Modelling Count Variables: A Comparative Analysis of two Discretization Techniques

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ABSTRACT

Background: Different discretization methods have been proposed to provide a better fit to count observations with characteristics resembling a given continuous distribution. This is done to provide discrete distribution with characteristics resembling a chosen continuous distribution. This study compares discretization through survival function and mixed Poisson processes.

Methodology: The Ailamujia distribution is extended using the cubic rank transmutation map. The shapes and some moment based properties of the continuous distribution are obtained. Two discretized versions of the distribution obtained are unimodal and skewed, depicting characteristics of the continuous distribution. Parameters of the new discrete distributions are estimated using the method of maximum likelihood, and both AIC and chi-square are used for model comparison.

Results: Real-life assessment using five count data shows that the two propositions provide a better fit than the three competing distributions considered. Also, discretization through the mixed Poisson process offers a better fit than the survival function technique.

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Conclusion: Various moment-based mathematical properties of the discretization through the mixed Poisson process are easily obtainable and hence, can be easily characterized.

Keywords: Discretization, survival function, mixed Poisson distribution, Ailamujia distribution.

1. INTRODUCTION

Some real lifetime data are discrete in observation even when they are primarily continuous in the real sense [1]. The discretization procedure was developed to improve efficiency in modelling count observations with shapes similar to a specific continuous distribution. The process involves using different mathematical concepts to derive discrete analogous to continuous distributions. Different approaches to discretizing a continuous distribution have been developed [2–4]. Among the prominent techniques for achieving this is the survival function of the continuous distribution, as was first used on the Weibull distribution [5–6]. A detailed survey of recent introductions in the discretization process has been reported [7].

If a continuous random variable has its CDF (distribution function) given as G_x and S_x is its survival function indexed with parameter vector Θ, the PMF (probability mass function) of a new discrete random variable P_x is obtained [8–9] as:

$$P_x = S_x - S_{x+1}$$

where

$$S_x = 1 - G_x$$

An advantage of this technique is that the survival function for discretized count distributions has a functional form resembling its corresponding continuous distributions [10]. Leveraging on these advantages, many notable continuous distributions have been discretized. Among these are the discrete Weibull [5, 11, 12]; discrete Rayleigh [9]; the discrete Lindley distribution [13]; the discrete Lomax distribution [14]; the discrete generalized exponential distribution [15]; discrete Marshall-Olkin Weibull [16]; discrete normal [8]; and discrete Pareto and discrete Burr [17].

Another technique of obtaining new discrete distribution involves utilizing the mixed Poisson [18] concept. The process involves assuming a continuous distribution with positive supports for the Poisson parameter. In most cases, the newly obtained discrete distribution's shape mimics the continuous distribution assumed for the parameter. Other notable characteristics of this distribution are presented in [19–21]. The procedure has received patronage in modelling datasets from actuary science in particular and dispersed observations in general [22–27]. Among many of the obtained discrete distributions in this paradigm include the mixed Poisson Lindley [28] and its generalizations [24, 29]. Another very popular application is the mixed Poisson-gamma distribution [18] which turned out to be a form of the negative binomial distribution with $p = \left(\frac{1}{1+\beta}\right)$. Different extensions of this distribution pervade literature [30, 31] with applications in diverse fields of studies [32–34].

Suppose discrete random variable *N* has the Poisson distribution with parameter *X*. Also, if *X* is assumed to follow a continuous random distribution with positive supports (0, ∞) with PDF denoted with g_x, a new discrete distribution is obtained in the mixed Poisson architecture by solving for the unconditional distribution for *N* in:

$$P_n = \int_0^\infty \frac{x^n e^{-x}}{n!} . g_x \, dx$$

Different distributions have been proposed for the choice of g_x [35]. The shape of g_x has a resemblance with the shape of the obtained discrete distribution from the process [36, 37].

In this study, a new continuous distribution is obtained using the cubic rank transmutation map [38] to extend the Ailamujia distribution [39]. Both the survival function [9] and the mixed Poisson [18] approaches of discretization are compared using the obtained continuous distribution.

2. AILAMUJIA DISTRIBUTION

The Ailamujia distribution [39] has been used to model lifetime observations that are skewed and unimodal [40]. Several authors [41–44] have obtained an improved version of the distribution using different compounding techniques. The distribution function for the Ailamujia distribution is defined as:

$$F_x = 1 - (1 + \beta x)e^{-\beta x}, \qquad \beta > 0 \tag{1}$$

Since introducing the quadratic transmutation map [45], many cubic transmutation maps that extend any baseline distribution pervade literature. The distribution function of the cubic rank transmutation (CRT) map of [38] is given as:

$$G_x = cF_x + (k - c)F_x^2 + (1 - k)F_x^3\,, \qquad c \in [0,1], \;\; k \in [-1,1], \tag{2}$$

2.1 Cubic Rank Transmuted Ailamujia Distribution

Some baseline distributions that have been extended using (2) include the inverse Rayleigh distribution [46], Gumbel distribution [47, 48], modified Burr III Pareto distribution [49], inverse Weibull distribution [50], Gompertz and Frechet distributions [48].
Inserting (1) into (2) gives the cubic rank transmuted Ailamujia (CRTA) distribution with CDF, PDF, and survival function respectively obtained as:

$$G_x = c\left(1-(1+\beta x)e^{-\beta x}\right) + (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 + (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 \tag{3}$$

$$g_x = \beta^2 x e^{-\beta x}\left(3-c-k+2(c+2k-3)(1+\beta x)e^{-\beta x} - 3(k-1)(1+\beta x)^2\left(e^{-2\beta x}\right)\right) \tag{4}$$

$$S_x = 1 - c\left(1-(1+\beta x)e^{-\beta x}\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 \tag{5}$$

Figure 1 shows different shapes of the PDF for the CRTA distribution for different parameter combinations. The figure reveals that the distribution is unimodal with positive skewness.

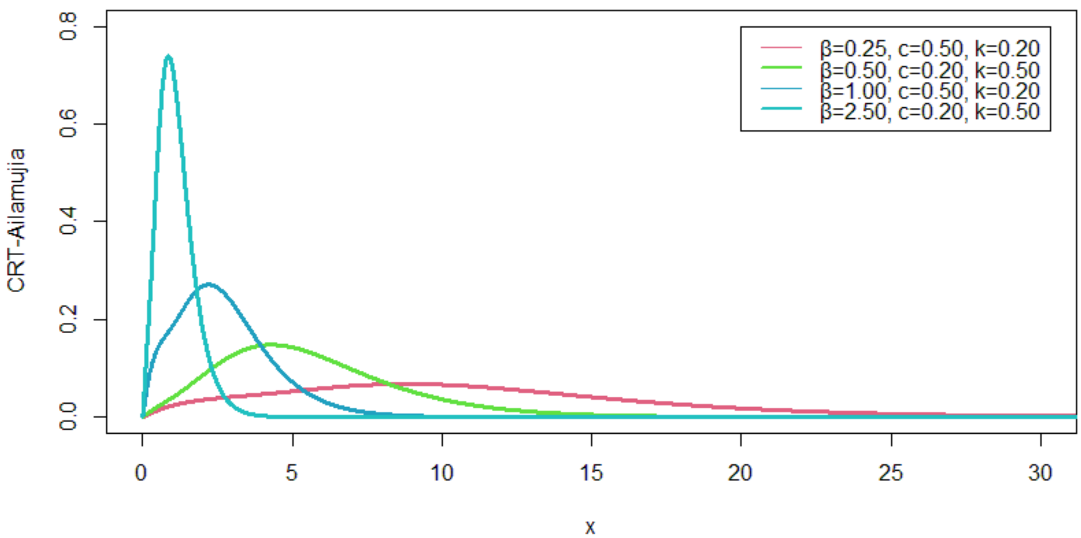

Figure 1. Shapes of PDF for CRTA distribution

2.2 Moments of the CRTA Distribution

Proposition 1. If a random variable X has a CRTA distribution, then the r*th* moment is obtained as:

$$\begin{aligned} E(x^r) = (3-c-k)(r+1)r! + \frac{2(c+2k-3)(r+1)r!}{4(2\beta)^r}\left(1+\frac{\beta(r+2)}{2\beta}\right) \\ - \frac{3(k-1)(r+1)r!}{9(3\beta)^r}\left(1+\frac{2\beta(r+2)}{3\beta}+\frac{(r+3)(r+2)}{(3\beta)^2}\right) \end{aligned} \tag{6}$$

Proof:

$$\begin{aligned} E(x^r) &= \int_0^\infty x^r g_x \, dx \\ &= \int_0^\infty x^r \left(\beta^2 x e^{-\beta x}\left(3-c-k+2(c+2k-3)(1+\beta x)e^{-\beta x} \right.\right. \\ &\qquad \left.\left. - 3(k-1)(1+\beta x)^2\left(e^{-2\beta x}\right)\right)\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \beta^2 x^{r+1} e^{-\beta x} \Big(3 - c - k + 2(c + 2k - 3)(1 + \beta x)e^{-\beta x} \\
&\qquad\qquad - 3(k - 1)(1 + \beta x)^2 \left(e^{-2\beta x}\right)\Big)\, dx \\
&= \beta^2 \int_0^\infty (3 - c - k)x^{r+1} e^{-\beta x} + 2(c + 2k - 3)\left(x^{r+1} e^{-2\beta x} + \beta x^{r+2} e^{-2\beta x}\right) \\
&\qquad\qquad - 3(k - 1)\left(x^{r+1} e^{-3\beta x} + 2\beta x^{r+2} e^{-3\beta x} + x^{r+3} e^{-3\beta x}\right) dx \\
&= \beta^2 \Bigg[\int_0^\infty (3 - c - k)x^{r+1} e^{-\beta x}\, dx + \int_0^\infty 2(c + 2k - 3)\left(x^{r+1} e^{-2\beta x} + \beta x^{r+2} e^{-2\beta x}\right) dx \\
&\qquad\qquad - \int_0^\infty 3(k - 1)\left(x^{r+1} e^{-3\beta x} + 2\beta x^{r+2} e^{-3\beta x} + x^{r+3} e^{-3\beta x}\right) dx\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \beta^2 \Bigg[(3 - c - k)\frac{(r + 1)r!}{\beta^2} + \frac{2(c + 2k - 3)(r + 1)r!}{(2\beta)^{r+2}}\left(1 + \frac{\beta(r + 2)}{2\beta}\right) \\
&\qquad\qquad - \frac{3(k - 1)(r + 1)r!}{(3\beta)^{r+2}}\left(1 + \frac{2\beta(r + 2)}{3\beta} + \frac{(r + 3)(r + 2)}{(3\beta)^2}\right)\Bigg] \\
&= (3 - c - k)(r + 1)r! + \frac{2(c + 2k - 3)(r + 1)r!}{4(2\beta)^r}\left(1 + \frac{\beta(r + 2)}{2\beta}\right) \\
&\qquad\qquad - \frac{3(k - 1)(r + 1)r!}{9(3\beta)^r}\left(1 + \frac{2\beta(r + 2)}{3\beta} + \frac{(r + 3)(r + 2)}{(3\beta)^2}\right)
\end{aligned}
$$

Hence, the first four moments of the CRTA distribution are obtained as:

$$m_1 = \frac{32(1 - k) + 9\beta^2(15c + 22k - 37) + 216\beta^3(3 - c - k)}{108\beta^3} \tag{7}$$

$$m_2 = \frac{160(1 - k) + 3\beta^2(243c + 398k - 641) + 1944\beta^3(3 - c - k)}{324\beta^4} \tag{8}$$

$$m_3 = \frac{320(1 - k) + \beta^2(1701c + 2986k - 4687) + 7776\beta^5(3 - c - k)}{324\beta^5} \tag{9}$$

$$m_4 = \frac{560(1 - k) + 15\beta^2(243c + 446k - 689) + 29160\beta^6(3 - c - k)}{243\beta^6} \tag{10}$$

3.0 DISCRETIZED TRANSMUTED AILAMUJIA DISTRIBUTION

Proposition 2: With the distribution function of the CRTA distribution obtained in (3) and the corresponding survival function obtained in (5), the discretized CRTA distribution (DCTA) is obtained as:

$P_x = S_x - S_{x+1},\ x = 0,1,2,\ldots$

Hence,

$$
\begin{aligned}
P_x &= c\left((1 + \beta x)e^{-\beta x} - 1\right) - (k - c)\left(1 - (1 + \beta x)e^{-\beta x}\right)^2 - (1 - k)\left(1 - (1 + \beta x)e^{-\beta x}\right)^3 \\
&\quad + c\left(1 - (1 + \beta(x + 1))e^{-\beta(x+1)}\right) + (k - c)\left(1 - (1 + \beta(x + 1))e^{-\beta(x+1)}\right)^2 \\
&\quad + (1 - k)\left(1 - (1 + \beta(x + 1))e^{-\beta(x+1)}\right)^3, \qquad x = 0,1,2,\ldots
\end{aligned} \tag{11}
$$

Special cases:

i. When $k = 1$, equation (11) becomes the DCTA I:

$$P_x = c\left((1+\beta x)e^{-\beta x} - 1\right) - (1-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (1-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2, \qquad x = 0,1,2,\ldots \tag{12}$$

ii. When $k = c$, equation (11) becomes the DCTA II

$$P_x = c\left((1+\beta x)e^{-\beta x} - 1\right) - (1-c)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (1-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3, \qquad x = 0,1,2,\ldots \tag{13}$$

iii. When $c = 0$, equation (11) becomes the DCTA III:

$$P_x = -k\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + k\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3, \qquad x = 0,1,2,\ldots \tag{14}$$

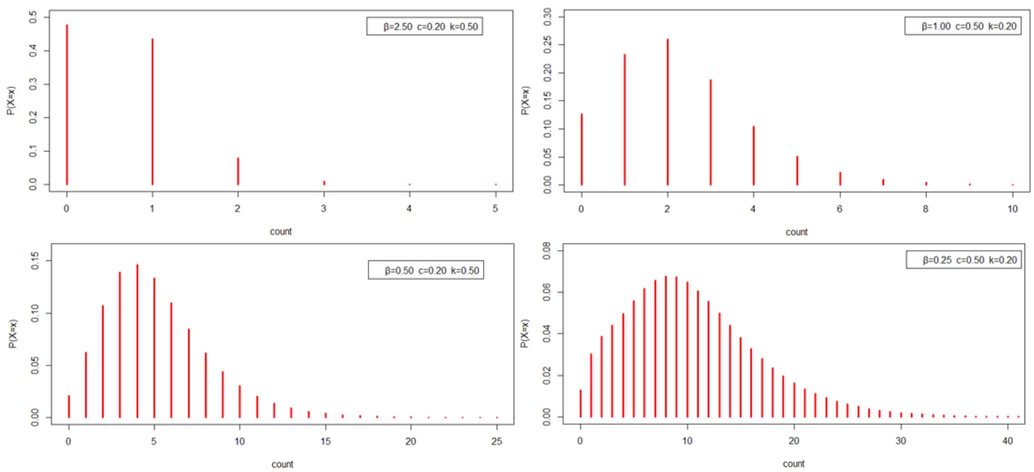

Figure 2. Shapes of PMF for DCTA distribution

The PMF of the DCTA distribution for different combinations of parameters show positive skewness, unimodality and resembles the shapes of the PDF of the CRTA distribution in figure 1.

If S_x is the survival function of the CRTA distribution, the distribution function (CDF) and the survival function for the DCTA distribution [10, 16] are obtained from:

$F(x) = 1 - S_x + P_x$

$S(x) = 1 - F(x) + P_x$

Hence, the CDF and survival functions are obtained as:

$$F(x) = c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3, \qquad x = 0,1,2,\ldots \tag{15}$$

$$S(x) = 1 - c\left(1-(1+\beta x)e^{-\beta x}\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3, \quad x = 0,1,2,\ldots \tag{16}$$

3.1 Moments of the DCTA Distribution

Proposition 3. If a random variable X has a CRTA distribution, then the r*th* moment of the DCTA distribution is obtained as:

$$E(x^r) = \sum_{x=0}^{\infty} x^r \Big[c\left((1+\beta x)e^{-\beta x} - 1\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3 \Big] \tag{15}$$

Proof:

$$E(x^r) = \mu_r' = \sum_{x=0}^{\infty} x^r P_x$$

$$= \sum_{x=0}^{\infty} x^r \Big[c\left((1+\beta x)e^{-\beta x} - 1\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3 \Big], \qquad r = 1,2,\ldots$$

In particular, the mean of the distribution is obtained from:

$$\mu_1' = \Big[c\sum_{x=0}^{\infty} x\left((1+\beta x)e^{-\beta x} - 1\right) - (k-c)\sum_{x=0}^{\infty} x\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\sum_{x=0}^{\infty} x\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\sum_{x=0}^{\infty} x\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\sum_{x=0}^{\infty} x\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\sum_{x=0}^{\infty} x\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3 \Big]$$

In general, there is no close form for the moments of the DCTA distribution.

3.2 MLE of the DCTA distribution

Proposition 4: Given that $x_1, x_2, \ldots, x_n$ are random samples of size *n* drawn from the DCTA distribution, the log-likelihood function for the distribution is obtained as

$$\mathcal{L} = \prod_{i=1}^{n} P_{x_i} = \prod_{i=1}^{n} \Big(c\left((1+\beta x)e^{-\beta x} - 1\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3 \Big)$$

$$\ell = log\,\mathcal{L} = \sum_{i=1}^{n} log\Big(c\left((1+\beta x)e^{-\beta x} - 1\right) - (k-c)\left(1-(1+\beta x)e^{-\beta x}\right)^2 - (1-k)\left(1-(1+\beta x)e^{-\beta x}\right)^3 + c\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right) + (k-c)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^2 + (1-k)\left(1-(1+\beta(x+1))e^{-\beta(x+1)}\right)^3 \Big)$$

Estimating the parameters (c, k, β) denoted with $(\hat{c}, \hat{k}, \hat{\beta})$ involves solving a system of non-linear equations. In this study, the *optimr* package [51] in the *R-language* [52] is used to obtain the estimates. A similar approach of parameter estimation was used in similar propositions [16].

4.0 MIXED POISSON CRTA DISTRIBUTION

Proposition 5. If $N \sim Poisson(X)$, where the PDF of X is given in equation (4), then the probability mass function (PMF) of the mixed Poisson CRTA distribution (MCTA) is obtained as:

$$P_n = \beta^2 \left[\frac{(3-c-k)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} - \frac{3(k-1)(n+1)}{(1+3\beta)^{n+4}}\left((1+3\beta)^2 + 2\beta(1+3\beta)(n+2) + \beta^2(n+3)(n+2)\right)\right] \tag{16}$$

Proof:

$$\begin{aligned}
P_n &= \int_0^\infty \frac{x^n e^{-x}}{n!} . g_x \, dx \\
&= \int_0^\infty \frac{x^n e^{-x}}{n!} \beta^2 x e^{-\beta x} \Big(3-c-k+2(c+2k-3)(1+\beta x)e^{-\beta x} \\
&\qquad - 3(k-1)(1+\beta x)^2\left(e^{-2\beta x}\right)\Big)\, dx \\
&= \frac{\beta^2}{n!}\Bigg[(3-c-k)\int_0^\infty x^{n+1} e^{-(1+\beta)x} dx + 2(c+2k-3)\int_0^\infty x^{n+1} e^{-(1+2\beta)x}(1+\beta x) dx - 3(k \\
&\qquad - 1)\int_0^\infty x^{n+1} e^{-(1+3\beta)x}(1+2\beta x+\beta^2 x^2)\, dx\Bigg] \\
&= \frac{\beta^2}{n!}\Bigg[\frac{(3-c-k)(n+1)n!}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)n!}{(1+2\beta)^{n+2}}\left(1+\frac{\beta(n+2)}{1+2\beta}\right) \\
&\qquad - \frac{3(k-1)(n+1)n!}{(1+3\beta)^{n+2}}\left(1+\frac{2\beta(n+2)}{1+3\beta}+\frac{\beta^2(n+3)(n+2)}{(1+3\beta)^2}\right)\Bigg] \\
&= \beta^2\Bigg[\frac{(3-c-k)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} \\
&\qquad - \frac{3(k-1)(n+1)}{(1+3\beta)^{n+4}}\left((1+3\beta)^2+2\beta(1+3\beta)(n+2)+\beta^2(n+3)(n+2)\right)\Bigg]
\end{aligned}$$

Special cases:

i. When $k = 1$, equation (16) becomes the MCTA I:

$$P_n = \beta^2 \left[\frac{(2-c)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c-1)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}}\right], \qquad x = 0,1,2,\ldots \tag{17}$$

ii. When $k = c$, equation (16) becomes the MCTA II:

$$P_n = \beta^2 \left[\frac{(3-2c)(n+1)}{(1+\beta)^{n+2}} + \frac{6(c-1)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} - \frac{3(c-1)(n+1)}{(1+3\beta)^{n+4}}\left((1+3\beta)^2 + 2\beta(1+3\beta)(n+2) + \beta^2(n+3)(n+2)\right)\right], \quad x = 0,1,2,\ldots \tag{18}$$

iii. When $c = 0$, equation (11) becomes the MCTA III:

$$P_n = \beta^2 \left[\frac{(3-c-k)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} \right.$$
$$\left. - \frac{3(k-1)(n+1)}{(1+3\beta)^{n+4}} \left((1+3\beta)^2 + 2\beta(1+3\beta)(n+2) + \beta^2(n+3)(n+2)\right) \right], \tag{19}$$
$$x = 0,1,2,\ldots$$

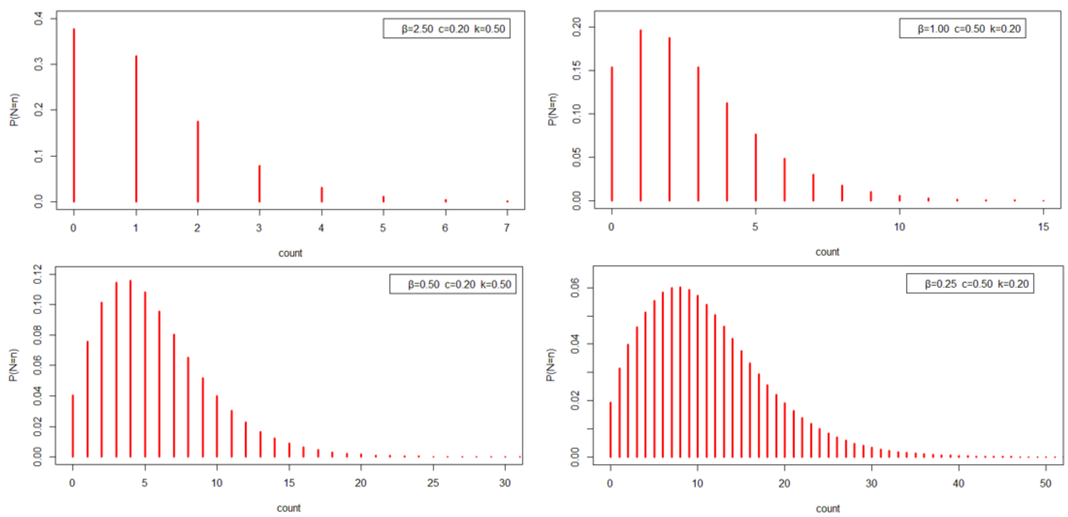

Figure 3. Shapes of PMF for the MCTA distribution

Figure 3 shows that the shapes of the MCTA distribution resemble the shapes of the PDF of the CRTA distribution. The shapes suggest that the distribution can model unimodal and positively skewed count observations.

4.1 Moment-Generating Function of the MCTA Distribution

Proposition 6. Given that g_n is the mixing distribution of a random variable *N* with the CRTA distribution, the probability generating function (PGF) of MCTA distribution is defined as:

$$P_n(z) = \int_0^\infty e^{n(z-1)} g_n \, dn$$
$$= \int_0^\infty e^{n(z-1)} \beta^2 n e^{-\beta n} \Big(3-c-k+2(c+2k-3)(1+\beta n)e^{-\beta n}$$
$$- 3(k-1)(1+\beta n)^2 \left(e^{-2\beta n}\right)\Big)\, dn$$
$$= \beta^2 \left[(3-c-k) \int_0^\infty n e^{-(1+\beta-z)n} dn + 2(c+2k-3) \int_0^\infty n e^{-(1+2\beta-z)n}(1+\beta n) dn - 3(k \right.$$
$$\left. - 1) \int_0^\infty n e^{-(+1+3\beta-z)n}(1+\beta n)^2 dn \right]$$
$$= \beta^2 \left[\frac{(3-c-k)}{(1+\beta-z)^2} + 2(c+2k-3) \int_0^\infty \left(n e^{-(1+2\beta-z)n} + \beta n^2 e^{-(1+2\beta-z)n}\right) dn - 3(k \right.$$
$$\left. - 1) \int_0^\infty n e^{-(+1+3\beta-z)n}(1+2\beta n + \beta^2 n^2) dn \right]$$

$$= \beta^2 \left[\frac{(3-c-k)}{(1+\beta-z)^2} + 2(c+2k-3)\left(\frac{1}{(1+2\beta-z)^2} + \frac{2}{(1+2\beta-z)^3} \right) - 3(k-1)\left(\frac{1}{(1+3\beta-z)^2} + \frac{4\beta}{(1+3\beta-z)^3} + \frac{6\beta^2}{(1+3\beta-z)^4} \right) \right]$$

Hence the PGF of the MCTA distribution is:

$$P_x(z) = \beta^2 \left[\frac{(3-c-k)}{(1+\beta-z)^2} + 2(c+2k-3)\left(\frac{1}{(1+2\beta-z)^2} + \frac{2}{(1+2\beta-z)^3} \right) - 3(k-1)\left(\frac{1}{(1+3\beta-z)^2} + \frac{4\beta}{(1+3\beta-z)^3} + \frac{6\beta^2}{(1+3\beta-z)^4} \right) \right] \quad (20)$$

Also, the moment generating function for the PMF in (16) is obtained by replacing z with e^t in (20). This is given as:

$$M_x(t) = \beta^2 \left[\frac{(3-c-k)}{(1+\beta-e^t)^2} + 2(c+2k-3)\left(\frac{1}{(1+2\beta-e^t)^2} + \frac{2}{(1+2\beta-e^t)^3} \right) - 3(k-1)\left(\frac{1}{(1+3\beta-e^t)^2} + \frac{4\beta}{(1+3\beta-e^t)^3} + \frac{6\beta^2}{(1+3\beta-e^t)^4} \right) \right] \quad (21)$$

From (21), the first four raw moments for the MCTA distribution are obtained as:

$$m_1 = \frac{2\beta(295-81c-106k) + 81(c+2k-3)}{108\beta^2} \quad (22)$$

$$m_2 = \frac{3\beta^2(295-81c-106k) + \beta(2399-729c-698k) + 243(c+2k-3)}{162\beta^3} \quad (23)$$

$$m_3 = \frac{6\beta^3(295-81c-106k) + 12\beta^2(1321-405c-430k) + 8\beta(2279-729c-578k) + 1215(c+2k-3)}{324\beta^4} \quad (24)$$

$$m_4 = \frac{18\beta^4(295-81c-106k) + 24\beta^3(4745-1458c-1586k) + 18\beta^3(20905-6723c-6406k) + 5\beta(55603-18225c-14050k) + 10935(c+2k-3)}{972\beta^5} \quad (25)$$

Hence, the variance for the distribution can be obtained from:

$$Var = m_2 - (m_1)^2$$

The index of dispersion, skewness and kurtosis for the MCTA distribution can be obtained using the moment-based relationships [53] respectively as:

$$DI = \frac{Var}{m_1}$$

$$S_k = \frac{m_3 - 3m_2m_1 + 2(m_1)^3}{(Var)^{\frac{3}{2}}}$$

$$Ku = \frac{m_4 - 4m_3m_1 + 6m_2(m_1)^2 - 3(m_1)^4}{(Var)^2}$$

4.2 MLE of the MCTA distribution

Proposition 7: Given that $n_1, n_2, \ldots, n_k$ are random samples of size *k* drawn from the MCTA distribution, the log-likelihood function for the distribution is obtained as

$$\mathcal{L} = \prod_{i=1}^{k} P_{n_i} = \prod_{i=1}^{k} \left(\beta^2 \left[\frac{(3-c-k)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} - \frac{3(k-1)(n+1)}{(1+3\beta)^{n+4}}\left((1+3\beta)^2 + 2\beta(1+3\beta)(n+2) + \beta^2(n+3)(n+2)\right) \right] \right)$$

$$\ell = log\,\mathcal{L} = \sum_{i=1}^{k} log\left(\beta^2\left[\frac{(3-c-k)(n+1)}{(1+\beta)^{n+2}} + \frac{2(c+2k-3)(n+1)(1+4\beta+n\beta)}{(1+2\beta)^{n+3}} - \frac{3(k-1)(n+1)}{(1+3\beta)^{n+4}}\left((1+3\beta)^2 + 2\beta(1+3\beta)(n+2) + \beta^2(n+3)(n+2)\right)\right]\right)$$

The parameter estimates of $(\hat{\beta}, \hat{c}, \hat{k})$ is obtained using the *optimr* package [51] in the *R-language* [52].

5.0 APPLICATIONS

The proposed distributions in this study are compared with (EDW) the exponentiated discrete Weibull distribution [17], (DMOG) the discrete Marshall-Olkin generalized exponential distribution [54], and (DBX) the discrete Bur XII distribution [55].

Table 1. PMF of the competing distribution

Distribution	PMF
EDW	$P_x = \left(1-\beta^{(x+1)^k}\right)^c - \left(1-\beta^{x^k}\right)^c$
DMOG	$P_x = \frac{k(1-(1-\beta^x)^c)}{k+(1-k)(1-\beta^x)^c} - \frac{k\left(1-\left(1-\beta^{(x+1)}\right)^c\right)}{k+(1-k)(1-\beta^{(x+1)})^c}$
DBX	$P_x = \beta^{\log(1+x^c)} - \beta^{\log(1+(x+1)^c)}$

Five real-life datasets are utilized to compare the new propositions and other competing distributions. The first dataset represents yeast cell counts per square, while the second dataset is the counts of the European red mites on apple leaves. Both data have been previously used in new propositions involving count distributions [16, 26, 56]. The third dataset is the frequency of epileptic seizures previously used on other discrete distributions [25, 57]. The fourth dataset on the number of mistakes in copying groups of random digits has been used to model various count distributions [25, 26, 28, 58]. The last dataset represents the number of strikes in a UK coal mining industries from 1948-1959 [16, 59].

RESULTS AND DISCUSSION

The maximum likelihood estimation technique using various non-linear algorithms that come with the *optimr* package in the *R-language* is used to obtain the estimates. The results obtained are presented in Tables 2 – 6. The Akaike Information Criterion (AIC) and the chi-square goodness of fits are used for model comparisons.

Table 2. Results of data on yeast cell counts per square

X	*Freq.*	*DCTA*	*DCTA I*	*DCTA II*	*DCTA III*	*MCTA*	*MCTA I*	*MCTA II*	*MCTA III*	*EDW*	*DMOG*	*DBX*
0	128	127.9	120.6	118.7	114.7	126.8	124.1	123.7	121.8	143.0	127.3	127.8
1	37	37.3	51.8	54.2	62.3	40.3	45.7	46.1	48.9	26.5	41.0	42.5
2	18	17.3	11.5	11.4	8.7	14.5	12.9	12.9	12.8	10.1	13.0	10.2
3	3	3.7	2.5	2.2	1.1	4.2	3.3	3.2	2.8	4.2	3.9	3.4
4	1	0.6	0.5	0.4	0.1	1.0	0.8	0.8	0.6	1.8	1.2	1.4
	β	2.06	1.79	1.93	2.33	5.73	3.82	4.21	5.14	0.56	0.30	0.19
	c	2.80	1.44	1.15	2.05	4.14	1.32	1.05	1.84	0.47	0.74	1.72
	k	-1.44				-5.54				1.44	1.58	
	χ^2	0.32	9.05	11.72	25.32	1.44	3.69	3.90	5.81	9.23	2.14	4.36
	AIC	343.04	349.48	351.68	365.78	344.39	344.74	344.85	346.34	344.62	345.82	349.61

Table 3. Results of data on the European red mites on Apple leaves

X	*Freq.*	*DCTA*	*DCTA I*	*DCTA II*	*DCTA III*	*MCTA*	*MCTA I*	*MCTA II*	*MCTA III*	*EDW*	*DMOG*	*DBX*
0	38	38.2	28.8	27.5	22.1	38.2	32.7	32.3	30.1	44.3	37.2	38.1
1	17	16.6	28.0	28.5	36.9	16.3	23.4	23.5	25.5	13.2	19.1	22.0
2	10	10.9	13.4	14.2	14.7	11.5	12.6	12.9	13.9	7.6	11.1	8.5
3	9	7.7	5.7	6.0	4.5	7.4	6.2	6.3	6.3	4.8	6.1	4.0
4	3	3.9	2.4	2.4	1.3	3.8	2.9	2.9	2.6	3.2	3.2	2.2
5	2	1.7	1.0	0.9	0.4	1.7	1.3	1.3	1.0	2.1	1.6	1.3

6	1	0.7	0.4	0.3	0.1	0.7	0.6	0.5	0.4	1.5	0.8	0.9
	β	1.13	1.06	1.16	1.46	2.38	1.54	1.70	2.08	0.78	0.50	0.39
	c	2.36	1.35	1.07	1.87	4.34	1.31	1.04	1.81	0.49	0.61	1.79
	k	-0.89				-6.01				1.48	2.16	
	χ^2	0.72	12.71	14.69	37.52	0.93	5.16	5.52	9.26	4.63	1.45	5.01
	AIC	240.89	249.73	251.51	273.43	241.09	243.34	243.55	246.78	241.96	242.77	250.21

Table 4. Results of data on the number of epileptic seizures

X	*Freq.*	*DCTA*	*DCTA I*	*DCTA II*	*DCTA III*	*MCTA*	*MCTA I*	*MCTA II*	*MCTA III*	*EDW*	*DMOG*	*DBX*
0	126	124.9	94.5	89.9	61.1	125.6	113.3	111.8	99.7	151.4	121.0	128.9
1	80	81.9	114.9	114.9	148.3	81.0	97.1	97.4	105.1	64.4	93.1	109.0
2	59	56.6	69.7	72.9	86.0	58.8	62.7	63.7	70.8	40.8	58.3	45.3
3	42	42.0	36.4	38.5	35.1	39.7	36.3	37.0	39.2	27.5	34.3	21.9
4	24	24.3	18.2	18.8	13.2	23.3	19.9	20.1	19.6	19.1	19.6	12.3
5	8	12.0	9.0	8.8	4.8	12.2	10.5	10.5	9.2	13.5	11.0	7.6
6	5	5.4	4.4	4.0	1.7	5.8	5.5	5.4	4.2	9.6	6.1	5.1
7	4	2.3	2.1	1.8	0.6	2.6	2.8	2.7	1.9	6.9	3.4	3.6
8	3	1.0	1.0	0.8	0.2	1.1	1.4	1.3	0.8	4.9	1.9	2.7
	β	1.01	0.87	0.95	1.20	1.78	1.20	1.30	1.54	0.80	0.55	0.52
	c	1.85	1.32	1.05	1.82	2.42	1.20	1.00	1.83	0.65	1.20	2.19
	k	-0.48				-2.92				1.44	1.17	
	χ^2	0.16	27.16	30.21	110.00	0.15	6.64	6.98	16.71	19.81	4.57	29.11
	AIC	1192.43	1215.20	1219.95	1317.17	1191.94	1195.45	1195.73	1206.71	1192.83	1196.78	1248.64

Table 5. Results of data on the number of mistakes in copying groups of random digits

X	*Freq.*	*DCTA*	*DCTA I*	*DCTA II*	*DCTA III*	*MCTA*	*MCTA I*	*MCTA II*	*MCTA III*	*EDW*	*DMOG*	*DBX*
0	35	35.0	29.1	28.1	25.4	34.9	31.2	31.0	29.8	38.9	34.5	34.9
1	11	10.9	20.4	21.2	26.2	11.4	17.2	17.4	18.8	9.8	13.9	14.8
2	8	8.4	7.1	7.5	6.6	7.9	7.2	7.3	7.7	4.9	6.3	5.0
3	4	3.9	*2.3*	2.3	1.4	3.8	2.7	2.8	2.6	2.7	2.9	2.1
4	2	1.3	0.7	0.7	0.3	1.4	1.0	1.0	0.8	1.5	1.3	1.1
	β	1.47	1.36	1.47	1.83	3.87	2.26	2.50	3.06	0.69	0.45	0.29
	c	2.55	1.38	1.10	1.92	6.95	1.31	1.04	1.78	0.48	0.82	1.61
	k	-1.16				-10.88				1.40	1.17	
	χ^2	0.29	10.68	12.54	27.11	0.23	4.91	5.17	7.48	2.30	1.69	3.68
	AIC	149.95	156.16	157.45	169.06	149.93	152.02	152.14	153.77	151.50	152.50	155.26

Table 6. Results of data on the number of strikes in UK coal mining industries

X	*Freq.*	*DCTA*	*DCTA I*	*DCTA II*	*DCTA III*	*MCTA*	*MCTA I*	*MCTA II*	*MCTA III*	*EDW*	*DMOG*	*DBX*
0	46	46.0	46.1	46.2	46.3	46.0	46.0	46.0	46.0	76.3	55.1	53.2
1	76	74.8	75.0	74.2	74.6	75.2	75.2	75.0	74.7	45.1	62.8	72.8
2	24	27.1	26.9	27.8	26.9	26.6	26.5	26.8	27.2	20.1	26.7	17.9
3	9	6.4	6.4	6.3	6.5	6.5	6.5	6.5	6.5	8.5	8.2	5.9
4	1	1.3	1.3	1.2	1.4	1.4	1.4	1.3	1.3	3.5	2.3	2.5
	β	1.86	1.84	1.93	1.84	5.82	5.76	6.24	6.86	0.59	0.27	0.55
	c	0.03	-0.03	0.27	0.96	-3.82	-3.97	-2.46	-7.79	1.34	2.38	3.86
	k	0.84				0.64				1.66	1.66	
	χ^2	1.27	1.20	1.52	1.22	1.08	1.08	1.12	1.22	39.72	6.14	6.18
	AIC	380.96	378.93	379.14	378.94	377.70	378.73	378.77	378.80	381.58	386.09	390.01

The parameter estimates, goodness of fit statistics, observed frequencies and expected frequencies when each proposition and competing distributions are assumed are presented in Tables 2 – 6. For dataset I and II, presented in Tables 2 and 3, the discretized cubic rank transmuted Ailamujia distribution (DCTA) has the least chi-square and AIC while the mixed Poisson cubic rank transmuted distribution (MCTA) provides the second best fit for both datasets.

The MCTA gives the best fit for datasets III, IV, and V, as shown in Tables 4 – 6. The second best fit for the three datasets is obtained when the DCTA is assumed. The MCTA and the DCTA provide a better fit than the three considered competing distributions. In most cases, the two-parameter special cases of the MCTA provide a relatively better fit to the dataset when compared with the special cases of the DCTA.

CONCLUSION

This study introduces two discrete versions of the continuous cubic rank transmuted Ailamujia distribution. The first version is obtained using the survival function of the continuous distribution. For the second version, the parameter of the classical Poisson distribution is assumed to follow the cubic rank transmuted Ailamujia distribution in the mixed Poisson architecture.

Both proposed distributions are unimodal and positively skewed. Five real-life count datasets are used to assess the flexibility of the new propositions. Comparisons are made between the two discretization techniques and three other discrete distributions. Parameters of the distributions are estimated using the method of maximum likelihood, and both AIC and chi-square are used for model comparison.

In all cases, the two propositions provide a better fit than the three competing distributions considered. Also, from the five real-life applications, the discretization through the mixed Poisson process provides a better fit than the survival function technique. Also, moment-based mathematical properties of the discretization through the mixed Poisson process are easily obtainable and hence, can be easily characterized.

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