# Berinde-Type generalized $\alpha - \beta - \psi$ contractive mappings in partial metric spaces and some related fixed points


## Abstract

Aims/ objectives: The objective of this paper is to introduce the notion of generalized $\alpha - \beta - \psi$ contractive mappings involving rational expressions and establish existence and uniqueness of fixed points of Berinde type generalized $\alpha - \beta - \psi$ contractive mappings in the context of partial metric spaces . Additionally, we provide an example in support of our results.



*Keywords: Generalized $\alpha - \beta - \psi$ contractive mappings, partial metric spaces, generalized almost contractions.*

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## 1 Introduction

In 1992, Matthews ([1],[2]) introduced a very interesting generalization of the metric space known as partial metric space (PMS) in which the self distance not required to be zero and proved the partial metric version of Banach fixed point theorem. Samet et al. [3] extended and generalized the Banach contraction principle by introducing a new class of contractive type mappings known as $\alpha - \psi$ contractive type mappings. Karapinar and Samet [4] generalized the $\alpha - \psi$ contractive type mappings and established various fixed point theorems. Also Karapinar et al. [15] shown some results in partial metric spaces using rational type expressions. Recently, Alsulami H. et al.[5] generalized the concept of $\alpha - \psi$ rational type contractive mappings and proved fixed point theorems for such class of mappings. Berinde [6, 7] presented the idea of almost contraction in metric spaces. Further, Aydi H. et al. [8] considered generalized Berinde type contraction in setting of partial metric spaces and established some fixed point theorems. Such type of contractions are also known as generalized almost contractions in the literature. On the other hand, Chandok [10] introduced the concept of $(\alpha, \beta)$-admissible mappings and presented some results. Recently, Masmali et al. [11] presented a new class of contractive pair of mappings called $\alpha-\beta-\psi-\varphi$ contraction and generalized $\alpha-\beta-\psi-\varphi$ contractive pair of mappings and studied various fixed point theorems for such mappings in digital metric spaces.

**Definition 1.1.** [4] Let $\Psi$ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ such that

(i) $\psi$ is nondecreasing;

(ii) $\sum_{n=1}^{\infty} \psi^n < \infty$ for all $t > 0$, where $\psi^n$ is the $n^{th}$ iterate of $\psi$

These functions are known as (c)-comparison functions in the literature. We can easily verify that if $\psi$ is (c)-comparison function then $\psi(t) < t$ for any $t > 0$.

**Definition 1.2.** [17] Let $\Theta$ be the family of functions $\vartheta : [0,\infty) \to [0,\infty)$ such that

(i) $\vartheta$ is continuous;

(ii) $\vartheta(t) = 0$ if and only if $t = 0$

Recently, Samet et al. [3] introduced the concepts of $\alpha - \psi$ contractive type mappings and $\alpha$-admissible mappings and proved following results:

**Definition 1.3.** [3] Let $(\mathcal{X}, \tilde{d})$ be a metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self mapping. $\mathcal{T}$ is said to be an $\alpha - \psi$ contractive mappings if there exists two functions $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(\delta,\eta) d(\mathcal{T}\delta, \mathcal{T}\eta) \leq \psi(d(\delta,\eta))$$

for all $\delta, \eta \in \mathcal{X}$.

**Definition 1.4.** [3] Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$. $\mathcal{T}$ is said to $\alpha$-admissible if

$$\alpha(\delta,\eta) \geq 1 \Rightarrow \alpha(\mathcal{T}\delta, \mathcal{T}\eta) \geq 1$$

for all $\delta, \eta \in \mathcal{X}$.

**Theorem 1.1.** *[3] Let $(\mathcal{X}, \tilde{d})$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an $\alpha - \psi$ contractive mappings satisfying the conditions:*

*(i) $\mathcal{T}$ is $\alpha$-admissible;*

*(ii) There exists $\delta_0 \in \mathcal{X}$ such that $\alpha(\delta_0, \mathcal{T}\delta_0) \geq 1$;*

*(iii) $\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed point.

**Theorem 1.2.** *[3] Let $(\mathcal{X}, \tilde{d})$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an $\alpha - \psi$ contractive mappings satisfying the conditions:*

*(i) $\mathcal{T}$ is $\alpha$-admissible;*

*(ii) There exists $\delta_0 \in \mathcal{X}$ such that $\alpha(\delta_0, \mathcal{T}\delta_0) \geq 1$;*

*(iii) If $\{\delta_n\}$ is a sequence in $\mathcal{X}$ such that $\alpha(\delta_n, \delta_{n+1}) \geq 1$ for all $n$ and $\delta_n \to \delta \in \mathcal{X}$ as $n \to \infty$, then $\alpha(\delta_n, \delta) \geq 1$.*

Then $\mathcal{T}$ has a fixed point. Further, Samet et al. [3] added the condition that for all $\delta, \eta \in \mathcal{X}$, there exists $\mu \in \mathcal{X}$ such that $\alpha(\delta, \mu) \geq 1$ and $\alpha(\eta, \mu) \geq 1$ to hypotheses of above theorems to assure the uniqueness of the fixed point.

**Definition 1.5.** [1] Let $\mathcal{X}$ be a non-empty set. A function $\rho : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ is said to be a partial metric on $\mathcal{X}$ if the following conditions hold:

(i) $\delta = \eta$ if and only if $\rho(\delta,\delta) = \rho(\eta,\eta) = \rho(\delta,\eta)$ ;

(ii) $\rho(\delta,\delta) \leq \rho(\delta,\eta)$;

(iii) $\rho(\delta, \eta) = \rho(\eta, \delta)$ :

(iv) $\rho(\delta, \eta) \leq \rho(\delta, \mu) + \rho(\mu, \eta) - \rho(\mu, \mu)$. for all $\delta, \eta, \mu \in \mathcal{X}$.

The set $\mathcal{X}$ equipped with the metric $\rho$ defined above is called a partial metric space and it is denoted by $(\mathcal{X}, \rho)$ (in short PMS).

**Example 1.3.** *[13] Let $\mathcal{X} = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define $\rho([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then $(\mathcal{X}, \rho)$ is a partial metric space.*

**Example 1.4.** *[13] Let $\mathcal{X} = [0, \infty)$ and define $\rho(\delta, \eta) = \max\{\delta, \eta\}$. Then $(\mathcal{X}, \rho)$ is a partial metric space.*

**Lemma 1.5.** *[1] Let $(\mathcal{X}, \rho)$ be a partial metric space.*

*(a) A sequence $\{\delta_n\}$ in $(\mathcal{X}, \rho)$ converges to a point $\delta \in \mathcal{X}$ if and only if*

$$\rho(\delta, \delta) = \lim_{n \to \infty} \rho(\delta_n, \delta),$$

*(b) A sequence $\{\delta_n\}$ in $(\mathcal{X}, \rho)$ is a Cauchy sequence if $\lim_{m,n \to \infty} \rho(\delta_n, \delta_m)$ exists and finite ,*

*(c) $(\mathcal{X}, \rho)$ is complete if every Cauchy $\{\delta_n\}$ in $\mathcal{X}$ converges to a point $\delta \in \mathcal{X}$, such that*

$$\rho(\delta, \delta) = \lim_{m,n \to \infty} \rho(\delta_m, \delta_n) = \lim_{n \to \infty} \rho(\delta_n, \delta) = \rho(\delta, \delta).$$

**Lemma 1.6.** *([12],[1],[2]) Let $\rho$ be a partial metric on $\mathcal{X}$, then the functions $\tilde{d}_\rho, \tilde{d}_m : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that*

$$\tilde{d}_\rho(\delta, \eta) = 2\rho(\delta, \eta) - \rho(\delta, \delta) - \rho(\eta, \eta)$$

*and*

$$\begin{aligned} \tilde{d}_m(\delta, \eta) =& \max\{\rho(\delta, \eta) - \rho(\delta, \delta), \rho(\delta, \eta) - \rho(\eta, \eta)\} \\ =& \rho(\delta, \eta) - \min\{\rho(\delta, \delta), \rho(\eta, \eta)\} \end{aligned}$$

*are metric on $\mathcal{X}$. Furthermore $(\mathcal{X}, \tilde{d}_\rho)$ and $(\mathcal{X}, \tilde{d}_m)$ are metric spaces. It is clear that $\tilde{d}_\rho$ and $\tilde{d}_m$ are equivalent.*
*Let $(\mathcal{X}, \rho)$ be a partial metric space. Then*

1. *A sequence $\{\delta_n\}$ in $(\mathcal{X}, \rho)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{X}, \tilde{d}_\rho)$,*
2. *$(\mathcal{X}, \rho)$ is complete if and only if the metric space $(\mathcal{X}, \tilde{d}_\rho)$ is complete. Moreover $\lim_{n \to \infty} \tilde{d}_\rho(\delta_n, \delta) = 0 \Leftrightarrow \rho(\delta, \delta) = \lim_{n \to \infty} \rho(\delta_n, \delta) = \lim_{n,m \to \infty} \rho(\delta_n, \delta_m)$.*

**Lemma 1.7.** *[9, 16] Assume that $\delta_n \to \mu$ as $n \to \infty$ in a partial metric space $(\mathcal{X}, \rho)$ such that $\rho(\mu, \mu) = 0$ Then $\lim_{n \to \infty} \rho(\delta_n, \eta) = \rho(\mu, \eta)$ for every $\eta \in \mathcal{X}$.*

**Lemma 1.8.** *[14, 18] Let $(\mathcal{X}, \rho)$ be a partial metric space.*

1. *if $\rho(\delta, \eta) = 0$ then $\delta = \eta$ ,*
2. *If $\delta \neq \eta$ then $\rho(\delta, \eta) > 0$.*

**Definition 1.6.** [10] Let $\mathcal{T}$ be a self mapping on $\mathcal{X}$ and let $\alpha, \beta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be functions. We say $\mathcal{T}$ is an $\alpha - \beta$ admissible mapping if $\alpha(\delta, \eta) \geq 1$ and $\beta(\delta, \eta) \geq 1$ implies $\alpha(\mathcal{T}\delta, \mathcal{T}\eta) \geq 1$ and $\beta(\mathcal{T}\delta, \mathcal{T}\eta) \geq 1$ for all $\delta, \eta \in \mathcal{X}$.

## 2 Main Results

**Definition 2.1.** Let $(\mathcal{X}, \rho)$ be a partial metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a given self map. We say that $\mathcal{T}$ is a generalized $\alpha - \beta - \psi$ contractive mapping if there exists two functions $\alpha, \beta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\psi \in \Psi$ such that for all $\delta, \eta \in \mathcal{X}$ we have

$$\alpha(\delta, \mathcal{T}\delta)\beta(\eta, \mathcal{T}\eta)\rho(\mathcal{T}\delta, \mathcal{T}\eta) \leq \psi(M(\delta, \eta))$$

Where

$$M(\delta, \eta) = \max\left\{\rho(\delta, \eta), \rho(\eta, \mathcal{T}\eta)\frac{1 + \rho(\delta, \mathcal{T}\delta)}{1 + \rho(\delta, \eta)}, \rho(\delta, \mathcal{T}\delta)\frac{1 + \rho(\delta, \mathcal{T}\delta)}{1 + \rho(\delta, \eta)}\right\}$$

**Definition 2.2.** Let $(\mathcal{X}, \rho)$ be a complete partial metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\alpha, \beta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be the mappings. We say that $\mathcal{X}$ is $(\alpha - \beta)$ regular if $\{\delta_n\}$ is a sequence in $\mathcal{X}$ such that $\delta_n \to \delta \in \mathcal{X}$, $\alpha(\delta_n, \delta_{n+1}) \geq 1$, $\beta(\delta_n, \delta_{n+1}) \geq 1$ for all $n$, there exists a subsequence $\{\delta_{n(k)}\}$ of $\{\delta_n\}$ such that $\alpha(\delta_{n(k)}, \delta_{n(k)-1}) \geq 1$, $\beta(\delta_{n(k)}, \delta_{n(k)-1}) \geq 1$ and $\alpha(\delta, \mathcal{T}\delta) \geq 1$, $\beta(\delta, \mathcal{T}\delta) \geq 1$ for all $k$.

**Theorem 2.1.** *Let $(\mathcal{X}, \rho)$ be a complete partial metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be self mapping. Suppose $\alpha, \beta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be the mappings satisfying the conditions:*

*(i) $\mathcal{T}$ is $\alpha - \beta$ admissible;*

*(ii) $\mathcal{T}$ is generalized $\alpha - \beta - \psi$ contractive;*

*(iii) There exists $\delta_0 \in \mathcal{X}$ such that $\alpha(\delta_0, \mathcal{T}\delta_0) \geq 1$ and $\beta(\delta_0, \mathcal{T}\delta_0) \geq 1$;*

*(iv) $\mathcal{T}$ is continuous;*

*(v) There exists $L \geq 0$ such that*

$$\alpha(\delta, \mathcal{T}\delta)\beta(\eta, \mathcal{T}\eta)\rho(\mathcal{T}\delta, \mathcal{T}\eta) \leq \psi(M(\delta, \eta)) + L\vartheta(N(\delta, \eta))$$

*where*

$$N(\delta, \eta) = \min\{\tilde{d}_m(\delta, \mathcal{T}\delta), \tilde{d}_m(\delta, \mathcal{T}\eta), \tilde{d}_m(\eta, \mathcal{T}\delta)\}$$

*for all $\delta, \eta \in \mathcal{X}$ and $\vartheta \in \Theta$. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

*Proof.* Let $\delta_0$ be an arbitrary point such that $\alpha(\delta_0, \mathcal{T}\delta_0) \geq 1$ and $\beta(\delta_0, \mathcal{T}\delta_0) \geq 1$. Suppose we have a sequence $\{\delta_n\}$ in $\mathcal{X}$ such that $\delta_{n+1} = \mathcal{T}\delta_n$ for all $n \in \mathbb{N}$.
If $\delta_n = \delta_{n+1}$ for some $n \in \mathbb{N}$, then $\delta_n$ is a fixed point of $\mathcal{T}$ and the existence part of the proof is finished. Suppose $\delta_n \neq \delta_{n+1}$ for every $n \in \mathbb{N}$. Since $\mathcal{T}$ is $\alpha - \beta$ admissible, so

$$\alpha(\mathcal{T}\delta_0, \mathcal{T}\delta_1) = \alpha(\delta_1, \delta_2) \geq 1$$
$$\alpha(\mathcal{T}\delta_1, \mathcal{T}\delta_2) = \alpha(\delta_2, \delta_3) \geq 1$$

and using induction we have $\alpha(\delta_n, \delta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Similarly, we have $\beta(\delta_n, \delta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, from $(v)$ we have

$$\begin{aligned}\rho(\delta_n, \delta_{n+1}) = \rho(\mathcal{T}\delta_{n-1}, \mathcal{T}\delta_n) &\leq \alpha(\delta_{n-1}, \mathcal{T}\delta_{n-1})\beta(\delta_n, \mathcal{T}\delta_n)\rho(\mathcal{T}\delta_{n-1}, \mathcal{T}\delta_n) \\ &\leq \psi(M(\delta_{n-1}, \delta_n)) + L\vartheta(N(\delta_{n-1}, \delta_n)) \qquad (2.1)\end{aligned}$$

where

$$
\begin{aligned}
M(\delta_{n-1},\delta_n) &= \max\left\{\rho(\delta_{n-1},\delta_n), \rho(\delta_n,\mathcal{T}\delta_n)\frac{1+\rho(\delta_{n-1},\mathcal{T}\delta_{n-1})}{1+\rho(\delta_{n-1},\delta_n)}, \rho(\delta_{n-1},\mathcal{T}\delta_{n-1})\frac{1+\rho(\delta_{n-1},\mathcal{T}\delta_{n-1})}{1+\rho(\delta_{n-1},\delta_n)}\right\}\\
&= \max\left\{\rho(\delta_{n-1},\delta_n), \rho(\delta_n,\delta_{n+1})\frac{1+\rho(\delta_{n-1},\delta_n)}{1+\rho(\delta_{n-1},\delta_n)}, \rho(\delta_{n-1},\delta_n)\frac{1+\rho(\delta_{n-1},\delta_n)}{1+\rho(\delta_{n-1},\delta_n)}\right\}\\
&= \max\left\{\rho(\delta_n,\delta_{n+1}), \rho(\delta_{n-1},\delta_n)\right\} \qquad (2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
N(\delta_{n-1},\delta_n) &= \min\{\tilde{d}_m(\delta_{n-1},\mathcal{T}\delta_{n-1}), \tilde{d}_m(\delta_{n-1},\mathcal{T}\delta_n), \tilde{d}_m(\delta_n,\mathcal{T}\delta_{n-1})\}\\
&= \min\{\tilde{d}_m(\delta_{n-1},\delta_n), \tilde{d}_m(\delta_{n-1},\delta_{n+1}), \tilde{d}_m(\delta_n,\delta_n)\}\\
&= 0 \qquad (2.3)
\end{aligned}
$$

From (2.1), (2.2), (2.3) and using the fact that $\vartheta(t) = 0$ if and only if $t = 0$, we get that

$$\rho(\delta_n,\delta_{n+1}) \leq \psi(\max\{\rho(\delta_n,\delta_{n+1}), \rho(\delta_{n-1},\delta_n)\})$$

Now, if $\rho(\delta_n,\delta_{n+1}) > \rho(\delta_{n-1},\delta_n)$ then

$$\rho(\delta_n,\delta_{n+1}) \leq \psi(\rho(\delta_n,\delta_{n+1})) < \rho(\delta_n,\delta_{n+1})$$

which is a contradiction since $\rho(\delta_n,\delta_{n+1}) > 0$ by lemma 1.8, therefore

$$\rho(\delta_n,\delta_{n+1}) \leq \psi(\rho(\delta_{n-1},\delta_n))$$

for all $n$. Continuing this process and using induction, we obtain

$$\rho(\delta_n,\delta_{n+1}) \leq \psi^n(\rho(\delta_0,\delta_1)) \qquad (2.4)$$

On the other hand, since

$$\rho(\delta_n,\delta_n) \leq \rho(\delta_n,\delta_{n+1})$$

and

$$\rho(\delta_{n+1},\delta_{n+1}) \leq \rho(\delta_n,\delta_{n+1})$$

Then from (2.4) we have

$$\rho(\delta_n,\delta_n) \leq \psi^n(\rho(\delta_0,\delta_1)) \qquad (2.5)$$

and

$$\rho(\delta_{n+1},\delta_{n+1}) \leq \psi^n(\rho(\delta_0,\delta_1)) \qquad (2.6)$$

Thus, we have

$$
\begin{aligned}
\tilde{d}_\rho(\delta_n,\delta_{n+1}) &= 2\rho(\delta_n,\delta_{n+1}) - \rho(\delta_n,\delta_n) - \rho(\delta_{n+1},\delta_{n+1})\\
&\leq 2\rho(\delta_n,\delta_{n+1}) + \rho(\delta_n,\delta_n) + \rho(\delta_{n+1},\delta_{n+1})\\
&\leq 4\psi^n(\rho(\delta_0,\delta_1))
\end{aligned}
$$

So far $m \geq 1$ we have

$$\tilde{d}_\rho(\delta_n, \delta_{n+m}) \leq \tilde{d}_\rho(\delta_n, \delta_{n+1}).......\tilde{d}_\rho(\delta_{n+m-1}, \delta_{n+m})$$
$$\leq \sum_{i=n}^{n+m-1} \psi^i(\rho(\delta_0, \delta_1))$$

Again since $\psi$ is a (c)-comparison function, it follows that

$$\sum_{i=0}^{\infty} \psi^i(\rho(\delta_0, \delta_1)) < \infty$$

This implies that $\{\delta_n\}$ is a Cauchy sequence in the metric space $(\mathcal{X}, \tilde{d}_\rho)$ which is complete. Therefore the sequence $\{\delta_n\}$ is convergent in the space $(\mathcal{X}, \tilde{d}_\rho)$. This implies that there exists $\nu \in \mathcal{X}$ such that $\tilde{d}_\rho(\delta_n, \nu) \to 0$ as $n \to +\infty$. Again from Lemma 1.6, we get

$$\rho(\nu, \nu) = \lim_{n\to\infty} \rho(\delta_n, \nu) = \lim_{m,n\to\infty} \rho(\delta_n, \delta_m)$$

Moreover, since $\{\delta_n\}$ is a Cauchy sequence in the metric space $(\mathcal{X}, \tilde{d}_\rho)$ we get

$$\lim_{m,n\to\infty} \tilde{d}_\rho(\delta_n, \delta_m) = 0$$

In view of (2.5) we get

$$\lim_{n\to\infty} \rho(\delta_n, \delta_n) = 0.$$

Using the definition of $\tilde{d}_\rho$ we obtain that $\lim_{m,n\to\infty} \rho(\delta_n, \delta_m) = 0$. Consequently, we follows that

$$\rho(\nu, \nu) = \lim_{n\to\infty} \rho(\delta_n, \nu) = \lim_{m,n\to\infty} \rho(\delta_n, \delta_m) = 0$$

As a result, $\{\delta_n\}$ is a Cauchy sequence in the complete partial metric space $(\mathcal{X}, \rho)$, and it is convergent to a point $\delta^*$ in $\mathcal{X}$.
As $\mathcal{T}$ is continuous, we have

$$\delta^* = \lim_{n\to\infty} \delta_{n+1} = \lim_{n\to\infty} \mathcal{T}\delta_n = \mathcal{T}\delta^*$$

□

In the following, we omit the continuity assumption of $\mathcal{T}$ in Theorem 2.1.

**Theorem 2.2.** *Let $(\mathcal{X}, \rho)$ be a complete partial metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be self mapping. Suppose $\alpha, \beta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be the mappings satisfying the conditions:*

*(i) $\mathcal{T}$ is $\alpha - \beta$ admissible;*

*(ii) $\mathcal{T}$ is generalized $\alpha - \beta - \psi$ contractive;*

*(iii) There exists $\delta_0 \in \mathcal{X}$ such that $\alpha(\delta_0, \mathcal{T}\delta_0) \geq 1$ and $\beta(\delta_0, \mathcal{T}\delta_0) \geq 1$;*

*(iv) $\mathcal{X}$ is $(\alpha - \beta)$ regular;*

*(v) There exists $L \geq 0$ such that*

$$\alpha(\delta, \mathcal{T}\delta)\beta(\eta, \mathcal{T}\eta)\rho(\mathcal{T}\delta, \mathcal{T}\eta) \leq \psi(M(\delta, \eta)) + L\vartheta(N(\delta, \eta))$$

*where*

$$N(\delta, \eta) = \min\{\tilde{d}_m(\delta, \mathcal{T}\delta), \tilde{d}_m(\delta, \mathcal{T}\eta), \tilde{d}_m(\eta, \mathcal{T}\delta)\}$$

*for all $\delta, \eta \in \mathcal{X}$ and $\vartheta \in \Theta$. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$. Further if $\mu$, $\mu_1$ are fixed points of $\mathcal{T}$ with $\alpha(\mu, \mathcal{T}\mu) \geq 1$, $\alpha(\mu_1, \mathcal{T}\mu_1) \geq 1$ and $\beta(\mu, \mathcal{T}\mu) \geq 1$, $\beta(\mu_1, \mathcal{T}\mu_1) \geq 1$, then $\mu = \mu_1$.*

*Proof.* From the proof of the Theorem 2.1, the sequence $\delta_n$ defined by $\delta_{n+1} = \mathcal{T}\delta_n$ is Cauchy in $\mathcal{X}$. Now, suppose that $\mathcal{X}$ is $(\alpha - \beta)$ regular. We have to show that $\mathcal{T}\delta^* = \delta^*$. Assume that $\rho(\mathcal{T}\delta^*, \delta^*) > 0$. Then as $\mathcal{X}$ is $(\alpha - \beta)$ regular there exists a subsequence $\{\delta_{n(k)}\}$ of $\{\delta_n\}$ such that $\alpha(\delta_{n(k)}, \delta_{n(k)+1}) \geq 1$, $\beta(\delta_{n(k)}, \delta_{n(k)+1}) \geq 1$ and $\alpha(\delta, \mathcal{T}\delta) \geq 1$, $\beta(\delta, \mathcal{T}\delta) \geq 1$ for all $k$.
Now in $(v)$ replacing $\delta$ by $\delta_{n(k)}$ and $\eta$ by $\delta^*$ we get

$$\begin{aligned}\rho(\delta_{n(k+1)}, \mathcal{T}\delta^*) = \rho(\mathcal{T}\delta_{n(k)}, \mathcal{T}\delta^*) &\leq \alpha(\delta_{n(k)}, \mathcal{T}\delta_{n(k)})\beta(\delta^*, \mathcal{T}\delta^*)\rho(\mathcal{T}\delta_{n(k)}, \mathcal{T}\delta^*) \\ &\leq \psi(M(\delta_{n(k)}, \delta^*)) + L\vartheta(N(\delta_{n(k)}, \delta^*))\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}M(\delta_{n(k)}, \delta^*) &= \max\left\{\rho(\delta_{n(k)}, \delta^*), \rho(\delta^*, \mathcal{T}\delta^*)\frac{1+\rho(\delta_{n(k)}, \mathcal{T}\delta_{n(k)})}{1+\rho(\delta_{n(k)}, \delta^*)}, \rho(\delta_{n(k)}, \mathcal{T}\delta_{n(k)})\frac{1+\rho(\delta_{n(k)}, \mathcal{T}\delta_{n(k)})}{1+\rho(\delta_{n(k)}, \delta^*)}\right\} \\ &= \max\left\{\rho(\delta_{n(k)}, \delta^*), \rho(\delta^*, \mathcal{T}\delta^*)\frac{1+\rho(\delta_{n(k)}, \delta_{n(k)+1})}{1+\rho(\delta_{n(k)}, \delta^*)}, \rho(\delta_{n(k)}, \delta_{n(k)+1})\frac{1+\rho(\delta_{n(k)}, \delta_{n(k)+1})}{1+\rho(\delta_{n(k)}, \delta^*)}\right\}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}N(\delta_{n(k)}, \delta^*) &= \min\{\tilde{d}_m(\delta_{n(k)}, \mathcal{T}\delta_{n(k)}), \tilde{d}_m(\delta_{n(k)}, \mathcal{T}\delta^*), \tilde{d}_m(\delta^*, \mathcal{T}\delta_{n(k)})\} \\ &= \min\{\tilde{d}_m(\delta_{n(k)}, \delta_{n(k)+1}), \tilde{d}_m(\delta_{n(k)}, \mathcal{T}\delta^*), \tilde{d}_m(\delta^*, \delta_{n(k)+1})\}\end{aligned} \tag{2.9}$$

Now, taking $n \to \infty$ in (2.8) and (2.9) we get

$$\lim_{n\to\infty} M(\delta_{n(k)}, \delta^*) = \rho(\delta^*, \mathcal{T}\delta^*) \tag{2.10}$$

and

$$\lim_{n\to\infty} N(\delta_{n(k)}, \delta^*) = 0 \tag{2.11}$$

Now, taking $n \to \infty$ in (2.7) and using (2.10), (2.11) and definitions $\psi, \vartheta$ we get

$$\rho(\delta^*, \mathcal{T}\delta^*) < \rho(\delta^*, \mathcal{T}\delta^*)$$

which is a contradiction. Therefore $\mathcal{T}\delta^* = \delta^*$ i.e. $\delta^*$ is a fixed point.
Further, suppose $\delta^*$ and $\eta^*$ be two fixed point of $\mathcal{T}$ such that $\alpha(\delta^*, \eta^*) \geq 1$, $\beta(\delta^*, \eta^*) \geq 1$ and $\rho(\delta^*, \eta^*) > 0$ then replacing $\delta$ by $\delta^*$ and $\eta$ by $\eta^*$ in $(v)$ we get

$$\begin{aligned}\rho(\delta^*, \eta^*) = \rho(\mathcal{T}\delta^*, \mathcal{T}\eta^*) &\leq \alpha(\delta^*, \mathcal{T}\delta^*)\beta(\eta^*, \mathcal{T}\eta^*)\rho(\mathcal{T}\delta^*, \mathcal{T}\eta^*) \\ &\leq \psi(M(\delta^*, \eta^*)) + L\vartheta(N(\delta^*, \eta^*))\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}M(\delta^*, \eta^*) &= \max\left\{\rho(\delta^*, \eta^*), \rho(\eta^*, \mathcal{T}\eta^*)\frac{1+\rho(\delta^*, \mathcal{T}\delta^*)}{1+\rho(\delta^*, \eta^*)}, \rho(\delta^*, \mathcal{T}\delta^*)\frac{1+\rho(\delta^*, \mathcal{T}\delta^*)}{1+\rho(\delta^*, \eta^*)}\right\} \\ &= \max\left\{\rho(\delta^*, \eta^*), \rho(\eta^*, \eta^*)\frac{1+\rho(\delta^*, \delta^*)}{1+\rho(\delta^*, \eta^*)}, \rho(\delta^*, \delta^*)\frac{1+\rho(\delta^*, \delta^*)}{1+\rho(\delta^*, \eta^*)}\right\} \\ &= \rho(\delta^*, \eta^*)\end{aligned} \tag{2.13}$$

and

$$N(\delta^*,\eta^*) = \min\{\tilde{d}_m(\delta^*,\mathcal{T}\delta^*), \tilde{d}_m(\delta^*,\mathcal{T}\eta^*), \tilde{d}_m(\eta^*,\mathcal{T}\delta^*)\}$$
$$= \min\{\tilde{d}_m(\delta^*,\delta^*), \tilde{d}_m(\delta^*,\eta^*), \tilde{d}_m(\eta^*,\delta^*)\}$$
$$= 0 \tag{2.14}$$

putting (2.13) and (2.14) in (2.12) and using the facts that $\psi(t) < t$ for $t > 0$ and $\vartheta(t) = 0$ if and only if $t = 0$ we get

$$\rho(\delta^*,\eta^*) \leq \psi(M(\delta^*,\eta^*)) + L\vartheta(N(\delta^*,\eta^*)) = \psi(\rho(\delta^*,\eta^*)) < \rho(\delta^*,\eta^*)$$

which is a contradiction. Hence $\mathcal{T}$ has a unique fixed point. This completes the proof. □

Following are consequences of the Theorem.

**Corollary 2.3.** *Let $(\mathcal{X},\rho)$ be a complete partial metric space. $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be self-mapping satisfying the condition*

$$\rho(\mathcal{T}\delta,\mathcal{T}\eta) \leq \psi(\rho(\delta,\eta)) \tag{2.15}$$

*For all $\delta,\eta\in\mathcal{X}$, $\psi\in\Psi$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.*

**Corollary 2.4.** *Let $(\mathcal{X},\rho)$ be a complete partial metric space. $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be self-mapping satisfying the condition*

$$\rho(\mathcal{T}\delta,\mathcal{T}\eta) \leq \psi(M(\delta,\eta)) \tag{2.16}$$

*For all $\delta,\eta\in\mathcal{X}$, $\psi\in\Psi$ and*
*Where*

$$M(\delta,\eta) = \max\left\{\rho(\delta,\eta), \rho(\eta,\mathcal{T}\eta)\frac{1+\rho(\delta,\mathcal{T}\delta)}{1+\rho(\delta,\eta)}, \rho(\delta,\mathcal{T}\delta)\frac{1+\rho(\delta,\mathcal{T}\delta)}{1+\rho(\delta,\eta)}\right\}$$

*Then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.*

**Example 2.5.** *Let $\mathcal{X} = [0,1]$ and $\rho(\delta,\eta) = \max\{\delta,\eta\}$. Then $(\mathcal{X},\rho)$ is a complete partial metric space. Consider the mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ defined by $\mathcal{T}(\delta) = \frac{\delta}{16}$ for all $\delta$ and $\psi,\vartheta:[0,\infty)\to[0,\infty)$ be such that $\psi(t) = \frac{t}{2}$ and $\vartheta(t) = t$. If we define the functions $\alpha,\beta:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ as*

$$\alpha(\delta,\eta) = \begin{cases} 2 & \delta > \eta \\ o & \textit{otherwise} \end{cases} \tag{2.17}$$

*and*

$$\beta(\delta,\eta) = \begin{cases} 4 & \delta > \eta \\ \frac{1}{2} & \textit{otherwise} \end{cases} \tag{2.18}$$

*We show that $\mathcal{T}$ is $\alpha-\beta$ admissible.*
*Let $\alpha(\delta,\eta)\geq 1$ and $\beta(\delta,\eta)\geq 1$. Then $\alpha(\mathcal{T}\delta,\mathcal{T}\eta) = \alpha(\frac{\delta}{16},\frac{\eta}{16})\geq 1$ and $\beta(\mathcal{T}\delta,\mathcal{T}\eta) = \beta(\frac{\delta}{16},\frac{\eta}{16})\geq 1$*

*Now,*

$$\alpha(\delta,\mathcal{T}\delta)\beta(\eta,\mathcal{T}\eta)\rho(\mathcal{T}\delta,\mathcal{T}\eta) = \alpha(\delta,\frac{\delta}{16})\beta(\eta,\frac{\eta}{16})\rho(\frac{\delta}{16},\frac{\eta}{16})$$
$$= \frac{\delta}{2} \tag{2.19}$$

*On the other side*

$$\begin{aligned}M(\delta,\eta) &= \max\left\{\rho(\delta,\eta),\rho(\eta,\mathcal{T}\eta)\frac{1+\rho(\delta,\mathcal{T}\delta)}{1+\rho(\delta,\eta)},\rho(\delta,\mathcal{T}\delta)\frac{1+\rho(\delta,\mathcal{T}\delta)}{1+\rho(\delta,\eta)}\right\}\\ &= \max\left\{\rho(\delta,\eta),\rho(\eta,\frac{\eta}{16})\frac{1+\rho(\delta,\frac{\delta}{16}))}{1+\rho(\delta,\eta)},\rho(\delta,\frac{\delta}{16})\frac{1+\rho(\delta,\frac{\delta}{16})}{1+\rho(\delta,\eta)}\right\}\\ &= \delta \end{aligned} \tag{2.20}$$

*and*

$$\begin{aligned}N(\delta,\delta) &= \min\{\tilde{d}_m(\delta,\mathcal{T}\delta),\tilde{d}_m(\delta,\mathcal{T}\eta),\tilde{d}_m(\eta,\mathcal{T}\delta)\}\\ &= \min\{\tilde{d}_m(\delta,\frac{\delta}{16}),\tilde{d}_m(\delta,\frac{\eta}{16}),\tilde{d}_m(\eta,\frac{\delta}{16})\}\end{aligned} \tag{2.21}$$

*Therefore*

$$\begin{aligned}\psi(M(\delta,\eta))+L\vartheta(N(\delta,\eta)) &= \psi(\delta)+L(\min\{\tilde{d}_m(\delta,\frac{\delta}{16}),\tilde{d}_m(\delta,\frac{\eta}{16}),\tilde{d}_m(\eta,\frac{\delta}{16})\})\\ &= \frac{\delta}{2}+L(\min\{\tilde{d}_m(\delta,\frac{\delta}{16}),\tilde{d}_m(\delta,\frac{\eta}{16}),\tilde{d}_m(\eta,\frac{\delta}{16})\})\end{aligned} \tag{2.22}$$

Now since $L(\min\{\tilde{d}_m(\delta,\frac{\delta}{16}),\tilde{d}_m(\delta,\frac{\eta}{16}),\tilde{d}_m(\eta,\frac{\delta}{16})\}) \geq 0$ for all $\delta,\eta \in \mathcal{X}$, and From (2.19) and (2.22) it is clear that it satisfies all the conditions of Theorem 2.1. Hence $\mathcal{T}$ has a fixed point, which in this case is $0$.

# 3 CONCLUSIONS

In this study, we introduced generalized $\alpha-\beta-\psi$ contrative mappings and presented some fixed point results of Berinde-type generalized $\alpha-\beta-\psi$ contrative mappings in the setting of partial metric spaces .We have provided an example to validate the findings.

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