UNDER PEER REVIEW

Original Research Article

The Beta Transmuted Standardized Half Logistic Distribution: Properties and Application

Abstract

In this paper, a new distribution called the Beta Transmuted Half Logistic Distribution is derived and studied using the Beta Transmuted-G distribution. The distribution generalizes the half logistic distribution for more flexibility. Then expressions for the moments, moment generating function, order statistic, survival function, and the hazard function were studied. Estimation of parameters of the model was done using the maximum likelihood estimation approach. Simulation studies were conducted to assess the performance of the estimates of the parameter. The distribution is applied to real-life dataset in which the results show that the distribution performed better than its submodels.

Key Words: Beta Transmuted-G Distribution; Half Logistic distribution; statistical properties; maximum likelihood estimation

1 Introduction

The half logistic, as proposed by Balakrishnan [5] is a truncated form of the logistic distribution. It is gradually gaining attention in terms of generalization. One of the attributes that makes it attractive is its non-decreasing monotone hazard rate for all values of its parameters. The distribution has been generalized such as [9],[6],[1],[8],[4],[2] e.t.c. These generalized distributions have been applied to real lifetime- related data.

The cumulative distribution function and the probability distribution function of the standardized half logistic distribution is given as in 1 and 2 respectively

$$q(x) = \frac{2e^x}{(1+e^x)^2} \quad x > 0 \tag{1}$$

$$Q(x) = 1 - \frac{2}{1+e^x} \tag{2}$$

The Beta Transmuted-H family of distribution as derived by [2] by extending the transmuted family of distribution of [10] with in introduction of two shape parameters(a and b). The beta transmuted family of distribution has the p.d.f and c.d.f as in 3 and 4 respectively.

$$f(x) = \frac{q(x)}{B(a,b)}(1+m-2mQ(x))((1+m)Q(x)-mQ^2(x))^{a-1}(1-[((1+m)Q(x)-mQ^2(x))])^{b-1} \tag{3}$$

$$F(x) = \frac{1}{B(a,b)}\int_0^{Q(x)(1+m-mQ(x))} j^{a-1}(1-j)^{b-1}dj \tag{4}$$

$$F(x) = I_j(a,b) = \frac{B(j;a,b)}{B(a,b)} \tag{5}$$

where $j = Q(x)(1+m-mQ(x))$ In application to distributions, some works have been done. Examples of such are Beta transmuted gumbel [7], Beta transmuted power distribution [3] e.t.c.

In this work, the beta transmuted family of distribution is applied to the half logistic distribution to obtain a new generalized form of the half logistic distribution. We are motivated to introduce the beta transmuted half logistic distribution which is is from the and it can be viewed as a suitable model for a real data applications compared with other competing lifetime distributions.

The remainder of this paper is organized as follows. In Section 2, the beta transmuted half logistic distribution is defined with further discussion on some of its sub-models. In Section 3, the mixture representation of the beta transmuted half logistic distribution is presented. Section 4 discusses the mathematical properties of the proposed family, including the survival function, hazard function, moments, moment generating function, quantiles, and order statistics. Estimation of parameters by the maximum likelihood method is performed in Section 5, while the performance of the estimators is assessed by simulation in Section 6. In Section 7, the distribution is used to analyze the real data. Finally, in Section 8, concluding remarks are made in our study.

2 Beta transmuted half logistic distribution

To obtain the beta transmuted half logistic distribution, insert 1 and 2 in 3 and 4 to have the p.d.f in equation(6) and the c.d.f in equation(7) respectively.

$$f(x) = \frac{2e^x}{B(a,b)(1+e^x)^2}\left[1+m-2m\left(\frac{e^x-1}{1+e^x}\right)\right]\left[(1+m)\left(\frac{e^x-1}{1+e^x}\right)-m\left(\frac{e^x-1}{1+e^x}\right)^2\right]^{a-1}$$

$$\left[1-\{(1+m)\left(\frac{e^x-1}{1+e^x}\right)-m\left(\frac{e^x-1}{1+e^x}\right)^2\}\right]^{b-1} \tag{6}$$

Figure 1 shows the plots of p.d.f of the beta transmuted standardized half logistic distribution with different parameter values. It tells that the distribution can fit data with different forms

$$F(x) = \frac{1}{B(a,b)}\int_0^{(\frac{e^x-1}{1+e^x})(1+m-m(\frac{e^x-1}{1+e^x}))} j^{a-1}(1-j)^{b-1}dj \tag{7}$$

The c.d.f can also be

$$F(x) = I_j(a,b) = \frac{B(j;a,b)}{B(a,b)} \tag{8}$$

where B(j;a,b) is an incomplete beta function and $j = (\frac{e^x-1}{1+e^x})(1+m-m(\frac{e^x-1}{1+e^x}))$

Sub-models of the beta transmuted half logistic distribution as dervied in 6 are

1. When a=1, b=1, m=0, then standardized half logistic distribution as in equation 2

2. When m=0, the Beta Half Logistic of Jose and Manoharan[3]

$$f(x) = \frac{2e^x}{B(a,b)(1+e^x)^2}\left(\frac{e^x-1}{1+e^x}\right)^{a-1}\left[1-\left(\frac{e^x-1}{1+e^x}\right)\right]^{b-1} \tag{9}$$

3. When a=1, b=1, the Transmuted Half logistic distribution is obtained

$$f(x) = \frac{2e^x}{(1+e^x)^2}\left[1+m-2m\left(\frac{e^x-1}{1+e^x}\right)\right] \tag{10}$$

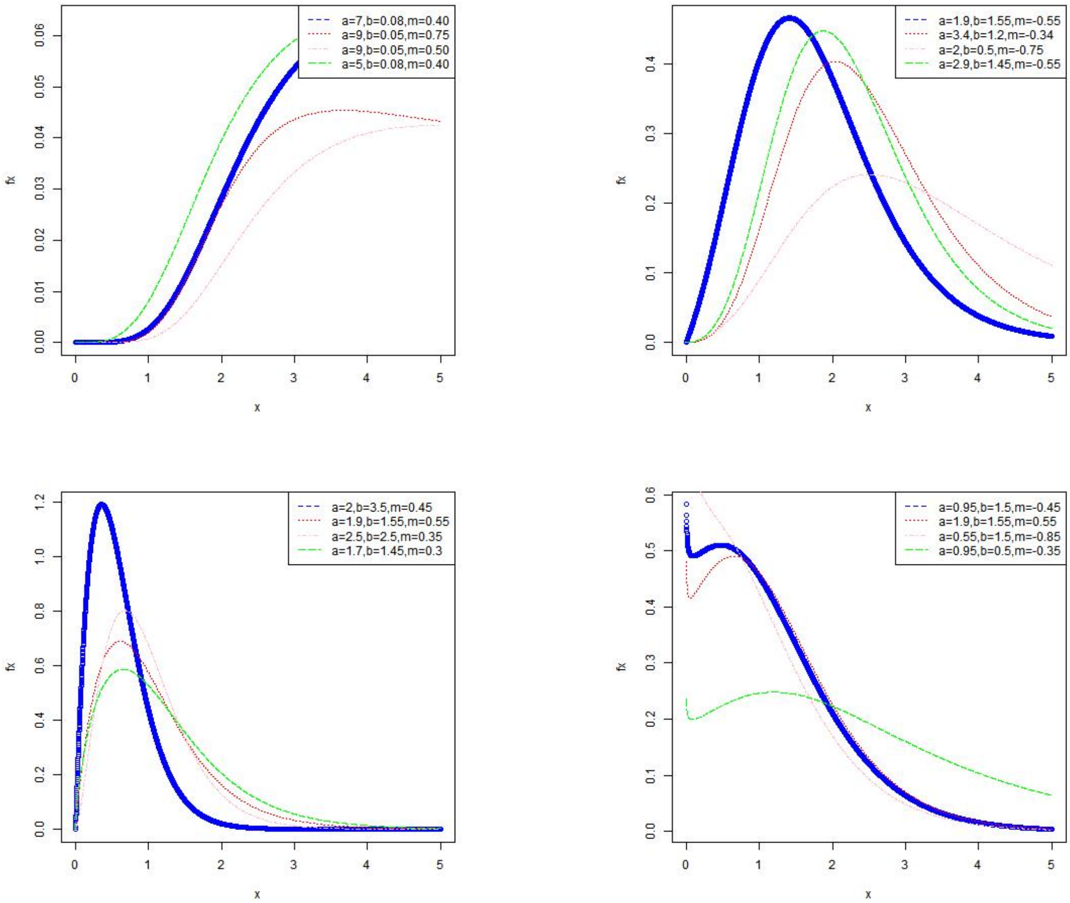

Figure 1: p.d.f curves of Beta Transmuted Standardized Half Logistic Distribution

3 Mixture representation

In this section, the series representations of the cdf and pdf of the beta transmitted half logistic distribution, which will be useful for studying its mathematical characteristics, is studied.

for $|z| < 1$

$$(1-z)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k!\Gamma(b-k)} z^k \tag{11}$$

Therefore, equation 7 becomes

$$F(x) = \frac{1}{B(a,b)} \int_0^{(\frac{e^x-1}{1+e^x})(1+m-m(\frac{e^x-1}{1+e^x}))} j^{a-1} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k!\Gamma(b-k)} j^k dj \tag{12}$$

which gives

$$F(x)=\frac{1}{B(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(b)}{k!\Gamma(b-k)(a+k)}\left[\left(\frac{e^x-1}{1+e^x}\right)\left(1+m-m\left(\frac{e^x-1}{1+e^x}\right)\right)\right]^{a+k} \tag{13}$$

Equivalently, equation 13 can be written as

$$F(x)=\frac{1}{B(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(b)}{k!\Gamma(b-k)(a+k)}\left[1-\left(\frac{2}{1+e^x}\right)\left(1-m+\frac{2m}{1+e^x}\right)\right]^{a+k} \tag{14}$$

Let

$$R(x)=\left[1-\left(\frac{2}{1+e^x}\right)\left(1-m+\frac{2m}{1+e^x}\right)\right]^{a+k} \tag{15}$$

$$R(x)=\sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(a+k+1)}{l!\Gamma(a+k-l+1)}\left(\frac{2}{1+e^x}\right)^l\left(1-m+\frac{2m}{1+e^x}\right)^l \tag{16}$$

then as

$$F(x)=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+l}\Gamma(a+k+1)\Gamma(b)}{k!l!\Gamma(a+k-l+1)\Gamma(b-k)(a-k)}\left(\frac{2}{1+e^x}\right)^l\left(1-m+\frac{2m}{1+e^x}\right)^l \tag{17}$$

Further Simplification gives

$$F(x)=\sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\frac{(-1)^{k+l+p}\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!l!p!\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)(a-k)}\left(\frac{2}{1+e^x}\right)^l$$

$$m^p\left(1-\frac{2}{1+e^x}\right)^p \tag{18}$$

and finally as

$$F(x)=\sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty}\frac{(-1)^{k+l+p+s}\Gamma(p+1)\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!p!s!\Gamma(p-s+1)\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)\Gamma(l-s+1)(a-k)}m^p$$

$$\left(1-\frac{2}{1+e^x}\right)^{p+s} \tag{19}$$

Therefore, differentiating 19 with respect to x,

$$f(x)=\sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty}\frac{(-1)^{k+l+p+s}\Gamma(p+1)\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!p!s!\Gamma(p-s+1)\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)(a-k)}m^p$$

$$(p+s)\left(\frac{2e^x}{(1+e^x)^2}\right)\left(1-\frac{2}{1+e^x}\right)^{p+s-1} \tag{20}$$

$$f(x) = \sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty} \frac{(-1)^{k+l+p+s}\Gamma(p+1)\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!p!s!\Gamma(p-s+1)\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)(a-k)}$$

$$m^p(p+s)g(x)G^{l+s-1}(x) \tag{21}$$

Expressing 21 as a mixture of Exp-G, it gives

$$f(x) = \sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty} d_{k,l,p,s}\delta_{p+s}(x) \tag{22}$$

where

$$\delta_\pi(x) = \pi g(x)G^{\pi-1}(x) \ is \ the \ Exp-G \ p.d.f \ , \ having \ the \ index \ power \ \pi > 0$$

and

$$d_{k,l,p,s} = \frac{(-1)^{k+l+p+s}\Gamma(p+1)\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!l!p!s!\Gamma(p-s+1)\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)(a-k)} m^p$$

Therefore, equation 19 can be expressed as a mixture of Exp-G as

$$F(x) = f(x) = \sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty} d_{k,l,p,s}\chi_{p+s}(x) \tag{23}$$

where

$$\chi_{l+s}(x) = G^{p+s}(x) \ is \ the \ Exp-G \ c.d.f \ , \ having \ the \ index \ power \ p+s > 0$$

4 Statistical properties

4.1 Survival function

The survival function is obtained from the c.d.f as

$$s(x) = 1 - F(x) \tag{24}$$

Inserting equation 7 in equation 24 it is

$$s(x) = 1 - I_j(a,b) = 1 - \frac{B(j;a,b)}{B(a,b)} \tag{25}$$

which gives

$$s(x) = \frac{B(a,b) - B(j;a,b)}{B(a,b)} \tag{26}$$

4.2 Hazard function

The hazard function is given as

$$h(x) = \frac{f(x)}{s(x)} \tag{27}$$

Inserting 6 and 24 in 27,

$$h(x) = \frac{1}{B(a,b) - B(j;a,b)} \left(\frac{2e^x}{(1+e^x)^2} [1 + m - 2m(\frac{e^x - 1}{1+e^x})][(1+m)(\frac{e^x - 1}{1+e^x}) - m(\frac{e^x - 1}{1+e^x})^2]^{a-1} \right)$$

$$*[1 - \{(1+m)(\frac{e^x - 1}{1+e^x}) - m(\frac{e^x - 1}{1+e^x})^2\}]^{b-1} \tag{28}$$

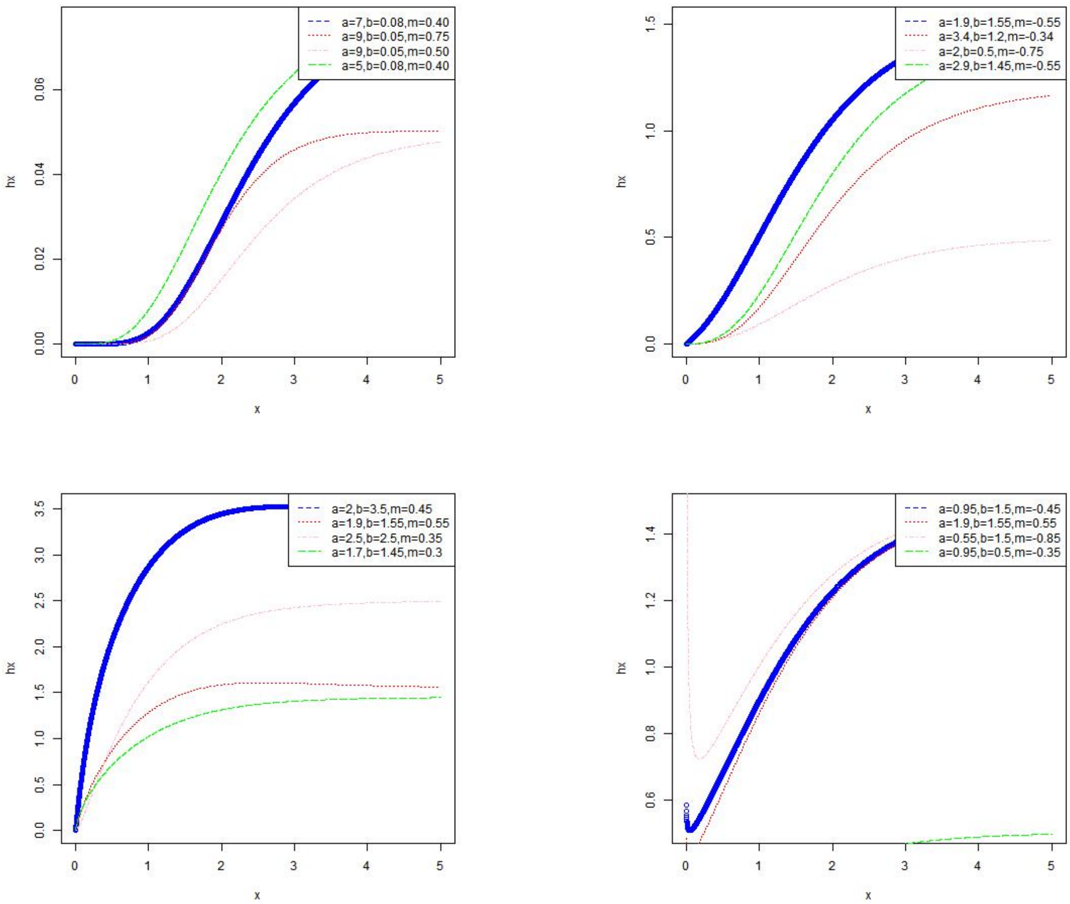

Figure 2: Hazard curves of Beta Transmuted Standardized Half Logistic Distribution

bution with different parameter values.

Table 1: Table displaying results of Moments when m= -1

a	b	E[X]	Variance	Skewness	Kurtosis
0.5	0.5	2.665	8.576	8.983	2.285
	1.0	1.282	1.192	8.41	1.968
	2	0.833	0.404	3.786	1.037
1.0	0.5	3.246	5.059	6.960	1.809
	1.0	2.341	2.642	5.476	1.389
	2	1.217	0.540	3.994	0.964
1.0	0.5	4.028	5.236	6.262	1.632
	1.0	2.910	2.204	4.925	1.354
	2	1.253	0.782	5.255	1.148

4.3 Quantile function

Quantiles are the points in a distribution that relates to the rank order of values. The quantile function of a distribution is the real solution of F(x_q)=q for 0≤ q≤ 1. Therefore, the quantile function of the Beta transmuted Half Logistic distribution is

$$X = H^{-1}\left[\frac{1+m-\sqrt{((m+1)^2+4mW)}}{2m}\right] \tag{29}$$

where W ∈ B(a, b), and

$$H^{-1}(u) = \ln\left(\frac{2}{1-u}-1\right)$$

4.4 Moments

The r-th moment of the distribution is

$$E[X^r] = \int_0^\infty x^r f(x)dx \tag{30}$$

Using equation 20, equation 30 becomes

$$f(x) = \sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\sum_{p=o}^{\infty}\sum_{s=o}^{\infty}\frac{(-1)^{k+l+p+s}\Gamma(p+1)\Gamma(a+k+1)\Gamma(b)\Gamma(l+1)}{k!p!s!\Gamma(p-s+1)\Gamma(a+k-l+1)\Gamma(b-k)\Gamma(l-p+1)(a-k)}m^p$$

$$\int_0^\infty x^r(p+s)\left(\frac{2e^x}{(1+e^x)^2}\right)\left(1-\frac{2}{1+e^x}\right)^{p+s-1}dx \tag{31}$$

Equation 31 is intractable. Therefore, the equation cant be solved explicitly. By the aid of R, moments of the the distribution are obtained, assigning values to the distribution parameters. The results are presented in table 1-4

Table 2: Table displaying results of Moments when m= -0.5

a	b	E[X]	Variance	Skewness	Kurtosis
0.5	0.5	2.374	8.545	9.052	2.310
	1.0	0.999	1.131	9.162	2.182
	2	0.567	0.339	4.547	1.389
1.0	0.5	2.945	5.086	6.931	1.809
	1.0	2.041	2.633	5.516	1.432
	2	0.917	0.506	4.308	1.192
2.0	0.5	3.729	5.276	6.216	1.617
	1.0	2.605	2.235	4.879	1.338
	2	1.439	0.791	5.273	1.181

Table 3: Table displaying results of Moments when m= 0.5

a	b	E[X]	Variance	Skewness	Kurtosis
0.5	0.5	1.767	7.104	10.812	2.631
	1.0	0.588	6.654	13.772	2.961
	2	0.289	0.127	6.510	1.890
1.0	0.5	2.148	4.270	8.175	2.136
	1.0	1.378	1.882	7.332	1.891
	2	0.498	0.233	5.737	1.707
2.0	0.5	2.819	4.686	6.973	1.835
	1.0	1.802	1.752	5.717	1.646
	2	0.856	0.456	7.411	1.766

Table 4: Table displaying results of Moments when m= 1

a	b	E[X]	Variance	Skewness	Kurtosis
0.5	0.5	1.140	2.346	8.608	2.262
	1.0	0.425	0.296	10.882	2.543
	2	0.216	0.070	6.232	1.836
1.0	0.5	1.438	1.436	6.522	1.748
	1.0	0.960	0.751	5.457	1.493
	2	0.370	0.123	5.361	1.613
2.0	0.5	1.860	1.494	5.708	1.495
	1.0	1.254	0.661	4.698	1.327
	2	0.628	0.218	6.016	1.461

4.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, ..., X_n$ be a random sample from the Beta Transmuted Half Logistic distributions. The pdf of ith order statistic, say $X_{i:n}$, can be written as

$$f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[1-F(x)]^{i-1} F(x)^{n-i} \tag{32}$$

which gives

$$f_{(i:n)}(x) = n\binom{n-1}{i-1} \frac{2e^x}{B(a,b)(1+e^x)^2} \left[1+m-2m\left(\frac{e^x-1}{1+e^x}\right)\right]$$

$$\left[(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\right]^{a-1} \left[1-\{(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\}\right]^{b-1}$$

$$\left[\frac{B(j;a,b)}{B(a,b)}\right]^{i-1} \left[1-\frac{B(j;a,b)}{B(a,b)}\right]^{n-i} \tag{33}$$

in which the first Order Statistics $X_{(1)}$ has the marginal p.d.f as

$$f_{(i:n)}(x) = n\frac{2e^x}{B(a,b)(1+e^x)^2} \left[1+m-2m\left(\frac{e^x-1}{1+e^x}\right)\right] \left[(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\right]^{a-1}$$

$$\left[1-\{(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\}\right]^{b-1} \left[1-\left(\frac{e^x-1}{1+e^x}\right)\right]^{b-1} \left[1-\frac{B(j;a,b)}{a,b}\right]^{n-i} \tag{34}$$

and the last order statistics having the marginal pdf as

$$f_{(i:n)}(x) = n\frac{2e^x}{B(a,b)(1+e^x)^2} \left[1+m-2m\left(\frac{e^x-1}{1+e^x}\right)\right] \left[(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\right]^{a-1}$$

$$\left[1-\{(1+m)\left(\frac{e^x-1}{1+e^x}\right) - m\left(\frac{e^x-1}{1+e^x}\right)^2\}\right]^{b-1} \left[\frac{B(j;a,b)}{a,b}\right]^{n-1} \tag{35}$$

5 Parameter estimation

In this section, estimation by the method of maximum likelihood estimation is discussed. Let $X_1, ..., X_n$ be a random sample from the beta transmuted Half Logistic distribution with observed values $x_1, ..., x_n$ and $\theta = (m, a, b)$ be parameter vector. Then sample likelihood and Log-Likelihood functions of Beta

Transmuted Half Logistic Distribution is obtained as

$$L(x/m,a,b)=\prod_{i=1}^{n}\frac{2e^{x}}{B(a,b)(1+e^{x})^{2}}[1+m-2m(\frac{e^{x}-1}{1+e^{x}})][(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}]^{a-1}$$

$$[1-\{(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}\}]^{b-1} \tag{36}$$

$$L(x/m,a,b)=\prod_{i=1}^{n}\left[\frac{2e^{x}}{B(a,b)(1+e^{x})^{2}}\right]\prod_{i=1}^{n}\left[1+m-2m\left(\frac{e^{x}-1}{1+e^{x}}\right)\right]$$

$$\prod_{i=1}^{n}\left[(1+m)\left(\frac{e^{x}-1}{1+e^{x}}\right)-m\left(\frac{e^{x}-1}{1+e^{x}}\right)^{2}\right]^{a-1}\prod_{i=1}^{n}[1-\{(1+m)\left(\frac{e^{x}-1}{1+e^{x}}\right)-m\left(\frac{e^{x}-1}{1+e^{x}}\right)^{2}\}]^{b-1} \tag{37}$$

Taking the log of L(x/m,a,b), and log(x/m,a,b)=l, then

$$l=\sum_{i=1}^{n}\ln\left(1+m-2m\left(\frac{e^{x}-1}{1+e^{x}}\right)\right)+(a-1)\sum_{i=1}^{n}\ln\left((1+m)\left(\frac{e^{x}-1}{1+e^{x}}\right)-m\left(\frac{e^{x}-1}{1+e^{x}}\right)^{2}\right)-$$

$$n\ln\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)+n\ln 2+\sum_{i=1}^{n}x_{i}-2\sum_{i=1}^{n}(1+e^{x})$$

$$+(b-1)\sum_{i=1}^{n}\ln\left(1-(1+m)\left(\frac{e^{x}-1}{1+e^{x}}\right)-m\left(\frac{e^{x}-1}{1+e^{x}}\right)^{2}\right) \tag{38}$$

Differentiating 38 with respect toθ to obtain the elements of score vector as below

$$\frac{\partial l}{\partial m}=\sum_{i=1}^{n}\frac{\frac{3e^{x}-1}{1+e^{x}}}{1+m-2m(\frac{e^{x}-1}{1+e^{x}})}+(a-1)\sum_{i=1}^{n}\frac{\frac{2e^{x}(e^{x}-1)}{1+e^{x}}}{(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}}$$

$$-(b-1)\sum_{i=1}^{n}\frac{\frac{2e^{x}(e^{x}-1)}{1+e^{x}}}{1-[(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}]} \tag{39}$$

$$\frac{\partial l}{\partial a}=\sum_{i=1}^{n}(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}-n(\frac{\Gamma'(a)}{\Gamma(a)}-\frac{\Gamma'(a+b)}{\Gamma(a+b)}) \tag{40}$$

$$\frac{\partial l}{\partial b}=\sum_{i=1}^{n}1-[(1+m)(\frac{e^{x}-1}{1+e^{x}})-m(\frac{e^{x}-1}{1+e^{x}})^{2}]-n(\frac{\Gamma'(b)}{\Gamma(b)}-\frac{\Gamma'(a+b)}{\Gamma(a+b)}) \tag{41}$$

Setting $\frac{\partial l}{d\theta}$ =0 , θ=(m,a,b) and solving the resulting equation to obtain $\hat{\theta}$ =($\hat{m}$,$\hat{a}$,$\hat{b}$) The solution of the system of equations 39, 40 are 41 are not in closed forms. The maximum likelihood estimates of each parameter is obtained numerically from the non-linear equations using computer programs.

For interval estimation and test of hypothesis on the parameters $(m,a,b,)$, we obtain a 3x3 unit information matrix

$$M = \begin{bmatrix} M_{m,m} & M_{m,a} & M_{m,b} \\ M_{m,a} & M_{a,a} & M_{m,b} \\ M_{m,b} & M_{a,b} & M_{b,b} \end{bmatrix}$$

The corresponding elements are calculated as minus the second derivates of l with respect to the parameters

Under conditions that are fulfilled for parameters, the asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ is $N_3(0, M(\hat{\theta})^{-1})$ distribution of θ can be used to construct approximate confidence intervals and confidence regions for the parameters and for the hazard and survival functions. The asymptotic normality is also useful for testing goodness of fit of the beta type I generalized half logistic distribution and for comparing this distribution with some of its special sub-models using one of these two well-known asymptotically equivalent test statistics-namely, the likelihood ratio statistic and Wald statistic. An asymptotic confidence interval with significance level ζ for each parameter θ_i is given by

$$ACI(\theta_i, 100(1-\zeta)) = \hat{\theta} - z_{\frac{\zeta}{2}}\sqrt{M^{\hat{\theta},\hat{\theta}}}, \theta + z_{\frac{\zeta}{2}}\sqrt{M^{\hat{\theta},\hat{\theta}}} \tag{42}$$

where $M^{\hat{\theta},\hat{\theta}}$ is the $i^t h$ diagonal element of $K_n(\hat{\theta})^{-1}$ for $i = 1, 2, 3$ and $z_{\zeta/2}$ is the quantile of the standard normal distribution.

6 Simulation studies

Here, the finite sample behaviors of the MLEs for the Beta Transmuted Half Logistic distribution is assessed by conducting simulation studies. Assessing the estimates, Average Estimates and Mean Square Error(MSE) were used. The simulation procedure is as follows;

For, values of θ=(a,b,m)

1. We generate a random sample of size n= 50,100,200,300 from the Beta Transmuted Half Logistic Distribution W

2. From (1), the observations $x_1, x_2, ...x_n$ following the Beta transmuted standardized half logistic distribution as

$$X = H^{-1}\left[\frac{1+m-\sqrt{((m+1)^2+4mW)}}{2m}\right] \tag{43}$$

$$H^{-1}(u) = \ln\left(\frac{2}{1-u} - 1\right)$$

Table 5: Table displaying simulation results for BTHL(0.5,0.8,0.2)

Sample size	Parameter	AE	Bias	MSE
50	m	0.2499	0.2501	0.3393
	a	0.8200	0.3197	0.2704
	b	0.2326	0.2610	0.1474
100	m	0.0925	0.4075	0.2997
	a	0.3251	0.1751	0.2596
	b	0.1016	0.3981	0.2442
200	m	0.1515	0.3485	0.2945
	a	0.7784	0.2781	0.1560
	b	0.2976	0.2021	0.1587
300	m	0.1403	0.3550	0.2567
	a	0.7850	0.2847	0.1536
	b	0.3280	0.1717	0.1567

where W $\in$ B(a, b)

3.The values from (2) is therefore replicated 10,000 times for samples sizes of n.

4.Using the values generated from (3),the maximum likelihood estimates of the parameters can be calculated.

5: The likelihood function of the model is maximized with respect to parameters a, b, m to obtain $\hat{a}, \hat{b}, \hat{m}$

6: If $\hat{\theta_{rl}}$ is a MLE of θ_l, such that l = 1, 2, 3(i.e. $\theta_1 = a$, $\theta_2 = b$, $\theta_3 = m$), based on sample k, $r = 1, ..., 10000$ then the average estimate (AE), MSE of $\hat{\theta_l}$ over the 10000 samples are given, respectively, by

$$\bar{\hat{\theta_l}} = \frac{1}{10000} \sum_{i=1}^{10000} \hat{\theta_{rl}}$$

$$MSE(\bar{\hat{\theta_l}}) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\theta_{rl}} - \theta_l)^2$$

The results of the simulation are displayed in table 5 and 6

From tables 5 and 6, it is revealed that the Average estimate is close to the assigned values of the parameters, therefore giving us small values for the bias. Furthermore, the Mean square errors of the parameters reduce as the sample size increases. This infers that the estimates of the parameters are efficient and consistent. Also, the maximum likelihood method of estimation is efficient for this model.

7 Application to real data

In this section, a real data set is applied using the beta transmuted standardized half logistic to illustrate its usefulness. The results obtained are compared to results from its submodels; beta half logistic(BHL)

Table 6: Table displaying simulation results for BTHL(0.1,3,2)

Sample size	Parameter	AE	Bias	MSE
50	m	0.0625	1.6359	4.3725
	a	3.0236	1.3222	4.1066
	b	2.0140	0.3138	1.8471
100	m	0.4075	1.6264	4.3247
	a	0.1757	1.2590	3.6222
	b	0.3981	0.2714	1.6912
200	m	0.0645	1.6339	4.3205
	a	2.9382	1.2369	3.3699
	b	1.9685	0.2683	1.6657
300	m	0.0725	1.6259	4.2390
	a	2.9320	1.2307	3.2994
	b	1.9625	0.2622	1.6388

and transmuted half logistic(THL) distributions. The dataset is on the survival times of Guinea Pigs

7.1 Survival Times of Guinea Pigs

This data shows survival time of 72 Guinea Pigs infected with virulent tubercle bacilli.The data set is culled from the work of Adeyinka and Olapade(2019) who had applied the transmuted Half Logistic to the data. The records of the survival times of the guinea pigs are as follows . 0.93, 1.24, 0.56, 0.44, 1.3, 0.72, 0.74, 0.77, 0.92, 0.1, 0.96, 1.13, 1.00, 2.16, 1.09, 1.07, 0.7, 0.08, 1.08, 1.63, 1.36, 2.51, 1.00, 1.15, 1.16, 1.20, 1.21, 1.22, 1.68, 4.02, 0.59, 1.34, 0.33, 1.39, 1.44, 4.32, 1.53, 1.22, 3.42, 1.08, 1.63, 1.59, 1.71, 1.72, 1.76, 1.83, 2.31, 1.96, 1.97, 2.02, 2.45, 2.15, 1.02, 2.22, 2.30, 1.95, 1.60, 2.13, 3.27, 2.53, 2.54, 2.54, 5.55, 2.93, 1.12, 2.40, 3.47, 3.61, 1.05, 1.46, 4.58, 2.78.

For model comparison, the log-likelihood(L), Akaike Information Criterion (AIC) and the Bayesian Information Criterion(BIC) statistics are used. The formula are as follows

7.2 Akaike Information Criterion (AIC)

The AIC is stated mathematically as

$$AIC = 2k - 2\ln(L) \tag{44}$$

where k is the number of parameters in the model and L is the maximized value of the likelihood function of the model.

The lower the AIC of the model indicates that the model performs better.

Table 7: Table displaying results of analysis of Survival Times of Guinea Pigs

Model	-L	AIC	BIC
BTHL	95.9	197.8	197.372
BHL	97.03	198.062	197.774
THL	97.97	199.94	197.77

Table 8: Table displaying results of analysis of Survival Times

Model	Parameter	Estimate	S.E
BTHL	a	-0.802	0.354
	b	1.260	0.434
	m	1.345	0.219
BHL	m	0.562	0.079
		2.279	0.420
THL	m	0.528	0.1365

7.3 Bayesian Information Criterion (BIC)

The Bayesian Information criterion also known as the Deviance Information Criterion is stated mathematically as

$$BIC = k\ln(n) - 2\ln(L) \tag{45}$$

where n is the number of observations, k is the number of parameters in the model and L is the maximized value of the likelihood function of the model.The lower the BIC of the model indicates that the model performs better. Table 7 reveals the comparative fit of the standardized BTHL the fits of the standardized distributions of BTHL distribution with the BHL, TrHL distributions. The figures in these tables show that the BTHL distribution has the lowest values for the -L, AIC, BIC statistics among the fitted distributions. We, therefore, conclude that the BTHL distribution is the best model for this data when compared to some other distributions. Table 8 gives the values of the estimates of the parameters of each model under consideration with their standard errors

8 Conclusion

In this work, a new probability distribution called the Beta transmuted standardized half logistic distribution was defined and studied. The new distribution generalizes the half logistic distribution by having additional three parameters, two shape parameters, and a transmuted parameter. The new distribution reduces to some existing distributions that have been studied in the past. Statistical properties of the new distribution were extensively studied and application to real data set revealed that the distribution performs better(produce better fit) than its submodels.

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