_{i,j,k}, t_{i',w,m} | i = 1, 2, \ldots, n + 1; i' = 1, 2, \ldots, n; j = 1, 2, \ldots, e_1; k = 1, 2, \ldots, d_1 - 1; w = 1, 2, \ldots, e_2; m = 1, 2, \ldots, d_2 - 1\}$;
$E = \{\{r_{i,j,d_1-1}, O\}, \{s_i, r_{i,j,1}\}, \{s_{i'}, t_{i',w,1}\}, \{s_{i'+1}, t_{i'+1,w,d_2-1}\}, \{r_{i,j,k}, r_{i,j,k+1}\}, \{t_{i',w,m}, t_{i',w,m+1}\} | i = 1, 2, \ldots, n + 1; i' = 1, 2, \ldots, n; j = 1, 2, \ldots, e_1; k = 1, 2, \ldots, d_1 - 2; w = 1, 2, \ldots, e_2; m = 1, 2, \ldots, d_2 - 2\}$.

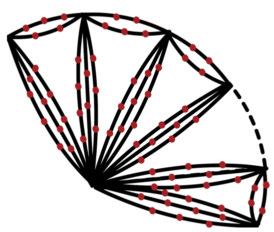

Fig. 3. $G_{d_1,d_2,e_1,e_2}(n)$

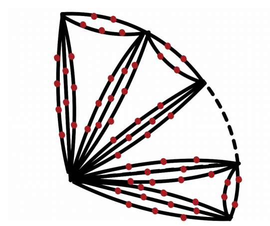

Fig. 4. $G_{d_1,d_2,e_1,e_2}(n - 1)$

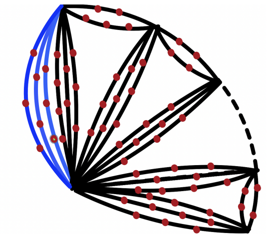

Fig. 5. $Z_{d_1,d_2,e_1,e_2}(n-1)$

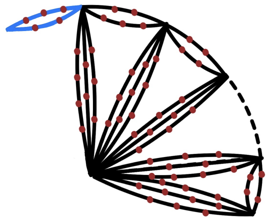

Fig. 6. $T_{d_1,d_2,e_1,e_2}(n-1)$

To simplify the proof, we introduce two new graphs $T_{d_1,d_2,e_1,e_2}(n)$ and $Z_{d_1,d_2,e_1,e_2}(n)$ (looking at Figure 4 and Figure 6). $T_{d_1,d_2,e_1,e_2}(n-1)$ is obtained by deleting $r_{1,i,j}, i = 1, 2, \ldots, e_1, j = 1, 2, \ldots, d_1 - 1$ from V and edges containing these vertices from E.

$Z_{d_1,d_2,e_1,e_2}(n-1)$ is obtained by deleting s_1 from $T_{d_1,d_2,e_1,e_2}(n-1)$'s vertex set and transforming edges $s_1t_{1,k,1}, k = 1, 2, \ldots, e_2$ to $Ot_{1,k,1}$ in $T_{d_1,d_2,e_1,e_2}(n-1)$'s edge set.

Definition 4. (vertex-division method) Let G be an undirected graph which contains $n+2$ vertices and u, v be two vertices of G. By choosing u and some other arbitrary vertices of G except v, a group of vertices is obtained, denoted by u-group, and then, we call the group which is composed of other vertices v-group.

Let $G_{i,u}$ be the graph obtained by deleting all vertices in v-group in G and $G_{i,v}$ be the graph obtained by deleting all vertices in u-group in G, where i means the ith way to form u-group and $i = 1, 2, \ldots, 2^n$.

Then we get a set of couples of graphs $I(G) = \{(G_{i,u}, G_{i,v}) | i = 1, 2, \ldots, 2^n\}$. Particularly, if $G_{i,u}$ is unconnected, $\tau(G_{i,u}) = 0$ since it has no tree, and so is $G_{i,v}$.We name the method as "**vertex-division method**"(short for "**VDM**").

3. Main Results

In order to make the following process of proofs more comprehensive and simplify the expression, we firstly introduce some lemmas:

Lemma 1. *We assume that an undirected and connected graph G has multiple edges. Then we can arbitrarily choose one edge from the graph. The computation of complexity of G can be divided into two parts. First, spanning trees don't contain the chosen edge. It can be denoted by the complexity of $G - e$,represented by $\tau(G - e)$. second, we consider that spanning trees have this edge. Then two vertices of this edge can merge together ,and we mark it as $\tau(G.e)$. Therefore, we can obtain:*

$$\tau(G) = \tau(G - e) + \tau(G.e)$$

Lemma 2. *Applying VDM to $C_{d,e}$, there are d^e elements of $I(C_{d,e})$ satisfying $\tau(C_{d,e\ i,u}) = \tau(C_{d,e\ i,v}) = 1$ for some $i \in \{1, 2, \ldots, 2^{e(d-1)}\}$. Without loss of general-*

ity, let these i *be* $1, 2, \ldots, d^e$*, respectively. Additionally,* $\tau(C_{d,e\ i,u})\tau(C_{d,e\ i,v}) = 0$*,for* $i = d^e + 1, d^e + 2, \ldots, 2^{e(d-1)}$*, which can be written as:*

$$\tau(C_{d,e\ i,u})\tau(C_{d,e\ i,v}) = \begin{cases} 1 \textit{ for } i = 1, 2, \ldots, d^e \\ 0 \textit{ for } i = d^e + 1, d^e + 2, \ldots, 2^{e(d-1)} \end{cases}$$

Proof. Considering the path $u, s_{i',1}, s_{i',2}, \ldots, s_{i',d-1}, v$ where $i' \in \{1, 2, \ldots, e\}, d \geq 3$, if $s_{i',j}, j \in \{1, 2, \ldots, d-2\}$ is included in v-group but $s_{i',k}, k \in \{j+1, j+2, \ldots, d-1\}$ is included in u-group (name this grouping as the mth way to distribute), it is easy to find that $C_{d,e\ m,u}$ will be unconnected by the definition of $C_{d,e}$ and VDM, and therefore $\tau(C_{d,e\ m,u})\tau(C_{d,e\ m,v}) = 0$.

Hence, if $\tau(C_{d,e\ n,u})\tau(C_{d,e\ n,v}) = 1$, its related u-group and v-group must satisfy that if $s_{a,b}$, where $a \in \{1, 2, \ldots, e\}, b \in \{1, 2, \ldots, d-1\}$ is included in u-group, then $s_{a,1}, s_{a,2}, \ldots, s_{a,b-1}$ are included in u-group. Similarly, if $s_{a,b}$ is included in v-group, then $s_{a,b+1}, s_{a,b+2}, \ldots, s_{a,d-1}$ are included in v-group. Since all vertices of $C_{d,e}$ should be included in either u-group or v-group, we can conclude a necessary condition of $\tau(C_{d,e\ m,u})\tau(C_{d,e\ m,v}) = 1$:

- If $s_{a,d-1}$ is included in u-group, then $s_{a,1}, s_{a,2}, \ldots, s_{a,d-1}$ are included in u-group.
- If $s_{a,1}$ is included in v-group, then $s_{a,1}, s_{a,2}, \ldots, s_{a,d-1}$ are included in v-group.
- If $s_{a,c}, c \in \{1, 2, , d-2\}$ is included in u-group and $s_{a,c+1}$ is included in v-group, then $s_{a,1}, s_{a,2}, \ldots, s_{a,c}$ are included in u-group and $s_{a,c+1}, s_{a,c+2}, \ldots, s_{a,d-1}$ are included in v-group.

In other words, $\tau(C_{d,e\ n,u})\tau(C_{d,e\ n,v}) = 1$ only if $C_{d,e\ n,u}$ and $C_{d,e\ n,v}$ can be considered as being obtained by deleting one segment from each path between u, v in $C_{d,e}$. By the definition of $C_{d,e}$, $C_{d,e\ n,u}$ and $C_{d,e\ n,v}$ are both trees and therefore $\tau(C_{d,e\ n,u}) = \tau(C_{d,e\ n,v}) = 1$.Thus, it is a sufficient and necessary condition of $\tau(C_{d,e\ n,u})\tau(C_{d,e\ n,v}) = 1$.

As each path between u, v in $C_{d,e}$ contains d fragments, there are totally d^e ways to choose one edge from each path. Since there is an one-to-one relationship between fragments and an element of $I(C_{d,e})$, we denote these elements by $1, 2, \ldots, d^e$ respectively and, similarly, denote other elements by $d^e+1, d^e+2, \ldots, 2^{e(d-1)}$. Then we get that

$$\tau(C_{d,e\ i,u})\tau(C_{d,e\ i,v}) = \begin{cases} 1 \text{ for } i = 1, 2, \ldots, d^e \\ 0 \text{ for } i = d^e + 1, d^e + 2, \ldots, 2^{e(d-1)} \end{cases}$$

holds for $d \geq 3$. It is easy to show that the formula also holds when $d = 1, 2$.

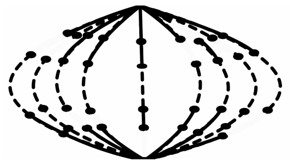

Fig. 7. One method making the vertices u, v not connected □

Lemma 3. *The number of spanning trees of a "C-graph" $C_{d,e}$ is ed^{e-1}.*

Proof. From the definition of spanning trees, it follows that there exists one path between u and v. Look at figure 8, the red path is connected and that is one case for spanning trees of "$C_{d,e}$".

We assume that there are e identical paths connecting u and v, each of these paths is divided into d parts. Except the red path which is chose arbitrarily, other points separate "C-graph" into 2 connected graphs such that u and v is not connected. Hence, we can deduce that the number of spanning trees of "C-graph" is ed^{e-1}. □

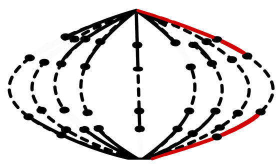

Fig. 8. A spanning tree of "C-graph"

Lemma 4. *Let G be an undirected and connected graph, and G' be a subgraph of G which contains $n+2$ vertices. Assume that in G', there are 2 vertices u, v such that any other vertices of G' are not linked to the rest of G except u and v. Applying VDM to G' and vertices u, v, we get $I(G') = \{(G'_{i,u}, G'_{i,v}) | i = 1, 2, \ldots, 2^n\}$. Let $G - G'$ be a new graph obtained by deleting all vertices of G' except u and v from G, and $G \cdot G'$ be a new graph obtained by merging u and v in $G - G'$. The number of spanning trees of G is*

$$\tau(G) = \tau(G')\tau(G \cdot G') + \tau(G - G') \sum_{i=1}^{2^n} \tau(G'_{i,u})\tau(G'_{i,v})$$

Proof. First, we need to prove a simple statement. The number of spanning trees of an undirected and connected graph H, which is composed of a subgraph H_1 and a tree T with only one vertex in T linked to H_1, is $\tau(H_1)$. Assume that some of edges in T may not be included in a spanning tree of H. since a tree must not have a cycle, some vertices in T must be separated which makes the graph not connected. This is a contradiction. So T must be included in any spanning tree of H, and therefore $\tau(H) = \tau(H_1)$.

Then we consider one case that there is a group of spanning trees of G which do not include any entire spanning tree of G'. Considering $G'_{i,u}$ and $G'_{i,v}, i \in \{1, 2, \ldots, 2^n\}$:

(i) $G'_{i,u}$ and $G'_{i,v}$ are both trees. By the statement above, the graph, which is consisted of $G - G'$, $G'_{i,u}$ and $G'_{i,v}$, has $\tau(G - G')$ spanning trees.

(ii) $G'_{i,u}$ and $G'_{i,v}$ are both arbitrary connected graphs. Then the combination of $G - G'$, $G'_{i,u}$ and $G'_{i,v}$ has $\tau(G'_{i,u})\tau(G'_{i,v})\tau(G - G')$ spanning trees since there are $\tau(G'_{i,u})\tau(G'_{i,v})$ ways to choose a spanning tree from $G'_{i,u}$ and $G'_{i,v}$ respectively. By continuing doing this for $i = 1, 2, \ldots, 2^n$, we find that there are totally $\sum_{i=1}^{2^n} \tau(G'_{i,u})\tau(G'_{i,v})$ spanning trees in this group.

In another case, if we choose a tree from G' instead of separating it as a part of spanning trees of G, another group of spanning trees will be found. Due to we have already selected a spanning tree of G', we merge vertices u and v, and the number of spanning trees of G which contain the chosen spanning tree, is equal to that of $G.G'$, that is, $\tau(G \cdot G')$. Considering G' has $\tau(G')$ spanning trees, we get that there are entirely $\tau(G')\tau(G \cdot G')$ trees .

Since a spanning tree of a graph must contain all its vertices and has no cycle, it is not difficult to deduce that all spanning trees of G are included in two cases which we have discussed above.

As a result, we get that

$$\tau(G) = \tau(G')\tau(G \cdot G') + \tau(G - G') \sum_{i=1}^{2^n} \tau(G'_{i,u})\tau(G'_{i,v})$$

If G' is a "C-graph", that is, $C_{d,e}$, then by Lemma 3, $\tau(G') = ed^{e-1}$. By Lemma 2, $\sum_{i=1}^{2^n} \tau(G'_{i,u})\tau(G'_{i,v}) = d^e$. Therefore,

$$\tau(G) = ed^{e-1}\tau(G \cdot G') + d^e\tau(G - G') \qquad \square$$

Lemma 5. *Let G be an undirected and connected graph and u be a vertex of G. If several cycles are added to u, denoted by $L_1, L_2, \ldots, L_n$, respectively, then the number of spanning trees of the new graph is $\tau(G)\prod_{i=1}^{n}\tau(L_i)$. To clarify the definition, let $G = (V_0, E_0)$ and $L_i = (V_i, E_i), i = 1, 2, \ldots, n$, where ($u_i$ is another notation of u)*

$$V_i = \{u_i, a_{i,1}, a_{i,2}, \ldots, a_{i,j_i} | j_i \in \mathbb{N}\}$$

and

$$E_i = \begin{cases} \{\{u, u_i\}\} & \text{for } j_i = 0 \\ \{\{u, a_{i,1}\}, \{a_{i,k}, a_{i,k+1}\}, \{a_{j_i}, u\} | k = 1, 2, \ldots, j_i - 1\} & \text{for } j_i \geq 1 \end{cases}$$

Let V, E be the vertex set and edge set of the new graph, then $V = \cup_{m=0}^{n} V_m$ and $E = \cup_{m=0}^{n} E_m$.

Proof. Assume the number of cycles in this graph is n. We particularly choose one cycle from them, and assume the cycle is divided into d segments by $d-1$ vertices. It is easy to understand that the number of spanning trees of this cycle is d(removing one segment of the cycle obtains a spanning tree and the cycle has d segments).

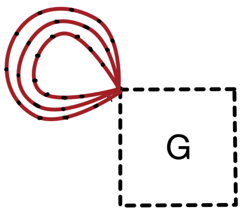

Fig. 9. The red part of this graphs is cycles

If there exists several cycles with the same vertex, then from **multiplication theorem**, we can get that the number of spanning trees of many cycles with the same vertex is the multiplication of the number of spanning trees of each cycle. We denoted it by $\prod_{i=1}^{n} \tau(L_i)$, where n represents n cycles, and L_i represents the ith cycle. Accordingly, the number of spanning trees of the graph is $\tau(G) \prod_{i=1}^{n} \tau(L_i)$.

Specially, when there are e cycles and each cycle have d segments. $\prod_{i=1}^{n} \tau(L_i) = d^e$. The total number of spanning trees is $d^e \tau(G)$. □

Next, we obtain results on the number of spanning trees of fan graphs under different circumstances. To start with, we consider fan graphs with multiple vertices.

4. Fan Graphs with Multiple Vertices

And write before Theorem 1, $G_{d_1,1,1,1}(n)$ is obtained by adding $d_1 - 1$ vertices into isosceles edges of the original $G(n)$ (See Figure 10).

Theorem 1. *The no.1 of spanning trees of $G_{d_1,1,1,1}(n)$ is* $\tau(G_{d_1,1,1,1}(n)) = \frac{1}{\sqrt{4d_1+1}} 2^{-n-1}[(2d_1 + 1 + \sqrt{4d_1 + 1})^{n+1} - (2d_1 + 1 - \sqrt{4d_1 + 1})^{n+1}]$

Proof. Obviously, the path from O to s_1 can be viewed as a "C-Graph" $C_{d_1,1}(O, s_1)$. According to Lemma 4, we can obtain the equation:

$$\tau(G_{d_1,1,1,1}(n)) = d_1 \tau(G_{d_1,1,1,1}(n-1) - C_{d_1,1}(O, s_1)) + \tau(G_{d_1,1,1,1} \cdot C_{d_1,1}(O, s_1))$$

where $G_{d_1,1,1,1} \cdot C_{d_1,1}(O, s_1)$ can be alternated by $Z_{d_1,1,1,1}(n-1)$ and $G_{d_1,1,1,1}(n-1) - C_{d_1,1}(O, s_1)$ can be alternated by $T_{d_1,1,1,})(n-1)$. Thus, we can transform it into

$$\tau(G_{d_1,1,1,1}(n)) = d_1\tau(T_{d_1,1,1,1}(n-1)) + \tau(Z_{d_1,1,1,1}(n-1)) \tag{1}$$

It is not difficult to find that $T_{d_1,1,1,1}(n-1)$ consists of a "C-Graph" $C_{1,1}(s_1, s_2)$ and $G_{d_1,1,1,1}(n-1)$. By Lemma 4, we have

$$\tau(T_{d_1,1,1,1}(n-1)) = \tau(G_{d_1,1,1,1}(n-1))$$

Subsequently,

$$\tau(Z_{d_1,1,1,1}(n-1)) = \tau(G_{d_1,1,1,1}(n)) - d_1\tau(G_{d_1,1,1,1}(n-1)) \tag{2}$$

and then we can get

$$\tau(Z_{d_1,1,1,1}(n-2)) = \tau(G_{d_1,1,1,1}(n-1)) - d_1\tau(G_{d_1,1,1,1}(n-2)) \tag{3}$$

Next, we consider $Z_{d_1,1,1,1}(n-1)$, and one of two paths from O to s_2 can be seen as $C_{1,1}(O, s_2)$. From Lemma 4 and Lemma 5, we get:

$$\begin{aligned}\tau(Z_{d_1,1,1,1}(n-1)) &= \tau(Z_{d_1,1,1,1}(n-1) - C_{1,1}(O, s_2)) + \tau(Z_{d_1,1,1,1}(n-1) \cdot C_{1,1}(O, s_2)) \\ &= \tau(G_{d_1,1,1,1}(n-1)) + d_1\tau(Z_{d_1,1,1,1}(n-2)) \end{aligned} \tag{4}$$

Now we substitute equation 2,3 into equation 4,

$$\tau(G_{d_1,1,1,1}(n)) = (2d_1+1)\tau(G_{d_1,1,1,1}(n-1)) - d_1^2\tau(G_{d_1,1,1,1}(n-2))$$

Hence, we get the iteration of $\tau(G_{d_1,1,1,1}(n))$.

Finally, we consider initial cases. That is, $\tau(G_{d_1,1,1,1}(1)) = 2d_1 + 1$ and $\tau(G_{d_1,1,1,1}(2)) = 3d_1^2 + 4d_1 + 1$. By putting the iteration we have already figured out and initial cases into Mathematica, we can attain spanning trees of $G_{d_1,1,1,1}(n)$. □

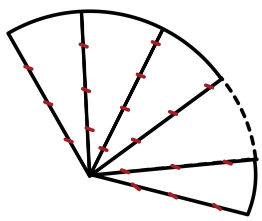

Fig. 10. $G_{d_1,1,1,1}(n)$

Corollary 1. *The spanning trees of $G_{1,d_2,1,1}(n)$ is* $\tau(G_{1,d_2,1,1}(n)) = \frac{1}{\sqrt{d_2}\sqrt{d_2+4}} 2^{-n-1}[(d_2+2+\sqrt{d_2+4}\sqrt{d_2})^{n+1} - (d_2+2-\sqrt{d_2+4}\sqrt{d_2})^{n+1}]$

Proceeding on similar lines as in the proof of Theorem 1, we can obtain the no.2 of spanning trees of $G_{1,d_2,1,1}(n)$.

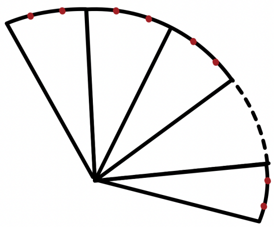

Fig. 11. $G_{1,d2,1,1}(n)$

Based on these cases above, we have

Theorem 2. *The spanning tree of* $G_{d_1,d_2,1,1}(n)$ *is* $\tau(G_{d_1,d_2,1,1}(n)) = \frac{1}{\sqrt{d_2}\sqrt{4d_1+d_2}} 2^{-n-1}[(2d_1+d_2+\sqrt{d_2}\sqrt{4d_1+d_2})^{n+1} - (2d_1+d_2-\sqrt{d_2}\sqrt{4d_1+d_2})^{n+1}]$

Proof. Similarly, the path which starts from O and ends with s_1 can be considered as a "C-Graph" $C_{d_1,1}(O,s_1)$. Therefore, by Lemma 4,

$$\tau(G_{d_1,d_2,1,1}(n)) = \tau(G_{d_1,d_2,1,1}(n)\cdot C_{d_1,1}(O,s_1)) + d_1\tau(G_{d_1,d_2,1,1}(n) - C_{d_1,1}(O,s_1))$$

where $G_{d_1,d_2,1,1}(n)\cdot C_{d_1,1}(O,s_1)$ represents $Z_{d_1,d_2,1,1}(n-1)$ and $G_{d_1,d_2,1,1}(n) - C_{d_1,1}(O,s_1)$ represents $T_{d_1,d_2,1,1}(n-1)$. Then we obtain:

$$\tau(G_{d_1,d_2,1,1}(n)) = \tau(Z_{d_1,d_2,1,1}(n-1)) + d_1\tau(T_{d_1,d_2,1,1}(n-1))$$

Apparently, $T_{d_1,d_2,1,1}(n-1)$ is composed of a "C-Graph" $C_{d_2,1}(s_1,s_2)$ and $G_{d_1,d_2,1,1}(n-1)$. By Lemma 4, we get:

$$\tau(T_{d_1,d_2,1,1}(n-1)) = \tau(G_{d_1,d_2,1,1}(n-1))$$

Subsequently, we get:

$$\tau(G_{d_1,d_2,1,1}(n)) = \tau(Z_{d_1,d_2,1,1}(n-1)) + d_1\tau(G_{d_1,d_2,1,1}(n-1)) \tag{1}$$

and,

$$\tau(G_{d_1,d_2,1,1}(n-1)) = \tau(Z_{d_1,d_2,1,1}(n-2)) + d_1\tau(G_{d_1,d_2,1,1}(n-2)) \tag{2}$$

When it comes to $Z_{d_1,d_2,1,1}(n-1)$, it is easy to see that the one of the two paths which start from O and end with s_2 can be considered as $C_{d_2,1}(O,s_2)$. Therefore, by Lemma 4 and Lemma 5,

$$\begin{aligned}\tau(Z_{d_1,d_2,1,1}(n-1)) &= \tau(Z_{d_1,d_2,1,1}(n-1)\cdot C_{d_2,1}(O,s_2)) + d_2\tau(Z_{d_1,d_2,1,1}(n-1) - C_{d_2,1}(O,s_2)) \\ &= d_1\tau(Z_{d_1,d_2,1,1}(n-2)) + d_2\tau(G_{d_1,d_2,1,1}(n-1))\end{aligned} \tag{3}$$

By equation (1), (2) and (3), we get:

$$\tau(G_{d_1,d_2,1,1}(n)) = (2d_1 + d_2)\tau(G_{d_1,d_2,1,1}(n-1)) - d_1^2\tau(G_{d_1,d_2,1,1}(n-2))$$

With $\tau(G_{d_1,d_2,1,1}(1)) = 2d_1 + d_2$ and $\tau(G_{d_1,d_2,1,1}(2)) = 3d_1^2 + 4d_1d_2 + d_2^2$, we obtain the formula by applying Mathematica. □

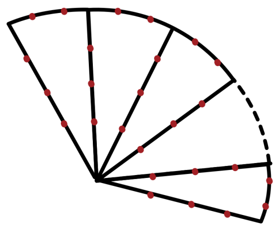

Fig. 12. $G_{d1,d2,1,1}(n)$

5. Fan Graphs with Multiple Edges

Theorem 3. *The spanning tree of* $G_{1,1,e_1,1}(n)$ *is* $\tau(G_{1,1,e_1,1}(n)) = \frac{\sqrt{e_1}}{\sqrt{e_1+4}}2^{-n-1}$ $[(e_1+2+\sqrt{e_1}\sqrt{e_1+4})^{n+1} - (e_1+2-\sqrt{e_1}\sqrt{e_1+4})^{n+1}]$

Proof. Like the process we display before, the e_1 edges which all start from O and end with s_1 can be considered as a "C-Graph" $C_{1,e_1}(O,s_1)$. Therefore, by Lemma 4,

$$\tau(G_{1,1,e_1,1}(n)) = e_1\tau(G_{1,1,e_1,1}(n)\cdot C_{1,e_1}(O,s_1)) + \tau(G_{1,1,e_1,1}(n) - C_{1,e_1}(O,s_1))$$

where $G_{1,1,e_1,1}(n) \cdot C_{1,e_1}(O,s_1)$ represents $Z_{1,1,e_1,1}(n-1)$ and $G_{1,1,e_1,1}(n) - C_{1,e_1}(O,s_1)$ represents $T_{1,1,e_1,1}(n-1)$. Then we obtain:

$$\tau(G_{1,1,e_1,1}(n)) = e_1\tau(Z_{1,1,e_1,1}(n-1)) + \tau(T_{1,1,e_1,1}(n-1))$$

Apparently, $T_{1,1,e_1,1}(n-1)$ is composed of an edge s_1s_2 and $G_{1,1,e_1,1}(n-1)$. By Lemma 1, we get:

$$\tau(T_{1,1,e_1,1}(n-1)) = \tau(G_{1,1,e_1,1}(n-1))$$

Subsequently, we get:

$$\tau(G_{1,1,e_1,1}(n)) = e_1\tau(Z_{1,1,e_1,1}(n-1)) + \tau(G_{1,1,e_1,1}(n-1)) \quad (1)$$

and,

$$\tau(G_{1,1,e_1,1}(n-1)) = e_1\tau(Z_{1,1,e_1,1}(n-2)) + \tau(G_{1,1,1,1}(n-2)) \quad (2)$$

When it comes to $Z_{1,1,e_1,1}(n-1)$, it is easy to see that all edges which start from O and end with s_2 can be divided into two groups. One of the two contains only a single edge Os_2 and the other one contains a "C-Graph" $C_{1,e_1}(O,s_2)$. Therefore, by Lemma 1 and Lemma 5,

$$\begin{aligned}\tau(Z_{1,1,e_1,1}(n-1)) &= \tau(Z_{1,1,e_1,1}(n-1)\cdot C_{1,e_1}(O,s_2)+\tau(Z_{1,1,e_1,1}(n-1)-C_{1,e_1}(O,s_2)) \\ &= \tau(Z_{1,1,e_1,1}(n-2))+\tau(G_{1,1,e_1,1}(n-1))\end{aligned} \tag{3}$$

By equation (1), (2) and (3), we get:

$$\tau(G_{1,1,e_1,1}(n)) = (e_1+2)\tau(G_{1,1,e_1,1}(n-1)) - \tau(G_{1,1,e_1,1}(n-2))$$

With $\tau(G_{1,1,e_1,1}(1)) = e_1^2+2e_1$ and $\tau(G_{1,1,e_1,1}(2)) = e_1^3+4e_1^2+3e_1$, we obtain the formula by applying Mathematica. This case has been calculated by the other method. [11] □

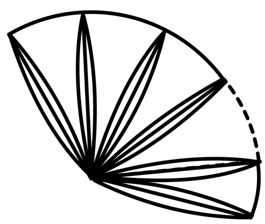

Fig. 13. $G_{1,1,e_1,1}(n)$

Corollary 2. *The spanning tree of* $G_{1,1,1,e_2}(n)$ *is* $\tau(G_{1,1,1,e_2}(n)) = \frac{1}{\sqrt{1+4e_2}}2^{-1-n}[(1+2e_2+\sqrt{1+4e_2})^{n+1}-(1+2e_2-\sqrt{1+4e_2})^{n+1}]$

After emulating the proving process of Theorem 4, we can finally get the number of spanning trees for$G_{1,1,1,e_2}(n)$.

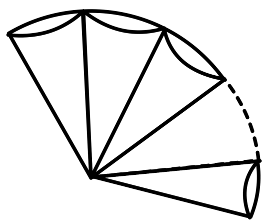

Fig. 14. $G_{1,1,1,e_2}(n)$

Theorem 4. *The spanning tree of* $G_{1,1,e_1,e_2}(n)$ *is* $\tau(G_{1,1,e_1,e_2}(n)) = \frac{\sqrt{e_1}}{\sqrt{e_1+4e_2}}2^{-1-n}[(e_1+2e_2+\sqrt{e_1}\sqrt{e_1+4e_2})^{n+1}-(e_1+2e_2-\sqrt{e_1}\sqrt{e_1+4e_2})^{n+1}]$

Proof. Vertices s_1,O(See figure3) and paths between them form a "C-Graph" as well. Keeping on applying Lemma 4, we can get:

$$\tau(G_{1,1,e_1,e_2}(n)) = e_1\tau(G_{1,1,e_1,e_2}(n) \cdot C_{1,e_1}(O, s_1)) + \tau(G_{1,1,e_1,e_2}(n) - C_{1,e_1}(O, s_1))$$

where $\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}) \cdot \mathrm{C}_{1,e_1}(\mathrm{O}, \mathrm{s}_1)$ represents $Z_{1,1,e_1,e_2}(n-1)$(See Figure 7) and $G_{1,1,e_1,e_2}(n) - C_{1,e_1}(O, s_1)$ represents $T_{1,1,e_1,e_2}(n-1)$(see Figure 9) in this case. Then we can obtain:

$$\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n})) = \mathrm{e}_1\tau(\mathrm{Z}_{1,1,e_1,e_2}(\mathrm{n}-1)) + \tau(\mathrm{T}_{1,1,e_1,e_2}(\mathrm{n}-1))$$

$\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1)$ and $\mathrm{T}_{1,1,e_1,e_2}(\mathrm{n}-1)$ is included in $\mathrm{C}_{1,e_2}(\mathrm{s}_1, \mathrm{s}_2)$. Using Lemma 4,

$$\tau(\mathrm{T}_{1,1,e_1,e_2}(\mathrm{n}-1)) = \mathrm{e}_2\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1))$$

Next,

$$\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n})) = \mathrm{e}_2\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1)) + \mathrm{e}_1\tau(\mathrm{Z}_{1,1,e_1,e_2}(\mathrm{n}-1)) \tag{1}$$

$$\tau(\mathrm{Z}_{1,1,e_1,e_2}(\mathrm{n}-1)) = \frac{\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n})) - \mathrm{e}_2\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1))}{\mathrm{e}_1} \tag{2}$$

and we can deduce that

$$\tau(\mathrm{Z}_{1,1,e_1,e_2}(\mathrm{n}-2)) = \frac{\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1)) - \mathrm{e}_2\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-2))}{\mathrm{e}_1} \tag{3}$$

Consider $Z_{1,1,e_1,e_2}(n-1)$, two different groups of edges ,which are edges between s_1 and s_2 and edges between O and s_2 in $G_{1,1,e_1,e_2}(n)$, consist of all edges between O and s_2. (In other words, by definition 3, all vertices on paths between O and s_2 in $Z_{1,1,e_1,e_2}(n-1)$ can be divided into 2 groups such that each of them form a "C-Graph" with O, s_2, that is, $C_{1,e_1}(O, s_2)$ and $C_{1,e_2}(O, s_2)$.) Now, we delete all edges between s_1 and s_2 in $G_{1,1,e_1,e_2}(n)$. Then we merge vertices O and s_2 in $Z_{1,1,e_1,e_2}(n-1)$, and cycles are accordingly generated. By applying Lemma 5, we can get

$$\begin{aligned}\tau(Z_{1,1,e_1,e_2}(n-1)) &= \tau(Z_{1,1,e_1,e_2}(n-1) - C_{1,e_2}(O, s_2)) + e_2\tau(Z_{1,1,e_1,e_2}(n-1) \cdot C_{1,e_2}(O, s_2)) \\ &= \tau(G_{1,1,e_1,e_2}(n-1)) + e_2\tau(Z_{1,1,e_1,e_2}(n-2)).\end{aligned} \tag{4}$$

By substituting equations (2) and (3) into equation (4) and simplifying the equation, we can get the iteration of $\tau(G_{1,1,e_1,e_2}(n))$. The iterative equation is:

$$\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n})) = (2\mathrm{e}_2 + \mathrm{e}_1)\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-1)) - \mathrm{e}_2^2\tau(\mathrm{G}_{1,1,e_1,e_2}(\mathrm{n}-2)) \tag{5}$$

Then, we find the initial case: $\tau(G_{1,1,e_1,e_2}(1)) = e_1(e_1 + 2e_2)$.

Based on $\tau(G_{1,1,e_1,e_2}(1))$, we can deduce the value of $\tau(G_{1,1,e_1,e_2}(2))$, that is: $\tau(\mathrm{G}_{1,1,e_1,e_2}(2)) = \mathrm{e}_1(\mathrm{e}_1 + \mathrm{e}_2)(\mathrm{e}_1 + 3\mathrm{e}_2)$ Combining the iterative equation and initial cases, the result of $\tau(G_{1,1,e_1,e_2}(n))$ can be easily computed by Mathematica. □

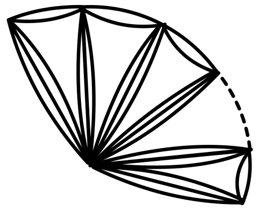

Fig. 15. $G_{1,1,e1,e2}(n)$

6. Fan Graphs with Multiple Vertices and Edges

The most complex case is that the value of d_1, d_2, e_1, e_2 are arbitrarily chosen from N^*. Thus, we have:

Theorem 5. *The number of spanning trees of* $G_{d_1,d_2,e_1,e_2}(n)$ *is* $\tau(G_{d_1,d_2,e_1,e_2}(n)) = \frac{\sqrt{e_1}}{\sqrt{d_2}\sqrt{d_2e_1+4d_1e_2}}2^{-1-n}d_1^{-1+e_1}[(d_1^{-1+e_1}d_2^{-1+e_2}(d_2e_1+2d_1e_2+\sqrt{d_2}\sqrt{e_1}\sqrt{d_2e_1+4d_1e_2}))^{n+1} - (d_1^{-1+e_1}d_2^{-1+e_2}(d_2e_1+2d_1e_2-\sqrt{d_2}\sqrt{e_1}\sqrt{d_2e_1+4d_1e_2}))^{n+1}]$

Proof. Obviously, vertices S_1,O(See figure3) and paths between them form a "C-Graph". By applying Lemma 4, we can get:

$$\tau(G_{d_1.d_2,e_1,e_2}(n)) = e_1d_1^{e_1-1}\tau(G_{d_1.d_2,e_1,e_2}(n)\cdot C_{d_1,e_1}(O,s_1)) \\ +d_1^{e_1}\tau(G_{d_1.d_2,e_1,e_2}(n)-C_{d_1,e_1}(O,s_1))$$

where $\mathrm{G_{d_1.d_2,e_1,e_2}(n)\cdot C_{d_1,e_1}(O,s_1)}$ represents $Z_{d_1,d_2,e_1,e_2}(n-1)$(See Figure 7) and $G_{d_1.d_2,e_1,e_2}(n)-C_{d_1,e_1}(O,s_1)$ represents $T_{d_1,d_2,e_1,e_2}(n-1)$(see Figure 9) in this case. Then we can obtain:

$$\tau(\mathrm{G_{d_1.d_2,e_1,e_2}(n)}) = \mathrm{e_1d_1^{e_1-1}}\tau(\mathrm{Z_{d_1.d_2,e_1,e_2}(n-1)}) + \mathrm{d_1^{e_1}}\tau(\mathrm{T_{d_1.d_2,e_1,e_2}(n-1)})$$

Apparently, $\mathrm{T_{d_1.d_2,e_1,e_2}(n-1)}$ is composed of $\mathrm{C_{d_2,e_2}(s_1,s_2)}$ and $\mathrm{G_{d_1,d_2,e_1,e_2}(n-1)}$. Using Lemma 4,

$$\tau(\mathrm{T_{d_1,d_2,e_1,e_2}(n-1)}) = \mathrm{e_2d_2^{e_2-1}}\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n-1)})$$

Subsequently,

$$\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n)}) = \mathrm{d_1^{e_1}e_2d_2^{e_2-1}}\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n-1)}) + \mathrm{e_1d_1^{e_1-1}}\tau(\mathrm{Z_{d_1.d_2,e_1,e_2}(n-1)}) \tag{1}$$

$$\tau(\mathrm{Z_{d_1,d_2,e_1,e_2}(n-1)}) = \frac{\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n)}) - \mathrm{d_1^{e_1}e_2d_2^{e_2-1}}\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n-1)})}{\mathrm{e_1d_1^{e_1-1}}} \tag{2}$$

and we can deduce that

$$\tau(\mathrm{Z_{d_1,d_2,e_1,e_2}(n-2)}) = \frac{\tau(\mathrm{G_{d_1,d_2,e_1,e_2}(n-1)}) - \mathrm{d_1^{e_1}e_2d_2^{e_2-1}}(\tau\mathrm{G_{d_1,d_2,e_1,e_2}(n-2)})}{\mathrm{e_1d_1^{e_1-1}}} \tag{3}$$

Consider $Z_{d_1,d_2,e_1,e_2}(n-1)$, all vertices on paths between O and s_2 in $Z_{d_1,d_2,e_1,e_2}(n-1)$ can be divided into 2 groups such that each of them form a "C-Graph" with O, s_2, that is, $C_{d_1,e_1}(O,s_2)$ and $C_{d_2,e_2}(O,s_2)$.) Now, we delete all edges between s_1 and s_2 in $G_{d_1,d_2,e_1,e_2}(n)$. Then we merge vertices O and s_2 in $Z_{d_1,d_2,e_1,e_2}(n-1)$, and cycles are accordingly generated. From Lemma 5, we can get

$$\begin{aligned}\tau(Z_{d_1,d_2,e_1,e_2}(n-1)) &= d_2^{e_2}\tau(Z_{d_1,d_2,e_1,e_2}(n-1) - C_{d_2,e_2}(O,s_2)) \\ &\quad + e_2 d_2^{e_2-1}\tau(Z_{d_1,d_2,e_1,e_2}(n-1)\cdot C_{d_2,e_2}(O,s_2)) \\ &= d_2^{e_2}\tau(G_{d_1,d_2,e_1,e_2}(n-1)) + e_2 d_2^{e_2-1} d_1^{e_1}\tau(Z_{d_1,d_2,e_1,e_2}(n-2)).\end{aligned} \tag{4}$$

After substituting equations (2) and (3) into equation (4) and simplify the equation, We can get the relationship between $\tau(G_{d_1,d_2,e_1,e_2}(n))$, $\tau(G_{d_1,d_2,e_1,e_2}(n-1))$ and $\tau(G_{d_1,d_2,e_1,e_2}(n-2))$. The iterative equation is:

$$\begin{aligned}\tau(G_{d_1,d_2,e_1,e_2}(n)) =& (2e_2 d_1^{e_1} d_2^{e_2-1} + e_1 d_1^{e_1-1} d_2^{e_2})\tau(G_{d_1,d_2,e_1,e_2}(n-1)) - \\ & e_2^2 d_1^{2e_1} d_2^{2e_2-2}\tau(G_{d_1,d_2,e_1,e_2}(n-2))\end{aligned} \tag{5}$$

Next, we return back to consider the initial case:$\tau(G_{d_1,d_2,e_1,e_2}(1)) = d_1^{-2+2e_1} d_2^{-1+e_2} e_1(d_2e_1 + 2d_1e_2)$.Based on $\tau(G_{d_1,d_2,e_1,e_2}(1))$, we can deduce the value of $\tau(G_{d_1,d_2,e_1,e_2}(2))$, that is: $\tau(G_{d_1,d_2,e_1,e_2}(2)) = d_1^{-3+3e_1} d_2^{-2+2e_2} e_1(d_2e_1 + d_1e_2)(d_2e_1 + 3d_1e_2)$.

Combining the iterative equation and initial cases, the result of $\tau(G_{d_1,d_2,e_1,e_2}(n))$ can be easily computed by Mathematica. □

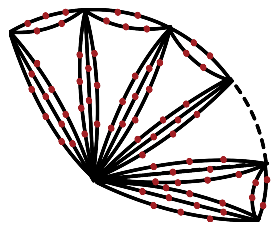

Fig. 16. $G_{d_1,d_2,e1,e2}(n)$

7. Analysis

After proving all theorems and by using Theorem 7, we then start to discover and analyse the impact that d_1, d_2, e_1 and e_2 respectively have on the number of spanning trees of $G_{d_1,d_2,e_1,e_2}(n)$. Additionally, we will use **Matrix Tree Theorem** to test the rationality of our results.

To begin with, we let one of these four parameters(d_1, d_2, e_1, and e_2) as a variable (greater than 0 and belongs to natural numbers). At the same time, we set other three parameters' value to be 2, thus, by using Matlab, we subsequently get four following subgraphs, where the Y-axis represents the number of spanning trees of $G_{d_1,d_2,e_1,e_2}(n)$, the X-axis represents the number of d_1, d_2, e_1 and e_2:

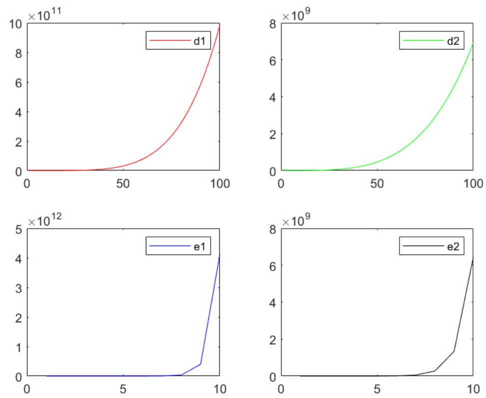

Fig. 17. Graphs of Four Factors d_1, d_2, e_1 and e_2

Firstly, we get a close look on two graphs at the bottom, which respectively are the graph of e_1 as variable and the graph of e_2 as variable. Apparently, when e_1 and e_2 are both equal to 10, the according value of Y-axies are approximately 4.0×10^{12} and 6.5×10^9.

We then observe graphs of d_1, d_2. When the value of d_1 and d_2 are both 100, the number of spanning trees of $G_{d_1,d_2,e_1,e_2}(n)$ is about 9.8×10^{11} and 6.9×10^9 correspondingly.

It is not difficult to find that when d_1 and d_2 are both 10 times the value of e_1 and e_2, values of Y-axies of d_1 and e_1 are nearly at the same order of magnitudes, so as values of Y-axies of d_2 and e_2. Therefore, we can simply conclude that graphs of d_1 and d_2 have the similar rising tendency, and the same goes for graphs of e_1 and e_2. Hence, in the next step, we will analyse and compare these graphs by integrating graphs of d_1 and d_2, e_1 and e_2.

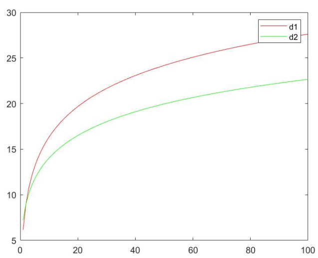

Fig. 18. Graphs of $d1$ and $d2$

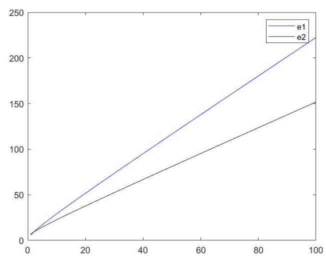

Fig. 19. Graphs of $e1$ and $e2$

To clarify, we use the natural logarithmic function to buffer these curves and then separate graphs of 4 parameters into 2 groups, where one of the two contains $d1, d2$ and another contains the rest. By observing Figure.18, the rate of increase of number of spanning trees caused by the change of parameters $d1$ and $d2$ respectively is shown. By comparing the two curves, we can see that the change of $d1$ has stronger influence on the number of spanning trees than $d2$.

Next, we analyse the influence of e_1 and e_2(see Figure.19). Similar to the analysis of d_1 and d_2, we use the natural logarithmic function for buffering (See Figure 19). By comparing two curves, it is obvious that the increase of e_1 has stronger effect on the number of spanning trees than the increase of e_2 does.

8. Conclusion

In summary, we have calculated the number of spanning trees of all kinds of graphs obtained from the fan graph. Furthermore, the new graph called "C-graph" and the recursive relation that we introduced greatly simplify the derivation process, and at the same time, it gives us a different viewpoint to identify the construction of general fan graph. After analysing the function we obtained from Theorem 7, we have drew the conclusion that edges play a major role in the number of spanning trees. Eventually, we have proved our results by applying Matrix Tree Theorem.

In the main result part of our article, it is easy to discover that Lemma 4 actually is the generalization of Lemma 1, and it has the advantage of directly addressing subgraphs of the original graph, which is more efficient than only working with one edge at a time. Similarly, when considering some complex graphs(such as wheel graphs), which are the composition of many subgraphs with the same structure, a general expression can be obtained by the method we employed in Lemma 4. Moreover, based on the general expression, the corresponding iterative relation and the number of spanning trees will be deduced.

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