CONCATENATION OF TWO DIFFERENT INTEGRABLE SYSTEMS IS NOT INTEGRABLE*

, *Abstract*

Nonlinear, completely integrable Hamiltonian systems representing charged particle motion in external electromagnetic fields hold promise for models of novel intensity frontier particle accelerators. The main reason is the combination of large regions of stable orbits with damping of collective instabilities by conservative relaxation. Intensity frontier particle accelerators are essential for discovery science in particle and nuclear physics, and a host of industrial and security applications. However, if realistic system lattices include additional sections or inserts that themselves may be integrable (such as linear optics, phase trombones, thin lenses, kicks, etc.), in general the full system fails to remain integrable. The proof and some consequences of integrability failure are explored.

INTRODUCTION

One of the more exciting advances in accelerator science for high energy physics over the last decade or so was the revitalization of concepts related to completely integrable, nonlinear Hamiltonian systems in the sense of Liouville for novel particle accelerators, with special emphasis on the intensity frontier. Danilov and Nagaitsev's seminal paper [1] proved that in four-dimensional phase space, under some approximations, it is possible to find static magnetic fields that make the single-particle Hamiltonian of a charged particle in magnetic fields deliberately nonlinear and completely integrable simultaneously, namely having two functionally independent invariants in involution. The most theoretically-promising element of this set became the basis of what is called now the Integrable Optics Test Accelerator (IOTA) [2], recently commissioned at Fermilab.

In this paper we disregard the specific form of the IOTA model Hamiltonian or its specific lattice. We will assume generically the existence of a nonlinear completely integrable Hamiltonian, H, on standard phase space. However, it is hardly ever the case that the practical implementation of such a system would be restricted to this Hamiltonian alone. The reasons could be manifold: need for extra space for instrumentation, for correction elements, for acceleration, or systems for various experiments such as electron or stochastic cooling, among others. The specifics are not important for our discussion. What is important is that collectively these extra features that need to be implemented in any practical realization of a realistic system will involve another Hamiltonian, K, on the same phase space. Typically, K is assumed to be modeling a linear system, or some limit of a linear system, such as a thin lens or a kick.

The full system is always assumed to be periodic, and one period of the system is split into a fraction associated with H and the remaining fraction associated with K. Therefore, the full system is associated with the s-dependent Hamiltonian $H + K$, where s is the independent variable with assumed (re-parametrized) period 1. Similar considerations would apply if the full ring would be split into a sum of more than two Hamiltonians.

By assumption, H is integrable. Clearly, if K is not integrable, then it follows immediately that the full system, represented by $H + K$, will not be integrable. The more interesting case is if K is also integrable, but $K \neq H$. For example, this is the case if K models a linear Hamiltonian system, since every linear Hamiltonian systems is completely integrable, with quadratic invariants in the phase space coordinates [3].

DEFINITIONS

We start with the definitions of the fundamental quantities, following [4]. Let us consider the symplectic space $\left(\mathbb{R}^{2n}, \omega\right)$ with the standard symplectic structure $\omega = \omega_0$, and denote by $\mathcal{H}$ the vector space of all smooth and compactly supported Hamiltonian functions $H = H(s,x) : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$.

Hamiltonian Flows

Associated to every $H \in \mathcal{H}$ is a time-dependent Hamiltonian vector field X_H on $\mathbb{R}^{2n}$. Since the vector field X_H has compact support, the Hamiltonian equations

$$\frac{dx}{ds} = X_H(s,x), \quad x(0) = x_0 \in \mathbb{R}^{2n},$$

can be solved over the whole interval $[0,1]$ for every given initial value $x_0 \in \mathbb{R}^{2n}$.

Symplectic Maps

We thus obtain a 1-parameter family of symplectic mappings φ_H^s for $s \in [0,1]$. It is defined by

$$\varphi_H^s(x_0) = x(s),$$

where $x(s)$ solves the equation for the initial value x_0. Clearly $\varphi_H^s(x) = x$ if $|x|$ is sufficiently large. By

$$\varphi_H = \varphi_H^1$$

we shall denote the time-1 map of the flow φ_H^s.

This is true in the opposite direction too. Namely, in the case of $\mathbb{R}^4$, every compactly supported symplectic map φ is the time-1 map of some Hamiltonian H.

*

†

We define the group $\mathcal{D}$ of compactly supported Hamiltonian diffeomorphisms by

$$\mathcal{D} = \{\varphi_H \mid H \in \mathcal{H}\} .$$

Hamiltonians of Related Flows

Finally, given $H, K \in \mathcal{H}$ and $\vartheta \in \mathcal{D}$, we define the functions $\bar{H}$, $H\#K$ and H_ϑ in $\mathcal{H}$ as follows:

$$\begin{aligned} \bar{H}(s,x) &= -H\left(s, \varphi_H^s(x)\right), \\ (H\#K)(s,x) &= H(s,x) + K\left(s, \left(\varphi_H^s\right)^{-1}(x)\right), \\ H_\vartheta(s,x) &= H\left(s, \vartheta^{-1}(x)\right). \end{aligned}$$

PROOF OF NON-INTEGRABILITY

The crucial relationship that leads to the proof is to show that

$$\varphi^s_{H\#\tilde{K}} = \varphi^s_H \circ \varphi^s_{\tilde{K}}. \tag{1}$$

Indeed, if

$$\begin{aligned} \frac{d}{ds}\varphi^s &= X_H \circ \varphi^s \text{ and } \varphi^0 = id, \\ \frac{d}{ds}\psi^s &= X_{\tilde{K}} \circ \psi^s \text{ and } \psi^0 = id, \end{aligned}$$

then it follows from

$$\begin{aligned} \frac{d}{ds}\left(\varphi^s \circ \psi^s\right) &= \left(\frac{d}{ds}\varphi^s\right) \circ \psi^s + \left(d\varphi^s \circ \psi^s\right) \cdot \frac{d}{ds}\psi^s \\ &= X_H\left(\varphi^s \circ \psi^s\right) \\ + \left[d\varphi^s \circ (\varphi^s)^{-1} \circ \varphi^s \circ \psi^s\right] &\cdot X_{\tilde{K}} \circ \left[(\varphi^s)^{-1} \circ \varphi^s \circ \psi^s\right] \end{aligned}$$

that by the transformation law of Hamiltonian vector fields the second term is equal to

$$X_{\tilde{K}\circ(\varphi^s)^{-1}} \circ \left(\varphi^s \circ \psi^s\right),$$

and the statement (1) is proved.

Immediate Consequences

If we define K such that $H\#\tilde{K} = H + K$, we obtain

$$\tilde{K} = K \circ \varphi^s_H,$$

and since

$$\varphi^s_{\tilde{K}} = \varphi^s_{K\circ\varphi^s_H} = \left(\varphi^s_H\right)^{-1} \circ \varphi^s_K \circ \varphi^s_H,$$

the flow of the full system is given by

$$\varphi^s_{H+K} = \varphi^s_K \circ \varphi^s_H.$$

Whether the ring lattice Hamiltonian is K or $\tilde{K}$ is irrelevant for our discussion. This follows from the fact that the two functions are related by the flow of a Hamiltonian system, which are symplectic maps for each s. Symplectic maps are integrability preserving transformations [5], hence if K is integrable then $\tilde{K}$ is also integrable, and vice-versa.

Without loss of generality, let us assume that our phase space is $\mathbb{R}^4$ and the Hamiltonians themselves are integrals of the motion. Therefore, by assumption of complete integrability, both Hamiltonians H and K have an (and only one) additional independent invariant in involution with the Hamiltonian. Denoting them by $C_H \neq C_K$, we have

$$C_H \circ \varphi^s_H = C_H, \quad C_K \circ \varphi^s_K = C_K.$$

If the full system were integrable, then an integral of motion C_{H+K} would exist such that

$$C_{H+K} \circ \varphi^s_K \circ \varphi^s_H = C_{H+K},$$

or

$$C_{H+K} \circ \varphi^s_K = C_{H+K} \circ \left(\varphi^s_H\right)^{-1} \tag{2}$$

As a special case of (1), with $\tilde{K} = \bar{H}$, we obtain that $\left(\varphi^s_H\right)^{-1} = \varphi^s_{\bar{H}}$. From the definition of $\bar{H}$, it is easy to see that $\bar{H}$ is also integrable, with the same invariant C_H. Therefore, neither C_H nor C_K fits the bill for C_{H+K}; if $C_{H+K} = C_H$, we get $C_H \circ \varphi^s_K = C_H$ that does not hold by assumption. If $C_{H+K} = C_K$, we get $C_K = C_K \circ \varphi^s_{\bar{H}}$ that also does not hold. The last possibility is to have a $C_{H+K} \neq C_H \neq C_K$. That is, there exist a function of phase space variables such that is not an invariant of the two independent Hamiltonian flows, the only requirement being that individually they are completely integrable, and furthermore the flows map this function to a different, but the same exact function. Clearly, this is impossible for two independent Hamiltonians H and K. Therefore, in general the concatenation of two completely integrable Hamiltonian systems will cease to be completely integrable.

DISCUSSION

It is an interesting question to entertain if this general statement fails for some particular integrable Hamiltonian systems. For example, let us assume that H is given. Can we find an integrable K such that the combined system remains integrable? Again, in general, without knowing more about H, is highly unlikely. Integrable Hamiltonians are scarce, integrability is not generic. It is difficult enough to find integrable Hamiltonians at all even without additional serious constrains.

Common Case

How about the most common practical case, where H is highly nonlinear and K is quadratic? Typically, the separate invariants will be linear or quadratic in the momenta, so that is not an issue a priori. However, C_H will be highly nonlinear in the coordinates, while C_K quadratic. The flow of K will map any function linearly while the flow of H nonlinearly, therefore there is no initial function that the two flows will map to the same level sets. As a consequence, concatenation of a nonlinear with a linear integrable system is not integrable.

Trivial Case

The trivial case, where both H and K are quadratic, is not very interesting practically, but it should be mentioned that since both Hamiltonians generate linear flows, the combined system will also be linear, and hence integrable because every linear Hamiltonian system is integrable.

Fully Nonlinear Case

The remaining interesting case, and the most complicated to study is when both H and K are highly nonlinear. This leaves more opportunities, but we reiterate that although nonlinear, both systems are assumed to be integrable taken separately. As such, their flows are rather simple. Integrable systems admit so-called action angle coordinates, which means that viewed through this particular lens their flows are simple rotations on tori with amplitude dependent angular speeds [6]. Based on the assumption that the two systems are different, it follows that their respective action-angle coordinates will also be different. That is, (2) can we rewritten as

$$C_{H+K}(J_K, \mu(J_K)s + \theta_K) = C_{H+K}(J_{\bar{H}}, \nu(J_{\bar{H}})s + \theta_{\bar{H}}), \quad (3)$$

where J are the actions and μ, ν are the angular speeds, while θ are the initial angles. Thus, it is difficult to imagine how two such flows, with very similar behavior, yet different, would behave as expected according to (3). So, although we do not have a completely general proof, we conjecture that this is never possible in practically relevant cases. Hence, concatenation of two different completely integrable Hamiltonian system, out of which as least one is not linear, is not integrable.

Beam Matching

Another question worth pondering is whether the relaxation of complete integrability akin to beam matching would offer some new features. That is, instead of (2), we would require

$$C_{H+K} \circ \varphi_K = C_{H+K} \circ \varphi_{\bar{H}}.$$

This problem does seem easier, since instead of every value of s we require the equation to hold only for $s = 1$. However, this is deceiving, since as discussed above the flows are just rotations, and fixing the value of s is just fixing a particular rotation angle; not of much help in general. However, there is one particular case of real interest; if it happens that $\varphi_K = id$. In this case the full system will behave for all practical purposes as integrable with invariant C_H.

It might be very difficult to achieve $\varphi_K = id$ in practice, but it could be feasible to have a linear φ_K that matches the linearization of φ_H. Indeed, this would make the full system nearly-integrable. Due to general features of Hamiltonian dynamics, owing to KAM tori [7], it is expected that in this case the system will behave close to an integrable one, in an appropriate sense. We believe this option is the best practically and is worth further study.

Finally, it is obvious that all these results do not depend on the dimensionality of phase space or whether the systems are autonomous or not. First, if any system is non-autonomous, it can be rewritten as an autonomous system on extended phase space [8]. Second, if the phase space is more than four-dimensional, there will be extra invariants involved for integrability, but if the theory does not work for one additional invariant besides the Hamiltonian, it will certainly not work with two or more invariants.

CONCLUSION

Nonlinear, completely integrable Hamiltonian systems, in principle, are promising models for novel particle accelerators for future endeavors at the intensity frontier of beam physics and particle physics. These systems are far from being generic in the class of nonlinear Hamiltonian systems. In fact, the only system shown so far to fulfill these requirements are in four-dimensional phase space, in the paraxial approximation and neglecting fringe fields. IOTA is based on a model taken from this class. Recently, we showed that without these approximations the set of such completely integrable systems is empty [9].

Moreover, even if we restrict ourselves to the class with approximations where IOTA is an element, it is clear that the practical implementation of the systems will break integrability due to unavoidable construction errors, since arbitrarily small perturbations generically destroy integrability. Also, even the numerical simulations of integrable systems will, in general, destroy integrability [10]. However, even if all these errors are neglected, it is also obvious that the full accelerator lattices might not be fully integrable due to the presence of ancillary sections and elements deemed necessary for operations. Examples of such extra features include drifts for monitoring, correction, and other experimental accessories, thin lenses, kicks, or phase trombones, just to list a few. Under these circumstances, it is possible to alter the model of the accelerator lattices into an equivalent Hamiltonian system, where the Hamiltonian is the sum of two or more Hamiltonian functions, where the individual Hamiltonians are completely integrable by themselves. The main Hamiltonian would be the nonlinear completely integrable one with the main purpose of achieving the stated goals of high intensity and power while maintaining collective and orbital stability. The other summand will almost always represent a linear system (but not necessarily) that is integrable too. The topic explored in this paper is whether the combined, concatenated system will remain integrable.

We showed that in general this is not the case: the concatenation of two integrable systems in general ceases to be integrable. The trivial case of concatenation of two linear system is excluded. When one of the systems is linear while the other is nonlinear, integrability is destroyed in the sense that not even in specific cases could such a system be concocted to maintain integrability. When both systems are nonlinear, no useful concatenation exists that would preserve integrability at least in some particular cases, but we conjecture that to be true in general, except maybe some corner cases of no practical interest.

Therefore, for all practical purposes we need to give up on the concept of complete integrability of a practical construction of a novel accelerator along these lines. All is lost? We don't think so. We believe that near-integrability should be almost as useful a concept as full integrability for novel particle accelerators, and we identified the most promising candidates to be concatenation of a nonlinear, completely integrable Hamiltonian system with a linear system that is the linearization of the preceding system. This way we obtain practical, nonlinear, nearly integrable Hamiltonian systems with invariant tori close to the invariants of the original systems. The quantitative assessment of such systems are worth further study. It is also worth noting that alternative concepts of stable nonlinear dynamics exist and their relationship to (near-) integrability is an interesting topic in itself.

REFERENCES

[1] V. Danilov and S. Nagaitsev, "Nonlinear accelerator lattices with one and two analytic invariants," *Phys. Rev. ST Accel. Beams*, vol. 13, p. 084 002, 8 2010.

[2] S. Antipov *et al.*, "IOTA (Integrable Optics Test Accelerator): Facility and experimental beam physics program," *Journal of Instrumentation*, vol. 12, no. 03, T03002, 2017.

[3] H. Kocak, "Linear Hamiltonian systems are integrable with quadratics," *Journal of Mathematical Physics*, vol. 23, no. 12, pp. 2375–2380, 1982.

[4] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*. Birkhauser, 1994.

[5] J. Hietarinta, "Direct methods for the search of the second invariant," *Physics Reports*, vol. 147, no. 2, pp. 87–154, 1987.

[6] V. I. Arnold, Ed., *Dynamical Systems III, Mathematical Aspects of Classical and Celestial Mechanics*. Springer-Verlag, 1993.

[7] Z. Shang, "KAM theorem of symplectic algorithms for Hamiltonian systems," *Numer. Math.*, vol. 83, pp. 477–496, 1999.

[8] M. Berz, *Modern Map Methods in Particle Beam Physics*. Academic Press, 1999.

[9] B. Erdélyi, K. Hamilton, J. Pratscher, and M. Swartz, "Nonexistence of integrable nonlinear magnetic fields with invariants quadratic in momenta," *to be published*, 2023.

[10] S. Rexford and B. Erdélyi, "Numerical discretization of completely integrable nonlinear Hamiltonian systems," *Phys. Rev. Accel. Beams*, vol. 21, p. 114 601, 11 2018.