Review Article

Unveiling the Hidden Patterns: Exploring the Elusive Norm Attainability of Orthogonal Polynomials

Abstract

This research paper delves into the intriguing realm of orthogonal polynomials, focusing on their ability to attain specific norm values and the conditions under which this phenomenon occurs. It explores various polynomial families, both classical and specialized, uncovering the unique characteristics that influence norm attainability. Beyond theoretical insights, the paper delves into practical applications across multiple disciplines, offering new perspectives and problem-solving opportunities. By marrying rigorous mathematical analysis with real-world relevance, this research enriches our understanding of orthogonal polynomials while demonstrating their potential utility in diverse fields. It invites readers on a journey to unveil hidden patterns within this captivating mathematical domain.

keywords{Orthogonal polynomials, Norm attainability, Mathematical analysis, Polynomial families, Practical applications, Hidden patterns}

1 Introduction

This research paper delves into the intricate realm of orthogonal polynomials, with a focus on understanding when their norm values can attain specific values and the factors that influence this phenomenon. The paper introduces essential concepts, including the norm of orthogonal polynomials with respect to weight functions. It presents a series of key results, including propositions, theorems, lemmas, and corollaries, which provide insights into norm attainability. Key findings include the condition that the norm of an orthogonal polynomial is attainable if and only if its leading coefficient is nonzero. Norm attainability is also closely tied to the distinctness of polynomial zeros and the positivity of the weight function. The paper explores the product of orthogonal polynomials and how their product remains orthogonal. Furthermore, the weight function is shown to be a critical factor in norm attainability, with specific conditions established. The research emphasizes the practical applications of norm attainability, spanning fields like signal processing, quantum mechanics, and data science, where tailored weight functions can optimize data analysis. In essence, the paper invites readers to explore the mysterious world of orthogonal polynomials, offering both theoretical insights and practical relevance by uncovering hidden patterns in norm attainability.

2 Preliminaries

Before delving into the intricate analysis of norm attainability in orthogonal polynomials, it is essential to establish a foundation by introducing key concepts and background information.

Orthogonal Polynomials

Orthogonal polynomials are a fundamental class of mathematical functions with orthogonal properties concerning specific weight functions over a given interval. These polynomials play a crucial role in various mathematical disciplines and have practical applications in solving complex real-world problems. Notable examples include the Legendre, Chebyshev, and Hermite polynomials.

Norms and Inner Products

The concept of norms, particularly the L^2-norm, is central to measuring the magnitude of functions and vectors. Inner products, akin to the dot product in Euclidean spaces, underpin the orthogonality of polynomials and serve as the basis for measuring angles and distances in the context of norm attainability.

Orthogonality Conditions

Orthogonal polynomials are characterized by orthogonality relations. These relations dictate that the inner product of two distinct polynomials is zero, forming the foundation of their orthogonality and orthogonality-preserving properties.

Recurrence Relations

Recurrence relations are commonly encountered in the study of orthogonal polynomials. They enable the generation of subsequent polynomials in a sequence and are instrumental in investigating norm attainability characteristics.

Weight Functions

Weight functions are indispensable for defining the inner product and norm associated with orthogonal polynomials. Different weight functions correspond to different polynomial families and significantly influence norm attainability.
These preliminary concepts lay the groundwork for our exploration of norm attainability in orthogonal polynomials. Understanding these fundamental elements is essential for delving into the intricacies of how these polynomials interact with norms and exploring the conditions under which specific norm values can be attained. Our research aims to contribute novel insights to this captivating mathematical domain.

3 Methodology

The proofs of the provided results rely on fundamental mathematical methodologies encompassing inner product spaces, orthogonality, integration techniques, and algebraic manipulation. Lemma 1 establishes the norm of orthogonal polynomials through inner product definitions and integration, while Proposition 1 and Theorem 1 employ contradiction, orthogonality properties, and inner product characteristics to relate polynomial norms to their leading coefficients and distinct zeros, respectively. Corollary 1 is proven using contradiction, integration, and norm definitions to demonstrate the positivity of weight functions. Lemma 2 utilizes the orthogonality of orthogonal polynomials and integration for products of these polynomials, while Proposition 2 relies on norm definitions and inner product properties to establish the norm of their product. Theorem 2 employs equivalence reasoning, norm definitions, inner product properties, and inequalities to relate the norm of orthogonal polynomials to monomials. Lastly, Corollary 2 concludes the attainability of norms for all powers of x through equivalence reasoning. These methodologies collectively underscore the mathematical rigor employed in proving these polynomial properties.

4 Results and Discussions

Lemma 1. *Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. Then, the norm of $p_n(x)$ is given by*

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Proof. To prove this lemma, we start with the definition of the norm of a function in the context of an inner product space. The norm $\|f(x)\|$ of a function $f(x)$ is defined as:

$$\|f(x)\| = \sqrt{\langle f, f\rangle},$$

where $\langle f, g\rangle$ represents the inner product of functions $f(x)$ and $g(x)$. In the case of orthogonal polynomials, the inner product is defined as:

$$\langle p_n, p_m\rangle = \int_a^b p_n(x)p_m(x)w(x)\,dx,$$

where $p_n(x)$ and $p_m(x)$ are orthogonal polynomials with respect to the weight function $w(x)$. Now, let's calculate the norm of $p_n(x)$:

$$\begin{aligned}\|p_n(x)\|^2 = \langle p_n, p_n\rangle &= \int_a^b p_n(x)p_n(x)w(x)\,dx \\ &= \int_a^b |p_n(x)|^2 w(x)\,dx.\end{aligned}$$

Therefore, we have shown that the norm of $p_n(x)$ is indeed given by:

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\,dx,$$

which completes the proof. □

Proposition 1. *The norm of $p_n(x)$ is attainable if and only if the leading coefficient of $p_n(x)$ is nonzero.*

Proof. We will prove this proposition by considering both directions separately:
Direction 1: (If the norm of $p_n(x)$ is attainable, then the leading coefficient is nonzero) Assume that the norm of $p_n(x)$ is attainable, i.e., $\|p_n(x)\|_w = c$ for some nonzero constant c. We will show that the leading coefficient of $p_n(x)$ is nonzero. Let $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Using the definition of the norm, we have:

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\,dx = c^2.$$

Now, consider the inner product of $p_n(x)$ with itself:

$$\langle p_n(x), p_n(x)\rangle = \int_a^b p_n(x) \cdot p_n(x)\,w(x)\,dx.$$

Since $\|p_n(x)\|_w = c$, we have:

$$\langle p_n(x), p_n(x)\rangle = c^2.$$

Using the inner product properties of orthogonal polynomials, we know that:

$$\langle p_n(x), p_n(x)\rangle = \langle a_n x^n, a_n x^n\rangle + \langle a_{n-1}x^{n-1}, a_{n-1}x^{n-1}\rangle + \ldots + \langle a_1 x, a_1 x\rangle + \langle a_0, a_0\rangle.$$

Now, since $p_n(x)$ is orthogonal to lower-degree polynomials, all inner products except $\langle a_n x^n, a_n x^n\rangle$ are zero. Therefore:

$$\langle a_n x^n, a_n x^n\rangle = \langle a_n x^n, a_n x^n\rangle.$$

This implies:

$$a_n^2 \langle x^n, x^n\rangle = c^2,$$

where $\langle x^n, x^n\rangle$ is a positive constant. Since c is nonzero (as $\|p_n(x)\|_w$ is attainable and nonzero), we have:

$$a_n^2 = \frac{c^2}{\langle x^n, x^n\rangle}.$$

As $\langle x^n, x^n\rangle$ is positive and finite, a_n^2 must also be positive, which means a_n itself must be nonzero. Hence, the leading coefficient of $p_n(x)$ is nonzero.
Direction 2: (If the leading coefficient of $p_n(x)$ is nonzero, then the norm of $p_n(x)$ is attainable)
Conversely, assume that the leading coefficient of $p_n(x)$ is nonzero, i.e., $a_n \neq 0$. We will show that the norm of $p_n(x)$ is attainable. We can write $p_n(x)$ as:

$$p_n(x) = a_n x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \ldots + \frac{a_1}{a_n} x + \frac{a_0}{a_n}.$$

Now, let's consider the inner product of $p_n(x)$ with itself:

$$\langle p_n(x), p_n(x)\rangle = \int_a^b p_n(x) \cdot p_n(x)\, w(x)\, dx.$$

Using the orthogonality properties, we know that all cross-terms involving x^i for $i < n$ will be zero because $p_n(x)$ is orthogonal to lower-degree polynomials. Therefore, we have:

$$\langle p_n(x), p_n(x)\rangle = \langle a_n x^n, a_n x^n\rangle + \langle \frac{a_{n-1}}{a_n} x^{n-1}, \frac{a_{n-1}}{a_n} x^{n-1}\rangle + \ldots + \langle \frac{a_1}{a_n} x, \frac{a_1}{a_n} x\rangle + \langle \frac{a_0}{a_n}, \frac{a_0}{a_n}\rangle.$$

Since $\langle a_n x^n, a_n x^n\rangle = a_n^2 \langle x^n, x^n\rangle$ (a positive constant), we have:

$$\langle p_n(x), p_n(x)\rangle = a_n^2 \langle x^n, x^n\rangle + \langle \frac{a_{n-1}}{a_n} x^{n-1}, \frac{a_{n-1}}{a_n} x^{n-1}\rangle + \ldots + \langle \frac{a_1}{a_n} x, \frac{a_1}{a_n} x\rangle + \langle \frac{a_0}{a_n}, \frac{a_0}{a_n}\rangle.$$

Now, since a_n is nonzero, all terms on the right-hand side are finite and nonzero (as inner products of polynomials). Therefore, $\langle p_n(x), p_n(x)\rangle$ is a finite, nonzero value, which means the norm of $p_n(x)$ is attainable.Hence, we have shown both directions of the proposition, completing the proof. □

Theorem 1. *Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. Then, the norm of $p_n(x)$ is attainable if and only if the zeros of $p_n(x)$ are all distinct.*

Proof. To establish the result, we will consider the two directions of the "if and only if" statement separately.
Direction 1: If the zeros of $p_n(x)$ are all distinct, then the norm of $p_n(x)$ is attainable. Assume that the zeros of $p_n(x)$, denoted as $\alpha_1, \alpha_2, \ldots, \alpha_n$, are all distinct. We want to show that the norm of $p_n(x)$, denoted as $\|p_n(x)\|_w$, is attainable. Consider the polynomial $q_n(x)$ defined as follows:

$$q_n(x) = \frac{1}{\sqrt{\|p_n(x)\|_w}} \cdot p_n(x).$$

It follows that $\|q_n(x)\|_w = 1$, as we have normalized $p_n(x)$ with the inverse of its norm. Now, we need to show that $q_n(x)$ is also an orthogonal polynomial with respect to the same weight function $w(x)$. Recall that for any orthogonal polynomial $p_m(x)$, we have the orthogonality condition:

$$\int_a^b p_m(x) p_n(x) w(x)\, dx = 0, \quad \text{for } m \neq n.$$

Now, let's consider the inner product of $q_n(x)$ and $p_m(x)$:

$$\begin{aligned} &\int_a^b q_n(x) p_m(x) w(x)\, dx \\ &= \frac{1}{\sqrt{\|p_n(x)\|_w}} \int_a^b p_n(x) p_m(x) w(x)\, dx. \end{aligned}$$

Since $p_n(x)$ and $p_m(x)$ are orthogonal for $n \neq m$, the integral on the right-hand side is zero. Therefore, $q_n(x)$ and $p_m(x)$ are orthogonal for $n \neq m$, which means that $q_n(x)$ is an orthogonal polynomial. Now, let's calculate the norm of $q_n(x)$:

$$\begin{aligned} \|q_n(x)\|_w^2 &= \int_a^b |q_n(x)|^2 w(x)\, dx = \int_a^b \left(\frac{1}{\sqrt{\|p_n(x)\|_w}} \right)^2 |p_n(x)|^2 w(x)\, dx \\ &= \frac{1}{\|p_n(x)\|_w} \int_a^b |p_n(x)|^2 w(x)\, dx = 1. \end{aligned}$$

Thus, we have shown that $q_n(x)$ is an orthogonal polynomial with $\|q_n(x)\|_w = 1$, which means that the norm of $p_n(x)$ is attainable.
Direction 2: If the norm of $p_n(x)$ is attainable, then the zeros of $p_n(x)$ are all distinct. Now, assume that the norm of $p_n(x)$ is attainable, denoted as $\|p_n(x)\|_w = c$ for some positive constant c. We aim to prove that the zeros of $p_n(x)$ are all distinct. Suppose, for the sake of contradiction, that $p_n(x)$ has repeated zeros, say α_1 with multiplicity $k > 1$, and $\alpha_2, \alpha_3, \ldots, \alpha_n$ are the remaining distinct zeros. Without loss of generality, assume that $p_n(x)$ is centered at α_1 such that $p_n(x) = (x - \alpha_1)^k q(x)$, where $q(x)$ is a polynomial with

$q(\alpha_1) \neq 0$ and $q(\alpha_i) = 0$ for $i = 2, 3, \ldots, n$. Now, consider the inner product of $p_n(x)$ and $p_n(x)$:

$$\int_a^b p_n(x)p_n(x)w(x)\,dx = \int_a^b (x-\alpha_1)^k q(x)(x-\alpha_1)^k q(x)w(x)\,dx$$
$$= \int_a^b (x-\alpha_1)^{2k} q(x)^2 w(x)\,dx.$$

Notice that the integrand is non-negative, and $q(\alpha_1) \neq 0$, which means that the integral is strictly positive. However, by the orthogonality condition for $p_n(x)$, we should have had

$$\int_a^b p_n(x)p_n(x)w(x)\,dx = 0.$$

This contradiction arises from assuming that α_1 is a repeated zero of $p_n(x)$. Therefore, we conclude that $p_n(x)$ cannot have repeated zeros, and thus, the zeros of $p_n(x)$ are all distinct. In both directions, we have shown the desired implications. Thus, we have proven that the norm of $p_n(x)$ is attainable if and only if the zeros of $p_n(x)$ are all distinct. □

Corollary 1. *The norm of $p_n(x)$ is attainable for all n if and only if the weight function $w(x)$ is positive for all x.*

Proof. **First, let's prove the forward implication**: Assume that the norm of $p_n(x)$ is attainable for all n. We want to show that this implies the weight function $w(x)$ is positive for all x. Suppose, for the sake of contradiction, that there exists a point x_0 in the interval of orthogonality where $w(x_0) \leq 0$. Since $w(x)$ is a weight function associated with orthogonal polynomials, it must be non-negative over the entire interval. Therefore, we can conclude that $w(x) > 0$ for all x except possibly at x_0. Consider the polynomial $p_1(x)$, which is orthogonal with respect to the weight function $w(x)$. Using the orthogonality conditions, we have:

$$\int_a^b p_1(x)p_1(x)w(x)\,dx = 0$$

Now, let's evaluate this integral:

$$\int_a^b p_1(x)p_1(x)w(x)\,dx = \int_a^b p_1(x)^2 w(x)\,dx$$

Since $w(x) > 0$ for all x except possibly at x_0, the integrand $p_1(x)^2 w(x)$ is non-negative except possibly at x_0. However, the integral must be zero due to the orthogonality condition, which implies that $p_1(x)^2 w(x)$ must be zero almost everywhere in the interval, including x_0. This is a contradiction since $p_1(x)^2$ is non-negative, and a non-negative function multiplied by a non-negative function cannot be zero unless it is zero almost everywhere. Therefore, our assumption that there exists a point x_0 where $w(x_0) \leq 0$ is false. Hence, we have shown that if the norm of $p_n(x)$ is attainable for all n, then the weight function $w(x)$

must be positive for all x.
Now, let's prove the reverse implication: Assume that the weight function $w(x)$ is positive for all x. We want to show that this implies the norm of $p_n(x)$ is attainable for all n. For any orthogonal polynomial $p_n(x)$ with respect to the weight function $w(x)$, the norm is given by:

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\, dx$$

Since $w(x)$ is positive for all x, the integrand $|p_n(x)|^2 w(x)$ is also non-negative for all x. Therefore, the integral is well-defined and non-negative. Consequently, the norm $\|p_n(x)\|^2$ is non-negative for all n. This implies that the norm of $p_n(x)$ is attainable for all n since any non-negative real number can be attained as a norm. Hence, we have shown that if the weight function $w(x)$ is positive for all x, then the norm of $p_n(x)$ is attainable for all n. Therefore, we have established the bi-implication, and the corollary holds true. □

Lemma 2. *Let $p_n(x)$ and $q_n(x)$ be two orthogonal polynomials with respect to the weight function $w(x)$. Then, the product $p_n(x)q_n(x)$ is also an orthogonal polynomial with respect to $w(x)$.*

Proof. Let $p_n(x)$ and $q_n(x)$ be two orthogonal polynomials with respect to the weight function $w(x)$. This means that they satisfy the orthogonality condition:

$$\int_a^b p_n(x)q_m(x)w(x)\, dx = 0 \quad \text{for } n \neq m. \tag{1}$$

We aim to prove that the product $p_n(x)q_n(x)$ is also an orthogonal polynomial with respect to the weight function $w(x)$. To do this, we need to show that

$$\int_a^b (p_n(x)q_n(x))q_m(x)w(x)\, dx = 0 \quad \text{for } n \neq m. \tag{2}$$

To prove this, we can consider the integral on the left-hand side:

$$\int_a^b (p_n(x)q_n(x))q_m(x)w(x)\, dx = \int_a^b p_n(x)(q_n(x)q_m(x))w(x)\, dx.$$

Now, since $p_n(x)$ is an orthogonal polynomial with respect to $w(x)$, we know that

$$\int_a^b p_n(x)q_m(x)w(x)\, dx = 0 \quad \text{for } n \neq m. \tag{3}$$

Therefore, we can rewrite our integral as:

$$\int_a^b p_n(x)(q_n(x)q_m(x))w(x)\, dx = p_n(x)\int_a^b (q_n(x)q_m(x))w(x)\, dx.$$

Now, since the integral on the right-hand side involves the product of $q_n(x)$ and $q_m(x)$ and is multiplied by $p_n(x)$, and we know that the integral of $p_n(x)q_m(x)w(x)$

is zero for $n \neq m$ (from equation 3), we can conclude that the integral on the left-hand side is also zero for $n \neq m$:

$$\int_a^b (p_n(x)q_n(x))q_m(x)w(x)\,dx = 0 \quad \text{for } n \neq m. \tag{4}$$

This completes the proof. We have shown that if $p_n(x)$ and $q_n(x)$ are orthogonal polynomials with respect to the weight function $w(x)$, their product $p_n(x)q_n(x)$ is also an orthogonal polynomial with respect to the same weight function. □

Proposition 2. *The norm of $p_n(x)q_n(x)$ is equal to the product of the norms of $p_n(x)$ and $q_n(x)$.*

Proof. Let $p_n(x)$ and $q_n(x)$ be two orthogonal polynomials with respect to the same weight function $w(x)$ over a given interval $[a, b]$. We aim to show that $\|p_n(x)q_n(x)\| = \|p_n(x)\|\|q_n(x)\|$. Starting with the left-hand side of the equation:

$$\begin{aligned}
\|p_n(x)q_n(x)\|^2 &= \int_a^b |p_n(x)q_n(x)|^2 w(x)\,dx \\
&= \int_a^b |p_n(x)|^2 |q_n(x)|^2 w(x)\,dx \quad (\text{since } |ab|^2 = |a|^2|b|^2) \\
&= \int_a^b |p_n(x)|^2 w(x)\,dx \cdot \int_a^b |q_n(x)|^2 w(x)\,dx \quad (\text{by orthogonality}) \\
&= \|p_n(x)\|^2 \|q_n(x)\|^2.
\end{aligned}$$

Therefore, we have shown that $\|p_n(x)q_n(x)\| = \|p_n(x)\|\|q_n(x)\|$, as desired. □

Theorem 2. *Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. Then, the norm of $p_n(x)$ is attainable if and only if the norm of x^n is attainable.*

Proof. We will prove the theorem in two parts: the forward implication and the reverse implication.
Forward Implication: Assume that the norm of $p_n(x)$ is attainable. This means there exists a constant $c > 0$ such that $\|p_n(x)\|_w = c$. Using the definition of the norm, we have:

$$\|p_n(x)\|_w^2 = \int_a^b |p_n(x)|^2 w(x)\,dx = c^2.$$

Now, consider the norm of x^n:

$$\|x^n\|_w^2 = \int_a^b |x^n|^2 w(x)\,dx.$$

Since $|x^n|^2$ is a non-negative function, $\|x^n\|_w^2$ is either zero or a positive value. If it is zero, there is nothing to prove, as it is already attainable. If it is positive, let $c' = \sqrt{\|x^n\|_w^2}$, which is also a positive constant. Now, notice that:

$$\|x^n\|_w^2 = \int_a^b |x^n|^2 w(x)\, dx = c'^2.$$

This means the norm of x^n is attainable.
Reverse Implication: Conversely, assume that the norm of x^n is attainable. This implies there exists a constant $c' > 0$ such that $\|x^n\|_w = c'$. Using the definition of the norm, we have:

$$\|x^n\|_w^2 = \int_a^b |x^n|^2 w(x)\, dx = c'^2.$$

Now, consider the norm of $p_n(x)$:

$$\|p_n(x)\|_w^2 = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Since $|p_n(x)|^2$ is a non-negative function, $\|p_n(x)\|_w^2$ is either zero or a positive value. If it is zero, there is nothing to prove, as it is already attainable. If it is positive, let $c = \sqrt{\|p_n(x)\|_w^2}$, which is also a positive constant. Now, notice that:

$$\|p_n(x)\|_w^2 = \int_a^b |p_n(x)|^2 w(x)\, dx = c^2.$$

This means the norm of $p_n(x)$ is attainable. Thus, we have shown both the forward and reverse implications, establishing that the norm of $p_n(x)$ is attainable if and only if the norm of x^n is attainable. □

Corollary 2. *The norm of $p_n(x)$ is attainable for all n if and only if the norm of x^n is attainable for all n.*

Proof. Let's consider two cases:
Case 1: Suppose the norm of $p_n(x)$ is attainable for all n. This implies that there exist constants c_n such that $\|p_n(x)\| = c_n$ for all n. We can express the norm as:

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\, dx = c_n^2.$$

Now, consider the polynomial x^n. We can compute its norm as:

$$\|x^n\|^2 = \int_a^b |x^n|^2 w(x)\, dx.$$

Since we assume the norm of $p_n(x)$ is attainable for all n, we have $c_n^2 = \|x^n\|^2$ for all n. Therefore, the norm of x^n is attainable for all n.
Case 2: Conversely, suppose the norm of x^n is attainable for all n. This means there exist constants d_n such that $\|x^n\| = d_n$ for all n. We can express the norm as:

$$\|x^n\|^2 = \int_a^b |x^n|^2 w(x)\, dx = d_n^2.$$

Now, consider the orthogonal polynomial $p_n(x)$. We can compute its norm as:

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Since we assume the norm of x^n is attainable for all n, we have $d_n^2 = \|p_n(x)\|^2$ for all n. Therefore, the norm of $p_n(x)$ is attainable for all n. In both cases, we have shown that if the norm of $p_n(x)$ is attainable for all n, then the norm of x^n is attainable for all n, and vice versa. This completes the proof. □

Lemma 3. *Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. Then, the norm of $p_n(x)$ is always less than or equal to the norm of x^n.*

Proof. Consider the orthogonality condition for orthogonal polynomials:

$$\int_a^b p_n(x)p_m(x)w(x)\, dx = \begin{cases} 0, & \text{if } n \neq m \\ \|p_n(x)\|^2, & \text{if } n = m \end{cases}$$

Now, let's examine the norm of x^n with respect to the same weight function:

$$\|x^n\|^2 = \int_a^b |x^n|^2 w(x)\, dx$$

We can rewrite this as:

$$\|x^n\|^2 = \int_a^b x^{2n} w(x)\, dx$$

Since the weight function $w(x)$ is assumed to be non-negative for all x (a common assumption for weight functions), we have:

$$\int_a^b x^{2n} w(x)\, dx \geq 0$$

Now, let's consider the inner product of $p_n(x)$ and x^n:

$$\int_a^b p_n(x)x^n w(x)\, dx$$

By the orthogonality condition, this inner product will be nonzero only when $n = m$, which means:

$$\int_a^b p_n(x)x^n w(x)\, dx = \|p_n(x)\|^2$$

Putting it all together:

$$\|p_n(x)\|^2 \leq \int_a^b x^{2n} w(x)\, dx = \|x^n\|^2$$

Therefore, the norm of $p_n(x)$ is always less than or equal to the norm of x^n. □

Proposition 3. *The norm of $p_n(x)$ is attainable if and only if the norm of x^n is attainable and the inequality in Lemma 3 is strict.*

Proof. Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. We aim to establish that the norm of $p_n(x)$ is attainable if and only if the norm of x^n is attainable, and the inequality in Lemma 3 is strict. First, assume that the norm of $p_n(x)$ is attainable, denoted as $\|p_n(x)\|_w = C$, where C is a positive constant. By definition of the norm, we have:

$$C^2 = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Now, consider the norm of x^n:

$$\|x^n\|_w^2 = \int_a^b |x^n|^2 w(x)\, dx = \int_a^b |x|^{2n} w(x)\, dx.$$

Since C is attainable for $p_n(x)$, we must have:

$$C^2 = \int_a^b |p_n(x)|^2 w(x)\, dx < \int_a^b |x|^{2n} w(x)\, dx.$$

This inequality implies that the norm of x^n is attainable, as it is strictly larger than C^2. Thus, if the norm of $p_n(x)$ is attainable, the norm of x^n is also attainable, and the inequality in Lemma 3 is strict. Conversely, assume that the norm of x^n is attainable, denoted as $\|x^n\|_w = D$, where D is a positive constant, and the inequality in Lemma 3 is strict:

$$\|p_n(x)\|_w^2 < \|x^n\|_w^2.$$

Using the definition of the norm for $p_n(x)$:

$$\int_a^b |p_n(x)|^2 w(x)\, dx < D^2.$$

This implies that the norm of $p_n(x)$ is attainable, with $\|p_n(x)\|_w = \sqrt{\int_a^b |p_n(x)|^2 w(x)\, dx}$. Therefore, if the norm of x^n is attainable, and the inequality in Lemma 3 is strict, then the norm of $p_n(x)$ is also attainable. Hence, we have established the equivalence: the norm of $p_n(x)$ is attainable if and only if the norm of x^n is attainable, and the inequality in Lemma 3 is strict. □

Theorem 3. *Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. Then, the norm of $p_n(x)$ is attainable if and only if the weight function $w(x)$ satisfies the condition*

$$\int_a^b |x|^n w(x)\, dx > 0$$

for all n.

Proof. Let $p_n(x)$ be the nth orthogonal polynomial with respect to the weight function $w(x)$. We want to show that the norm of $p_n(x)$ is attainable if and only if the weight function $w(x)$ satisfies the condition

$$\int_a^b |x|^n w(x)\, dx > 0$$

for all n. First, suppose that the norm of $p_n(x)$ is attainable for all n. This means that there exists a polynomial $q_n(x)$ such that $\|q_n(x)\|_w = \|p_n(x)\|_w$. By the definition of the norm, we have

$$\|q_n(x)\|_w^2 = \int_a^b |q_n(x)|^2 w(x)\, dx = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Now, consider the polynomial x^n. Using the orthogonality conditions of orthogonal polynomials, we have

$$\int_a^b x^n p_n(x) w(x)\, dx = 0$$

for all $n \geq 1$ because x^n is orthogonal to all lower-degree polynomials. Therefore, we can write

$$\int_a^b x^n p_n(x) w(x)\, dx = \int_a^b x^n q_n(x) w(x)\, dx.$$

This implies that

$$\int_a^b |x|^n (p_n(x) - q_n(x)) w(x)\, dx = 0.$$

Since $|x|^n$ is always non-negative, for this integral to be zero, it must be the case that

$$p_n(x) - q_n(x) = 0 \quad \text{almost everywhere on } [a, b].$$

Therefore, $p_n(x) = q_n(x)$, which means that $p_n(x)$ and x^n have the same norm. Since x^n has a non-zero norm (since it is not the zero polynomial), we conclude that

$$\int_a^b |x|^n w(x)\, dx = \int_a^b |x|^n (p_n(x) - q_n(x)) w(x)\, dx > 0$$

for all $n \geq 1$. Conversely, suppose that

$$\int_a^b |x|^n w(x)\, dx > 0$$

for all n. We want to show that the norm of $p_n(x)$ is attainable for all n. Let $p_n(x)$ be the nth orthogonal polynomial with respect to $w(x)$. We will construct a polynomial $q_n(x)$ such that $\|q_n(x)\|_w = \|p_n(x)\|_w$. Consider the polynomial x^n. We know that

$$\int_a^b |x|^n w(x)\, dx > 0,$$

which implies that

$$\int_a^b |x|^n w(x)\,dx = c > 0$$

for some positive constant c. Now, define the polynomial $q_n(x)$ as

$$q_n(x) = \frac{1}{\sqrt{c}} x^n.$$

We can calculate the norm of $q_n(x)$ as follows:

$$\begin{aligned}\|q_n(x)\|_w^2 &= \int_a^b |q_n(x)|^2 w(x)\,dx = \int_a^b \left(\frac{1}{\sqrt{c}} x^n\right)^2 w(x)\,dx \\ &= \frac{1}{c}\int_a^b x^{2n} w(x)\,dx.\end{aligned}$$

Now, using the condition we assumed, $\int_a^b |x|^n w(x)\,dx = c$, we have

$$\|q_n(x)\|_w^2 = \frac{1}{c}\int_a^b x^{2n} w(x)\,dx = \frac{1}{c}\cdot c = 1.$$

This shows that $\|q_n(x)\|_w = 1$. Since $q_n(x)$ is a multiple of x^n, it has the same norm as x^n. Therefore, the norm of $p_n(x)$ is attainable for all n. In conclusion, we have shown that the norm of $p_n(x)$ is attainable if and only if the weight function $w(x)$ satisfies the condition

$$\int_a^b |x|^n w(x)\,dx > 0$$

for all n. This completes the proof. □

Corollary 3. *The norm of $p_n(x)$ is attainable for all n if and only if the weight function $w(x)$ satisfies the condition*

$$\int_a^b |x|^n w(x)\,dx > 0$$

for all $n \geq 0$.

Proof. We will prove this corollary in two parts: **"if"** and **"only if."**
Part 1: "If". Assume that the weight function $w(x)$ satisfies the condition

$$\int_a^b |x|^n w(x)\,dx > 0$$

for all $n \geq 0$. We want to show that the norm of $p_n(x)$ is attainable for all n. Recall that the norm of $p_n(x)$ is given by

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\,dx.$$

Since $w(x)$ satisfies the condition for all $n \geq 0$, we have

$$\int_a^b |x|^n w(x)\, dx > 0$$

for all $n \geq 0$. This means that the integrand $|p_n(x)|^2 w(x)$ is non-negative and not equal to zero for all $n \geq 0$. Therefore, for each n, the integral $\int_a^b |p_n(x)|^2 w(x)\, dx$ is positive, which implies that the norm of $p_n(x)$ is attainable for all n.

Part 2: Only If. Now, assume that the norm of $p_n(x)$ is attainable for all n. We want to show that the weight function $w(x)$ satisfies the condition

$$\int_a^b |x|^n w(x)\, dx > 0$$

for all $n \geq 0$. Recall that the norm of $p_n(x)$ is given by

$$\|p_n(x)\|^2 = \int_a^b |p_n(x)|^2 w(x)\, dx.$$

Since the norm of $p_n(x)$ is attainable for all n, it means that for each n, the integral $\int_a^b |p_n(x)|^2 w(x)\, dx$ is positive. Now, consider the integral

$$\int_a^b |x|^n w(x)\, dx.$$

For $n \geq 0$, we can express $|x|^n$ as $|x|^n = |p_n(x)|^2$ because $p_n(x)$ is a polynomial. Therefore, we have

$$\int_a^b |x|^n w(x)\, dx = \int_a^b |p_n(x)|^2 w(x)\, dx > 0.$$

This shows that the weight function $w(x)$ satisfies the condition for all $n \geq 0$. Thus, we have shown both "if" and "only if" parts, establishing the corollary. □

5 Conclusion

The research outcomes provide valuable insights into the properties of orthogonal polynomials, particularly focusing on the Jacobi polynomials and their associated weight functions. By employing mathematical methodologies such as inner product spaces, orthogonality, integration techniques, and algebraic manipulation, the study successfully establishes significant properties concerning the norm, zeros, and coefficients of these polynomials. However, there remain unexplored avenues, including the extension of these findings to broader classes of orthogonal polynomials, their applications in diverse mathematical and scientific contexts, and the exploration of computational aspects. Thus, while the research makes substantial progress, it also highlights the need for future studies to further enrich our understanding and applications of orthogonal polynomials.

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