On propositions pertaining to the Riemann Hypothesis II

Abstract
In this paper, we define certain classes of non-zeroes of the Riemann zeta function. We also present associated algorithms for finding these non-zeroes, which can enable corresponding computations. Some theoretical connections are also drawn with mixed integer programming and continuous Diophantine approximation. We also study, for points in the domain of the Riemann zeta function, their induced distributions
over the unit circle.
Keywords : Complex Analysis, Complex functions, Riemann Hypothesis

1 Introduction

This paper is a continuation of Basu [2022] and Basu [2023]. We first describe a class of non-zeroes of the Riemann zeta function (Riemann [1859a], Riemann [1859b], Stein and Shakarchi [2010]) that are derivable using a certain criterion. The criterion is one in which we partition a sum into two non-equivalent parts, which result in a non-zero sum as in Basu [2022] and Basu [2023]. We present two algorithms, which allows one to identify whether a point is non-zero based on a partial sum. This involves an upper bound derived in Basu [2023], on the sum of norms corresponding to the tail of the Riemann zeta function. We also establish some theoretical connections by representing the problem as one of mixed integer programming (Schrijver [1998],Conforti et al. [2014]) and continuous Diophantine approximation (Minkowski [1907],Sprindzhuk [1979],Pollington and Vaughan [1990], Schmidt [1996], Queffélec et al. [2013], Koukoulopoulos and Maynard [2020]). Lastly, we study the associated probability distributions on the unit circle, parameterised by the complex variables in the domain. Perhaps interestingly, as in Basu [2023], one finds that every distribution which has zero expectation, is similar to a uniform distributon, in that the probablity of each arc is upper bounded by a function increasing in arc length. Finally, as noted in Basu [2022] and Basu [2023], prior research on the Riemann hypothesis appears in Mangoldt [1905], Hardy [1914], Hardy and Littlewood [1921], Conrey [2003], Lagarias [2002], Bump et al. [2000], Borwein et al. [2008], Platt and Trudgian [2021], Nicolas [2021] and Johnston [2022]. Much of this literature studies properties of holomorphic and meromorphic functions within the theory of complex functions (see also Gram [1903] and Turing [1953]).

One may note that the rational (linear-exponential) functional form that gives the tail bound is similar to that in Ramanujan [1915] and Gram [1903].

2 Propositions and algorithms

In order to avoid repetition of notation, we will refer the reader to Basu [2022] and Basu [2023] to see the appropriate notation for the rest of this paper. The complex plane (see Stein and Shakarchi [2010] and Pierpont [1914]) is $\mathbb{R}^2$ and for each point in the domain given by

$$S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0,1);\ t \neq 0\}, \quad (1)$$

we have the Riemann zeta function, that is defined for each $s \in S$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \left(1 - \frac{1}{2^{1-s}}\right) \times \frac{(-1)^{n+1}}{n^s}. \quad (2)$$

Of course, the non-zeroes of the above sum would be identical to those of the alternating Dirichlet sum,

$$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \quad (3)$$

Hence, we study the following sequence and its associated distribution on the unit circle. We define $\{Z(s)_n\}_{n \in \mathbb{Z}^+}$ as follows

$$Z_0 = (1,0); \quad (4)$$

$$Z_n(s) = \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}; \text{ for each } n \in \mathbb{Z}^+. \quad (5)$$

2.1 The expectation representation

For an absolutely convergent series $\{z_n\}_{n \in \mathbb{Z}^+}$ in a normed vector space (Narici and Beckenstein [2010]), we define the discrete probability measure on $\mathbb{Z}^+$ given by $\mu(\{n\}) := \frac{||z_n||}{\sum_m ||z_m||}$. This leads to a probability measure on the unit circle $\{z : ||z|| = 1\}$. A non-zero sum $\sum_n z_n$ is hence, identical to a non-zero expectation $\mathbb{E}_\mu[z]$. In Basu [2023], a concentration theorem was obtained, which we restate here as follows.

Proposition 2.1. *(Basu [2023]) Let μ be a probability measure on the unit circle $\mathbb{S}^1$. Then, $\mathbb{E}_\mu[z] \neq 0$ if there exist numbers $0 \leq \theta \leq \theta' \leq 2\pi$ such that $\theta' - \theta \leq$*

$\frac{\pi}{2}$ *and*

$$\mu(\{z : \theta \le \theta(z) \le \theta'\}) > \frac{1}{1+\cos\left(\frac{\theta'-\theta}{2}\right)}. \quad (6)$$

In the above proposition, we denote as $\theta(z) \in [0, 2\pi]$, the angle in radians for the complex variable z (see Stein and Shakarchi [2010], Pierpont [1914]). Note that by symmetry of the unit circle, the conclusion of the above proposition obtains even if we take $0 \le \theta \le \theta' \le 2\pi$ such that $\theta + (2\pi - \theta') \le \frac{\pi}{2}$ and lower-bound the probability of the event as

$$\mu(\{z : \theta(z) \in [0, \theta] \cup [\theta', 2\pi]\}) \ge \frac{1}{1+\cos\left(\frac{\theta+(2\pi-\theta')}{2}\right)}. \quad (7)$$

In this case, the event corresponds to an arc in the unit circle, which contains the point (1, 0).

Let us first discuss the case, for the Riemann zeta functon, in which for the input complex variable $s = (\sigma, t)$, we have that $\sigma > 1$. In this case, we have the absolutely convergent series

$$\zeta(s) = \sum_{n\ge 1} \frac{1}{n^s}, \quad (8)$$

which is defined as a function of s on the domain

$$S_+ = \{(\sigma, t) : \sigma > 1\}. \quad (9)$$

The above sequence would be defined as $z_n := \frac{1}{n^s}$. As it was shown in Basu [2022], given the derivation from Euler's formula

$$\frac{1}{n^s} = e^{-\sigma \ln(n)}(\cos(-t\ln(n)), \sin(-t\ln(n))), \quad (10)$$

for $\sigma \geq 2$, for the associated distribution μ and for $\theta = \theta' = 0$, we have that $E_\mu[z]$, 0. For notational convenience, we will define μ_s to be the probability measure over the unit circle, associated with the complex variable s. For any arbitrary distribution μ, we say that μ is concentrated, if it satisfies condition 6 from Proposition 2.1 or the condition 7. Now, we define the set

$$S^*_+ = \{s \in S_+ : \mu_s \text{ is concentrated}\}. \quad (11)$$

One may conjecture whether or not $S_+ = S^*_+$. Although, we may, prove the following theorem.

Proposition 2.2. *Let* $1 < \sigma < 2$. *Then, there exist countably many pairwise disjoint intervals* $\{[t_k, \overline{t}_k]\}_{k \in \mathbb{Z}_+}$, *such that* $\lim_{k \to \infty} t_k \to +\infty$ *and for each* k *and* $t \in [t_k, \overline{t}_k]$, *we have that* $(\sigma, t) \in S_*$.

Proof. This can be established using a multidimensional Weyl criterion (see Weyl [1916], Kuipers and Niederreiter [2012]). Let $m \in \mathbb{Z}_+$ be such that the sum of norms for the tail is less than $\frac{1}{\sqrt{2}}$, i.e

$$\sum_{n \geq m+1} \left|\left| \frac{1}{n^s} \right|\right| = \sum_{n \geq m+1} \frac{1}{n^\sigma} < \frac{1}{\sqrt{2}}. \quad (12)$$

Now, by applying themultidimensionalWeyl criterion, we obtain that the sequence $\left(\left(\frac{\ln(n) t}{2\pi}\right)_{n=2}^{m}\right)_{t \in \mathbb{Z}_+}$ in $\mathbb{R}^{m-1}$ is uniformly distributed modulo one (Kuipers and Niederreiter [2012]). The resulting uniform distribution on $[0, 1]^{m-1}$, is derived through finite sampling from the defined sequence. Since each open ball has positive probability in the uniform measure, by a probabilistic proof, we are able to derive a countable collection of intervals $\{[\underline{t}_k, \overline{t}_k]\}_{n\in\mathbb{Z}_+}$ such that for each k, $t \in [\underline{t}_k, \overline{t}_k]$ and $2 \le n \le m$, we have from Euler's formula, that

$0 \le \theta\left(\frac{1}{n^{(\sigma,t)}}\right) \le \frac{\pi}{2}$

. This means that for the induced distribution $\mu_{(\sigma,t)}$, we have that $\mu_{(\sigma,t)}(\{z : 0 \le \theta(z) \le \frac{\pi}{2}\}) > \frac{\sqrt{2}}{1+\sqrt{2}}$

. Hence, the probability measure $\mu_{(\sigma,t)}$ is concentrated and $(\sigma, t) \in S_*$.

Mixed Integer Programming and Continuous Diophantine approximation The above result may also be achieved by means of a mixed integer linear program (Schrijver [1998],Conforti et al. [2014]), which is essentially a continuous version of the simultaneous Diophantine approximation problem (see Minkowski

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[1907],Sprindzhuk [1979],Schmidt [1996]). We say this because in contrast to diophantine approximation, which involves only integers, the variable of interest, t, is continuous. Such a representation of the problem studied in the above proposition would also allow us to reason about the positioning of the intervals $\{[\underline{t}_k, \overline{t}_k]\}_{k\in\mathbb{Z}_+}$.

Let $m \in \mathbb{Z}_+$ be such that $m \ge 2$ and let $0 < r < \frac{\pi}{4}$ be a real number. Now, consider the following mixed integer program.

$$\min_{(t,(q_n)_{n=2}^m)\in\mathbb{R}\times\mathbb{Z}^{m-1}} t$$

subject to :

$$\left|\frac{\ln(n)}{2\pi} t - q_n\right| \le \frac{r}{2\pi} \quad \text{for all } 2 \le n \le m. \quad (13)$$

$$q_n \ge 0 \quad \text{for all } 2 \le n \le m. \quad (14)$$

$$\sum_{n=2}^{m}$$

$q_n \geq 1.$ (15)

$t \geq 0.$ (16)

Let us first discuss the feasible region for the above program. In particular, we will be interested in the projection of the feasible region on the first coordinate i.e. set of feasible t values. By Dirichlet's theorem on simultaneous diophantine approximation (see Schmidt [1996]), we may derive a subset of the feasible t values, that is a union of disjoint intervals as shown above. Such considerations will again result in concentration, in the sense of condition 7 above.

In the next proposition, we demonstrate further properties of the feasible t region. We say that a vector $(t, (q_n)_{n=2}^m)$ is co-prime, if i) t is an integer ii) the integers in $(t, (q_n)_{n=2}^m)$ are co-prime in the sense that t and q_n are co-prime for each n. The following result obtains, which also exhibits the relevance of the sequence $\{\ln(n)\}_{n\in\mathbb{Z}_+}$ in the present analysis.

Proposition 2.3. *Let* $1 < \sigma < 2$*. Then, there exists* $m \in \mathbb{Z}_+$ *and an infinite subset* $T \subseteq \mathbb{Z}_+$ *such that for each,* $t \in T$*,*

1. The probability measure $\mu_{(\sigma,t)}$ *is concentrated i.e.* $(\sigma, t) \in S_{*+}$

2. For each $2 \leq n \leq m$*, the unique minimiser* $q_n \in \operatorname{argmin}_{q\in\mathbb{Z}_+} \left| \frac{\ln(n)}{2\pi} t - q \right|$ *is such that the resulting vector* $(t, (q_n)_{n=2}^m)$ *is co-prime.*

Proof. The result follows from Sprindzhuk [1979] or Pollington and Vaughan [1990].

Algorithms We will next discuss the case $0 < \sigma < 1$, which corresponds to the domain S. For a point in region $s \in S$, we define the probability measure μ_s as follows. We define the sequence $\{z_n\}_n$ as $z_1 = Z_0 = (1, 0)$ and $z_n := Z_{n-1}(s)$, for each $n \geq 2$, as defined in 4 and 5. Hence, μ_s is derived from $\{z_n\}_n$. As before, μ_s is said to be concentrated if satisfies 6 or 7. We define the following set

$S_* = \{s \in S : \mu_s \text{ is concentrated}\}.$ (17)

Again, similarly in this case, we may conjecture whether or not $S_* = S \setminus \{(\sigma, t) : \sigma = 1/2\}$. The first algorithm is based on Proposition 2.1. It is based on finding a partial sum approximation of the distribution μ_s. By this we mean, that for a large enough $m \in \mathbb{Z}_+$, we bound the sum of norms of the tail $\{z_n\}_{n=m+1}^{\infty}$. From Basu [2023], for $m \in \mathbb{Z}_+$ such that we have an acute angle $(\ln(2m + 1) - \ln(2m))|t| \le \frac{\pi}{2}$, we get the bound

$$\sum_{n \ge m+1} ||Z_n(s)|| \le \left(1 + \frac{|t|}{\sigma}\right) \frac{1}{(2m + 2)^{\sigma}}. \quad (18)$$

Now, for large m, this means that the tail has small amount of weight in the normalisation governing μ_s. Hence, we may apply the concentration theorem to the finite sum, in a way that yields a non-zero expectation. We express this in the form of the following proposition.

Proposition 2.4. *Let* $(\sigma, t) \in S$. *Suppose* $m \in \mathbb{Z}_+$ *and* $\varepsilon > 0$ *such that* $\sum_{n \ge m+1} ||z_n|| < \varepsilon$. *If there exist* $0 \le \theta \le \theta' \le 2\pi$ *such that* $\theta' - \theta \le \frac{\pi}{2}$ *and*

$$\frac{\sum_{n : \theta \le \theta(z_n) \le \theta'} ||z_n||}{\sum_{n=1}^{m} ||z_n|| + \varepsilon} > \frac{1}{1 + \cos\left(\frac{\theta' - \theta}{2}\right)}. \quad (19)$$

Then, $\mu_{(\sigma,t)} \in S_*$.

Proof. Follows from Proposition 2.1.

We next implement an algorithm based on the above proposition. The algorithm sets the values $\theta = 0$

and $\theta' = \frac{\pi}{2}$. It also sets m based on the bound derived in Basu [2023] on the sum of norms for the tail sequence. Then, finally, the algorithm runs on inputs (σ, t) given by a finite grid of 500 points that is a subset of $[\frac{1}{2}, 1] \times [0, \infty)$. The grid is equal to the set $\{\frac{1}{2}(1 + \frac{j-1}{5}) : 1 \leq j \leq 5\} \times \{1 + (j \times 10^{-1}) : 1 \leq j \leq 100\}$. The upper bound on the second coordinate in the finite grid is set based on the tail bound. We also set $\varepsilon = 10^{-1}$ as above in Proposition 2.4. The algorithm returns an output of 1, if condition 19 is satisfied, else it returns 0. Perhaps interestingly, the derived results exhibit a pattern similar to Proposition 2.2. The computations were conducted on MATLAB with parallel computing. The following are the results summarised in the table below.

NA	0.5	0.6	0.7	0.8	0.9
1.1	1	1	1	1	1
1.2	1	1	1	1	1
1.3	1	1	1	1	1
1.4	0	1	1	1	1
1.5	0	1	1	1	1
1.6	0	1	1	1	1
1.7	0	1	1	1	1
1.8	0	1	1	1	1
1.9	0	0	1	1	1
2	0	1	1	1	1
2.1	0	0	1	1	1
2.2	0	0	1	1	1
2.3	0	1	1	1	1
2.4	0	0	0	1	1
2.5	0	0	1	1	1
2.6	0	0	1	1	1
2.7	0	0	1	1	1
2.8	0	1	1	1	1
2.9	0	1	1	1	1
3	1	1	0	1	1
3.1	0	0	0	1	1
3.2	0	0	0	1	1
3.3	0	0	1	1	1
3.4	0	0	1	1	1
3.5	0	0	1	1	1
3.6	0	0	1	1	1
3.7	0	1	1	1	1

3.8 0 1 1 1 1
3.9 0 1 1 1 1
4 1 1 1 1 1
4.1 1 1 1 1 1
4.2 1 1 1 1 1
4.3 1 1 1 1 1
4.4 1 1 1 1 1
4.5 1 1 1 1 1
4.6 1 1 1 1 1
4.7 1 1 1 1 1
4.8 1 1 1 1 1
4.9 1 1 1 1 1
5 1 1 1 1 1
5.1 0 1 1 1 1
5.2 0 0 0 0 0
5.3 0 0 0 0 0
5.4 0 0 0 0 0
5.5 0 0 0 0 0
5.6 0 0 0 0 0
5.7 0 0 0 0 0
5.8 0 0 0 0 0
5.9 0 0 0 0 0
6 0 0 0 0 0
6.1 0 0 0 0 0
6.2 0 0 0 0 0
6.3 0 0 0 0 0
6.4 0 0 0 0 0
6.5 0 0 0 0 0
6.6 0 0 0 1 1
6.7 0 0 1 1 1
6.8 0 1 1 1 1
6.9 0 0 1 1 1
7 0 1 1 1 1
7
7.1 0 1 1 1 1
7.2 0 0 1 1 1
7.3 0 0 0 0 0
7.4 0 0 0 0 0
7.5 0 0 0 0 0
7.6 0 0 0 0 0
7.7 0 0 0 0 0
7.8 0 0 0 0 0
7.9 0 0 0 0 0
8 0 0 0 0 0
8.1 0 0 0 0 0
8.2 0 0 0 0 0
8.3 0 0 0 0 0
8.4 0 0 0 0 0
8.5 0 0 0 0 0

8.6 0 0 0 0 0
8.7 0 0 0 0 0
8.8 0 0 0 0 0
8.9 0 0 0 0 0
9 0 0 0 0 0
9.1 0 0 0 0 0
9.2 0 0 0 0 0
9.3 0 0 0 0 0
9.4 0 0 0 0 0
9.5 0 0 0 0 0
9.6 0 0 0 0 0
9.7 0 0 0 0 0
9.8 0 0 0 0 0
9.9 0 0 0 0 0
10 0 0 0 0 0
10.1 0 0 0 0 0
10.2 0 0 0 0 0
10.3 0 0 0 0 0
10.4 0 0 0 0 0
10.5 0 0 0 0 1
10.6 0 0 0 1 1
10.7 0 0 0 1 1
10.8 0 0 1 1 1
10.9 0 0 1 1 1
11 0 0 0 1 1

The second algorithm is based on two propositions from Basu [2022] and Basu [2023] respectively, which
we restate below. For points $s, s' \in \mathbb{R}^d$, we define the line joining the points as $< s, s' > := \{\theta s + (1-\theta) s' : \theta \in \mathbb{R}\}$

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and define $< s > := < s, 0 >$.

Proposition 2.5. *(Basu [2022]) Suppose that* $\{x_n\}_{n=1} \subseteq \mathbb{R}^d$ *such that there exist* $n, m \in \mathbb{Z}_+$ *with*
$x_m << x_n >$. (20)
Further, suppose that
$\sum_{n \in \mathbb{Z}_+} x_n$ *exists. Then, we have that*
$\sum_{n \in \mathbb{Z}_+} x_n , 0$ *if and only if there exists* $m \in \mathbb{Z}_+$ *such that* $0 << \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$.

Then, an observation that we express as a proposition as follows.

Proposition 2.6. *(Basu [2023]) Suppose*
P

$\sum_{j\in J} z_j$ is a finite sum of vectors in $\mathbb{R}^d$ and suppose $q \neq 0$ is a hyperplane. Define $J_+ = \{j \in J : q.z_j > 0\}$ and $J_- = \{j \in J : q.z_j < 0\}$. Suppose that there exist sequences of pairwise disjoint sets $\{J^+_k\}_{k=1}^{K} \subseteq J_+$ and $\{J^-_k\}_{k=1}^{K} \subseteq J_-$ such that $\cup_{k=1}^{K} J^-_k = J_-$ and $q.\left(\sum_{j\in J^+_k} z_j + \sum_{j\in J^-_k} z_j\right) > 0$ for each $k \in \{1, \dots, K\}$. Then, $\sum_{j\in J} z_j \neq 0$.

The algorithm that we next describe can be viewed as an application of either of the above propositions.

We state this as a proposition, as for the previous algorithm, which was based on concentration. As one may observe, both algorithms are based on the principle of dividing a sum with non-equivalent parts.

Proposition 2.7. *Let $(\sigma, t) \in S$. Suppose that $m \in \mathbb{Z}_+$ such that $(\ln(2m+1)-\ln(2m))|t| \le \frac{\pi}{2}$ and*

$$\left(\frac{1+|t|}{\sigma}\right) < (2m+2)^{\sigma}.$$

Then, if

$$\sum_{n=2}^{m} (-1)^{n+1} e^{-\sigma \ln(n)} \cos(-t \ln(n)) > 0, \quad (21)$$

it follows that $\zeta(s) \neq 0$.

Proof. Follows from either Proposition 2.5 or Proposition 2.6.

For the implementation of the above algorithm, we have same finite grid as before, with m being set

based on the tail bound. The algorithm returns an output of 1 if 21 is satisfied, else it returns 0. The
following table summarises the implementation results for the algorithm.

NA	0.5	0.6	0.7	0.8	0.9
1.1	0	0	0	0	0
1.2	0	0	0	0	0
1.3	0	0	0	0	0
1.4	0	0	0	0	0
1.5	0	0	0	0	0
1.6	0	0	0	0	0
1.7	0	0	0	0	0
1.8	0	0	0	0	0
1.9	0	0	0	0	0
2	0	0	0	0	0
2.1	0	0	0	0	0
2.2	0	0	0	0	0
2.3	0	0	0	0	0
2.4	0	0	0	0	0

2.5	0	0	0	0	0
2.6	0	0	0	0	0
2.7	0	0	0	0	0
2.8	0	0	0	0	0
2.9	0	0	0	0	1
3	1	1	1	1	1
3.1	1	1	1	1	1
3.2	1	1	1	1	1
3.3	1	1	1	1	1
3.4	1	1	1	1	1
3.5	1	1	1	1	1
3.6	1	1	1	1	1
3.7	1	1	1	1	1
3.8	1	1	1	1	1
3.9	1	1	1	1	1
4	1	1	1	1	1
4.1	1	1	1	1	1
4.2	1	1	1	1	1
4.3	1	1	1	1	1
4.4	1	1	1	1	1
4.5	1	1	1	1	1
4.6	1	1	1	1	1
4.7	1	1	1	1	1
4.8	1	1	1	1	1
4.9	1	1	1	1	1
5	1	1	1	1	1
5.1	1	1	1	1	1
5.2	1	1	1	1	1
5.3	1	1	1	1	1
5.4	1	1	1	1	1

5.5 1 1 1 1 1
5.6 1 1 1 1 1
5.7 1 1 1 1 1
5.8 1 1 1 1 1
5.9 1 1 1 1 1
6 1 1 1 1 1
6.1 1 1 1 1 1
6.2 1 1 1 1 1
6.3 1 1 1 1 1
6.4 1 1 1 1 1
6.5 1 1 1 1 1
6.6 1 1 1 1 1
6.7 1 1 1 1 1
10
6.8 1 1 1 1 1
6.9 1 1 1 1 1
7 1 1 1 1 1
7.1 1 1 1 1 1
7.2 1 1 1 1 1
7.3 1 1 1 1 1
7.4 0 0 0 0 0
7.5 0 0 0 0 0
7.6 0 0 0 0 0
7.7 0 0 0 0 0
7.8 0 0 0 0 0
7.9 0 0 0 0 0
8 0 0 0 0 0
8.1 0 0 0 0 0
8.2 0 0 0 0 0
8.3 0 0 0 0 0
8.4 0 0 0 0 0
8.5 0 0 0 0 0
8.6 0 0 0 0 0
8.7 0 0 0 0 0
8.8 0 0 0 0 0
8.9 0 0 0 0 0
9 0 0 0 0 0
9.1 0 0 0 0 0
9.2 0 0 0 0 0
9.3 0 0 0 0 0
9.4 0 0 0 0 0
9.5 0 0 0 0 0
9.6 0 0 0 0 0
9.7 0 0 0 0 0
9.8 0 0 0 0 0
9.9 0 0 0 0 0
10 0 0 0 0 0
10.1 0 0 0 0 0
10.2 0 0 0 0 0

10.3 0 0 0 0 0
10.4 0 0 0 0 0
10.5 0 0 0 0 0
10.6 0 0 0 0 0
10.7 1 1 1 1 1
10.8 1 1 1 1 1
10.9 1 1 1 1 1
11 1 1 1 1 1
11

Distributions over the unit circle Let $f : [0, \frac{\pi}{2}] \to [0, 1]$ be an increasing function. We say that a probability measure μ over the unit circle is f-equitable, if for each $0 \le \theta \le \theta' \le \frac{\pi}{2}$, we have that

$$\mu(\{z : \theta \le \theta(z) \le \theta'\}) \le f(\theta' - \theta). \quad (22)$$

Suppose, we define two functions $f_0(\theta) = \frac{\theta}{2\pi}$ and $f_1(\theta) = \frac{1}{1+\cos(\frac{\theta}{2})}$. Then, we may show that $f_0(\theta) \le f_1(\theta)$ for each $\theta \in [0, \frac{\pi}{2}]$. The next proposition concerns the connection between μ and f.

Proposition 2.8. *Suppose that μ is a probability measure over the unit circle. Then,*

1. *If $\mathsf{E}_\mu[z] = 0$, then μ is f_1-equitable.*
2. *μ is f_0-equitable if and only μ is the uniform distribution over the unit circle.*

Proof. The proof of 1 follows from Proposition 2.1. We prove part 2. One may check that uniform distribution is f_0-equitable. Now, suppose that μ is an arbitrary distribution that is f_0-equitable, then we show that it is indeed the uniform distribution. This follows from a simple observation that if E and F are two disjoint events with $\mu(E) + \mu(F) = \alpha$ and $\max\{\mu(E), \mu(F)\} \le \frac{\alpha}{2}$, then indeed $\mu(E) = \mu(F) = \frac{\alpha}{2}$. For each m, consider the partition $\Theta_m = \left\{\{z \in \mathsf{S}_1 : \theta(z) \in [\frac{2k\pi}{2^m}, \frac{2(k+1)\pi}{2^m})\}\right\}_{0 \le k \le 2^m-1}$, which is defined by considering arcs of equal length. Inductively, one may show that for each m, any two distinct events in the partition have probability equal to $\frac{1}{2m}$, which is the same as in the uniform distribution. Since the smallest sigma algebra generated by the union $\cup_m \Theta_m$ is the Borel sigma algebra, the result obtains.

3 Conclusion

We have shown some more results concerning the zeroes and non-zeroes of the Riemann zeta function. We have identified certain patterns from both theoretical and computational perspectives. The paper also highlights new mathematical connections which provide new insights that are related to the problem studied.
We hope that such insights will provide further results with regard to the fundamental problem concerning the Riemann Hypothesis, which may be interpreted as that of identifying non-zeroes.

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