UNITARY QUASI-EQUIVALENCE AND PARTIAL ISOMETRY OPERATORS IN HILBERT SPACES

Abstract

Properties of almost similarity and unitary equivalence operators on different classes of operators have been established by various researchers in the recent past. On the other hand, unitary quasi- equivalence has been shown to preserve unitary, normality, hyponormality and binormality of operators. However, properties of unitary quasi-equivalence on partial isometric operators have not been fully established. This study therefore, determines properties of unitary quasi-equivalence on partial isometric operators.

Key words: Unitary quasi-equivalence, isometry, co-isometry, partial isometry, operator and Hilbert space.

1 Introduction

Let H be a complex Hilbert space and $B(H)$ be a set of bounded linear operators on Hilbert space H. $A \overset{u.q.e}{\sim} B$ and $A \overset{a.s}{\sim} B$ denotes unitary quasi-equivalent and almost similar operators respectively. An operator $V \in B(H)$ is defined as a structure preserving map. For $V \in B(H)$, $V^*$is called the adjoint of an operator V. Two operators $A, B \in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that the following conditions are satisfied.

$$A^*A = UB^*BU^*$$
$$AA^* = UBB^*U^*$$

[11] investigated unitary quasi-equivalence under the concept of near equivalence. This class of equivalence relation operators was later studied further by [5]. [7] established unitary quasi-equivalence to be an equivalence relation operator. [7], also established that unitary equivalence implies unitary quasi-equivalence. However, the converse is not necessary true unless the operators are similar normal. Thereafter Muteti (2014) determined that unitary quasi-equivalence preserve normality of operators. [9] further established that projection unitary quasi-equivalent operators imply that the operators are unitary equivalent. [9] also established that unitary quasi-equivalence preserves binormality and hyponormality of operators. [9], also established that unitary quasi-equivalence preserves unitary operator properties. That is, if two operators are unitarily quasi-equivalent and one is a unitary operator so is the other unitary. However, the same results have not been established for isometry, co-isometry and partial isometry operators. This study therefore determines properties of unitary quasi-equivalence on isometry, co-isometry and partial isometry operators.

2 Definitions and Terminologies

Definition 2.1: A Hilbert space H [1]. A Hilbert space H is a complete inner product space.
Definition 2.2: [3]. An operator $T \in B(H)$ is said to be:

i unitary if $TT^* = T^*T = I$

ii Isometry if $T^*T = I$.

iii Co-isometry if $TT^* = I$.

iv Partial isometry if T^*T is a projection or $T = TT^*T$.

From the above definitions, we have the following class inclusions:

i Unitary operators $\subseteq$ isometry operators $\subseteq$ partial isometry operators.

ii Unitary operators $\subseteq$ co-isometry operators $\subseteq$ partial isometry operators.

Definition 2.3: Unitary quasi-equivalence operators [5]. Two operators $A, B \in B(H)$ are said to be unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that:

$$A^*A = UB^*BU^*$$
$$AA^* = UBB^*U^*$$

3 Methodology

In achieving this objective, properties of partial isometry operators are useful. In addition, properties of unitary equivalence on partial isometry operators are also used in comparison. This study aimed to extend the following Lemmas to unitary quasi-equivalence.

Lemma 3.1: [6]**:** Let $A \in B(H)$ be a partial isometry and $B \in B(H)$ be any other operator such that, either $A = UBU^*$or $A = U^*BU$ where U is unitary, then $B \in B(H)$ is also a partial isometry

Lemma 3.2: [4]**.** Let $P, Q \in B(H)$ such that $P \overset{a.s}{\sim} Q$. If P^2 is a partial isometry and Q is self-adjoint, then Q^2 is also partially isometric.

4 Main Results

The following are results that were established.

Theorem 4.1: If $A, B \in B(H)$ are unitarily quasi-equivalent then A is isometry if and only if B is isometry.

Proof. Suppose that an operator A is isometry Since $A \overset{u.q.e}{\sim} B$, by (Kutkut, 1998), these operators satisfy the following conditions.

$$A^*A = UB^*BU^* \tag{1}$$

$$AA^* = UBB^*U^* \tag{2}$$

But A is isometry thus by definition, Halmos, (2017).

$$A^*A = I$$

Substituting equation (1) in $A^*A = I$

$$I = A^*A$$

$$= UB^*BU^*$$

thus,

$$UB^*BU^* = I$$

Pre-multiplying both sides of equation (3) with U^* and post multiplying with U
It implies

$$U^*UB^*BU^*U = U^*IU$$
$$U^*UB^*BU^*U = U^*U$$

Since an operator U is unitary, then by definition Halmos (2017), $U^*U = UU^* = I$ It implies

$$IB^*BI = I$$
$$B^*B = I$$

Thus B is isometry.
Conversely, suppose that an operator B is isometry, and thus by definition of isometry,

$$B^*B = I.$$

However, $A \overset{u.q.e}{\sim} B$ implies that equation (1) and (2) hold
Thus substituting $B^*B = I$ in equation (1),

$$A^*A = UB^*BU^*$$

$$= UIU^*$$

thus

$$A^*A = UU^*.$$

Since an operator U is unitary then

$$UU^* = I.$$

thus

$$A^*A = I$$

Thus, an operator A is isometry.

□

Theorem 4.2: If $V, W \in B(H)$ are unitarily quasi-equivalent operators, an operator V is co-isometry if and only if W is co-isometry.

Proof. Since $V \overset{u.q.e}{\sim} W$, then by definition (Kutkut, 1998),

$$V^*V = UW^*WU^* \tag{3}$$

$$VV^* = UWW^*U^* \tag{4}$$

Suppose V is co-isometry, then by definition, (Halmos, 2017).

$$< V^*x, V^*y > = < x, y >, \forall x, y \in H$$

$$VV^* = I$$

Substituting $VV^* = I$ in equation (4)

$$I = VV^*$$

$$= UWW^*U^*$$

thus

$$UWW^*U^* = I$$

Pre-multiplying both sides of equation (6) with U^* and post multiplying with U,

$$U^*UWW^*U^*U = U^*IU$$

thus

$$U^*UWW^*U^*U = U^*U$$

But $U^*U = UU^* = I$, since an operator U is unitary
it implies that,

$$IWW^*I = I$$
$$WW^* = I.$$

Thus W is co-isometry.
Conversely suppose that an operator W is co-isometry, Then by definition,

$$WW^* = I$$

Substituting $WW^* = I$ in (4) implies that

$$VV^* = UWW^*U^*$$

$$= UIU^*$$

$$VV^* = UIU^*$$
$$VV^* = UU^*$$

thus

$$VV^* = I\ .\ (\text{Since } UU^* = I)$$

Thus V is co-isometry. □

Remark 4.1: Results one and two implies that unitary quasi-equivalence preserves isometric and co-isometric properties of operators

Theorem 4.3: If $K, L \in B(H)$ are unitarily quasi-equivalent and K is partial isometry, then an operator L is also a partial isometry operator.

Proof. Since $K \overset{u.q.e}{\sim} L$, then by definition (Kutkut, 1998),

$$K^*K = UL^*LU^* \tag{5}$$

$$KK^* = ULL^*U^* \tag{6}$$

Since K is partial isometry, then by, Salhi & Zerovali. (2019), K^*K is a projection.
that is,

$$(K^*K)^2 = K^*K \tag{7}$$

But by equation (5)

$$K^*K = UL^*LU^*$$

Replacing K^*K with UL^*LU^* in equation (7). It imlies that

$$UL^*LU^* = (UL^*LU^*)^2$$

this implies that,

$$UL^*LU^* = UL^*LU^*.UL^*LU^*$$

But

$$U^*U = I$$

thus

$$UL^*LU^* = UL^*LIL^*LU^*$$
$$UL^*LU^* = UL^*LL^*LU^*$$

But

$$L^*LL^*L = (L^*L)^2$$

thus

$$UL^*LU^* = U(L^*L)^2U^* \tag{8}$$

Pre multiplying both sides of equation (8) with U^* and post multiplying with U, we get,

$$U^*UL^*LU^*U = U^*U(L^*L)^2U^*U$$

But

$$U^*U = I$$

Thus

$$IL^*LI = I(L^*L)^2I$$

implying that,

$$L^*L = (L^*L)^2$$

This implies that L^*L is a projection thus by definition of partial isometry Salhi & Zerovali, (2019) it means that L is partial isometry. □

Theorem 4.4: let $V, W \in B(H)$ be self-adjoint and unitary quasi-equivalence operators, if V^2 is partial isometry so is W^2 partial isometry.

Proof. Since $V \overset{u.q.e}{\sim} W$, the following conditions hold,

$$V^*V = UW^*WU^* \tag{9}$$

$$VV^* = UWW^*U^* \tag{10}$$

But V, W are self adjoint unitary quasi-equivalent operators, Nzimbi & Wanyonyi. (2020), established that V^2,W^2 are unitarily equivalent. That is by (9) and (10) and definition of self-adjoint operator,

$$V^*V = VV^*$$

$$= VV$$
$$= V^2$$

and

$$W^*W = WW^*$$

$$= WW$$
$$= W^2$$

thus

$$V^2 = UW^2U^*$$

Since V^2 is partial isometry then by definition of partial isometry,

$$V^2 = V^2(V^2)^*V^2 \tag{11}$$

but

$$V^2 = UW^2U^*$$

Replacing V^2 with UW^2U^* in equation (11)
it implies that,

$$UW^2U^* = UW^2U^*(UW^2U^*)^*UW^2U^*$$

But $(UW^2U^*)^* = UW^2U^*$
thus

$$UW^2U^* = UW^2U^*UW^{2^*}U^*UW^2U^*$$

it implies that,

$$UW^2U^* = UW^2I(W^2)^*IW^2U^*, (sinceU^*U = I)$$

thus

$$UW^2U^* = UW^2(W^2)^*W^2U^*. \tag{12}$$

Pre-multiplying both sides of equation (12) by U^* and post multiplying by U, We have :

$$U^*UW^2U^*U = U^*UW^2(W^2)^*W^2U^*U$$

But $U^*U = I$
thus

$$(IW)^2I = (IW)^2(W^2)^*W^2I$$

imlying that

$$W^2 = W^2(W^2)^*W^2 \tag{13}$$

Equation (13) implies that W^2 is a partial isometric operator. □

5 Conclusion

From the results established above, we can conclude that unitary quasi-equivalent operators preserves; isometry, co-isometry and partial isometric properties. That is if two operators are unitarily quasi-equivalent and one is isometry, co-isometry or partial isometry then so is the other operator isometry, co-isometry or partial isometry respectively.

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