Properties and Convergence Analysis of Orthogonal Polynomials, Reproducing Kernels, and Bases in Hilbert Spaces Associated with Norm-Attainable Operators

Original Research Article

Abstract

This research paper delves into the properties and convergence behaviors of various sequences of orthogonal polynomials, reproducing kernels, and bases within Hilbert spaces governed by norm-attainable operators. Through rigorous analysis, the study establishes the completeness of the sequences of monic orthogonal polynomials and orthonormal polynomials, highlighting their comprehensive representation and approximation capabilities in the Hilbert space. The paper also demonstrates the completeness and density attributes of the sequence of normalized reproducing kernels, showcasing its effective role in capturing the intrinsic structure of the space. Additionally, the research investigates the uniform convergence of these sequences, revealing their convergence to essential operators within the Hilbert space. Ultimately, these results contribute to both theoretical understanding and practical applications in various fields by providing insights into function approximation and representation within this mathematical framework.: **keywords**{norm-attainable operators, Hilbert spaces, orthogonal polynomials, reproducing kernels, completeness, density, uniform convergence, basis, approximation, function representation, mathematical analysis, convergence behavior.}

1 Introduction

The introduction of this research paper initiates an exploration into the intricate relationship between norm-attainable operators, Hilbert spaces, and their associated sequences of orthogonal polynomials, reproducing kernels, and bases. The study delves into the significance of this relationship within the context of functional analysis. With a focus on the unique properties of norm-attainable operators, the investigation aims to uncover the completeness, density, and convergence behaviors of these sequences, offering insights into the representation and approximation of functions. The analysis encompasses monic orthogonal and orthonormal polynomials, as well as normalized reproducing kernels, showcasing their convergence to fundamental operators in the Hilbert space. By combining theoretical understanding with practical implications, the research contributes to both the mathematical foundations and real-world applications of this dynamic framework.

2 Preliminaries

This research paper delves into the study of sequences of polynomials associated with norm-attainable operators on Hilbert spaces. These operators play a crucial role in functional analysis and have significant applications in various mathematical and scientific domains. Before delving into the main results and their proofs, it is essential to establish some foundational concepts. A norm-attainable operator is defined as an operator on a Hilbert space that can be approximated arbitrarily closely in the operator norm by finite-rank operators. The paper focuses on investigating the properties of three distinct sequences of polynomials: monic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernels. These sequences are intricately linked to the operator's properties and exhibit unique characteristics within the Hilbert space. The subsequent sections present a series of theorems and lemmas that elucidate the orthogonality, completeness, density, and uniform convergence properties of these polynomial sequences. The proofs rely on key concepts such as orthogonality, the reproducing property of kernels, and the uniform boundedness principle. By establishing these preliminary results, the paper lays the foundation for a deeper exploration of the intricate interplay between operators and polynomials in the context of Hilbert spaces.

3 Methodology

The presented series of results and their proofs establish fundamental properties of sequences of polynomials associated with a norm-attainable operator on a Hilbert space. These properties include the orthogonality of monic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernels with the constant function 1. Additionally, the completeness and density of these sequences in the Hilbert space are demonstrated. Furthermore, the uniform convergence of these sequences to the identity operator and the Dirac delta measure is established. The methodology employed involves utilizing known properties of Hilbert spaces, the reproducing property of kernels, and leveraging the characteristics of the norm-attainable operator. Through systematic reasoning and well-defined steps, the proofs establish the basis property of the sequences of polynomials, concluding their comprehensive understanding in the context of the given Hilbert space.

4 Results and Discussions

Lemma 4.1. *Let T be a norm-attainable operator on a Hilbert space H. Then the monic orthogonal polynomials with respect to T are orthogonal to the constant function 1.*

Proof. Let $P_n(x)$ be the monic orthogonal polynomial of degree n with respect to T. Then $\langle P_n(x), 1\rangle = 0$ for all $n \geq 0$. To prove this, we can use the fact that $P_n(x)$ is the unique polynomial of degree n that satisfies the following two conditions:
1. $P_n(x)$ is orthogonal to all polynomials of degree less than n.
2. $\langle P_n(x), P_n(x)\rangle = 1$.
Condition 1 implies that $\langle P_n(x), 1\rangle = 0$ for all $n \geq 1$. Condition 2 implies that $\langle P_0(x), 1\rangle = 0$. Therefore, the monic orthogonal polynomials with respect to T are orthogonal to the constant function 1. □

Lemma 4.2. *Let T be a norm-attainable operator on a Hilbert space H. Then the orthonormal polynomials with respect to T are orthogonal to the constant function 1.*

Proof. Let $Q_n(x)$ be the orthonormal polynomial of degree n with respect to T. Then

$$\langle Q_n(x), 1\rangle = 0$$

for all $n \geq 0$. To prove this, we can use the fact that $Q_n(x)$ is the unique polynomial of degree n that satisfies the following two conditions:
1. $Q_n(x)$ is orthogonal to all polynomials of degree less than n.
2. $\langle Q_n(x), Q_n(x)\rangle = \frac{1}{n!}$.
Condition 1 implies that $\langle Q_n(x), 1\rangle = 0$ for all $n \geq 1$. Condition 2 implies that $\langle Q_0(x), 1\rangle = 0$. Therefore, the orthonormal polynomials with respect to T are orthogonal to the constant function 1. Alternatively:
Let $Q_n(x)$ be the orthonormal polynomial of degree n with respect to T. Then

$$\langle Q_n(x), 1\rangle = \frac{\langle Q_n(x), TQ_n(x)\rangle}{\langle Q_n(x), Q_n(x)\rangle} = 0$$

for all $n \geq 0$. The first equality follows from the definition of the inner product. The second equality follows from the fact that $Q_n(x)$ is orthogonal to all polynomials of degree less than n. Therefore, the orthonormal polynomials with respect to T are orthogonal to the constant function 1. □

Lemma 4.3. *Let T be a norm-attainable operator on a Hilbert space H. Then the normalized reproducing kernels with respect to T are orthogonal to the constant function 1.*

Proof. Let $K_x(y)$ be the normalized reproducing kernel with respect to T, where $x, y \in H$. Then

$$\langle K_x(y), 1 \rangle = 0$$

for all $x, y \in H$. To prove this, we can use the fact that $K_x(y)$ is the unique function in H that satisfies the following two conditions:
1. $K_x(y)$ is the reproducing kernel for T, i.e., $\langle K_x(y), Tf \rangle = f(y)$ for all $f \in H$.
2. $\langle K_x(y), K_x(y) \rangle = 1$.
Condition 1 implies that $\langle K_x(y), 1 \rangle = \langle K_x(y), Ty \rangle = y(x)$ for all $y \in H$. Condition 2 implies that $\langle K_x(x), K_x(x) \rangle = 1$. Setting $y = x$ in the first equation, we get

$$\langle K_x(x), 1 \rangle = x(x) = \langle K_x(x), K_x(x) \rangle$$

Therefore, $\langle K_x(x), 1 \rangle = 0$ for all $x \in H$. Since $K_x(y)$ is uniquely determined by these two conditions, it follows that $\langle K_x(y), 1 \rangle = 0$ for all $x, y \in H$. □

Proposition 4.1. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T is complete in H.*

Proof. Let $f \in H$ be such that $\langle f, P_n(x) \rangle = 0$ for all $n \geq 0$. Then

$$\langle f, P_n(x) \rangle = \langle f, T^n \rangle$$

for all $n \geq 0$. This means that f is orthogonal to the range of T.Since T is norm-attainable, the range of T is dense in H. Therefore, $f = 0$. This shows that the sequence of monic orthogonal polynomials with respect to T is complete in H. □

Theorem 4.1. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T converges uniformly to the identity operator on H.*

Proof. Let $P_n(x)$ be the monic orthogonal polynomial of degree n with respect to T. Then

$$\langle P_n(x), P_m(x) \rangle = \delta_{nm}$$

for all $n, m \geq 0$. We can write the identity operator on H as

$$I = \sum_{n=0}^{\infty} P_n(x) \langle P_n(x), \cdot \rangle$$

To prove that the sequence of monic orthogonal polynomials converges uniformly to the identity operator, we need to show that

$$\lim_{n \to \infty} \|P_n(x) - I\| = 0$$

for all $x \in H$. Let $x \in H$. Then

$$\begin{aligned}\|P_n(x) - I\|^2 &= \|P_n(x)\|^2 + \|I\|^2 - 2\langle P_n(x), I\rangle \\ &= \|P_n(x)\|^2 + 1 - 2\langle P_n(x), P_n(x)\rangle \\ &= \|P_n(x)\|^2 - 1.\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\|P_n(x)\|^2 \leq \|P_n(x)\|\|x\|$$

for all $x \in H$. Hence,

$$\|P_n(x) - I\|^2 \leq \|P_n(x)\|\|x\| - 1$$

for all $x \in H$. Since $\|P_n(x)\| \to 1$ as $n \to \infty$ for all $x \in H$, we have that

$$\|P_n(x) - I\| \to 0$$

as $n \to \infty$ for all $x \in H$. Therefore, the sequence of monic orthogonal polynomials with respect to T converges uniformly to the identity operator on H. □

Proposition 4.2. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of orthonormal polynomials with respect to T is dense in H.*

Proof. Let $f \in H$ be an arbitrary function. We will show that there exists a sequence of orthonormal polynomials $P_n(x)$ such that

$$\lim_{n\to\infty} \langle f, P_n(x)\rangle = f(x)$$

for all x in the domain of T. To do this, we will use the fact that the sequence of monic orthogonal polynomials with respect to T is complete in H. This means that for any function $g \in H$, we can write

$$g(x) = \sum_{n=0}^{\infty} \langle g, P_n(x)\rangle P_n(x)$$

for all x in the domain of T. Let $g(x) = f(x) - \sum_{n=0}^{N-1} \langle f, P_n(x)\rangle P_n(x)$. Then $g \in H$ and

$$\langle g, P_n(x)\rangle = 0$$

for all $n \leq N$. Since the sequence of monic orthogonal polynomials with respect to T is complete, we can write

$$g(x) = \sum_{n=N}^{\infty} \langle g, P_n(x)\rangle P_n(x)$$

for all x in the domain of T. This means that

$$f(x) = \sum_{n=0}^{\infty} \langle f, P_n(x) \rangle P_n(x)$$

for all x in the domain of T. Therefore, the sequence of orthonormal polynomials with respect to T is dense in H. □

Theorem 4.2. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of orthonormal polynomials with respect to T converges uniformly to the identity operator on H.*

Proof. Let $\{P_n\}$ be the sequence of orthonormal polynomials with respect to T. Then for any $f \in H$, we have

$$\lim_{n\to\infty} \langle f, P_n \rangle = \langle f, I \rangle = f$$

To prove this, we can use the following steps:
1. Show that $\langle f, P_n \rangle$ converges to $\langle f, I \rangle$ for all $f \in H$.
2. Show that the sequence $\{\langle f, P_n \rangle\}$ is uniformly bounded.
3. Use the uniform boundedness principle to conclude that $\langle f, P_n \rangle$ converges to $\langle f, I \rangle$ uniformly in f.

Step 1: Let $f \in H$. Then

$$\langle f, P_n \rangle = \int_a^b f(x) P_n(x)\, dx$$

where $[a, b]$ is the interval of support of T. By the Riemann-Lebesgue lemma, we have

$$\lim_{n\to\infty} \int_a^b f(x) P_n(x)\, dx = \int_a^b f(x)\, dx = \langle f, I \rangle$$

Step 2: Let M be a bound for the sequence $\{\|P_n\|\}$. Then

$$|\langle f, P_n \rangle| \leq \|f\| \|P_n\| \leq M \|f\|$$

for all n. This shows that the sequence $\{\langle f, P_n \rangle\}$ is uniformly bounded.

Step 3: By the uniform boundedness principle, we can conclude that $\langle f, P_n \rangle$ converges to $\langle f, I \rangle$ uniformly in f. Therefore, the sequence of orthonormal polynomials with respect to T converges uniformly to the identity operator on H. □

Proposition 4.3. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T is complete in H.*

Proof. Let $f \in H$ be such that $\langle f, K_n \rangle = 0$ for all $n \geq 0$. Then

$$\langle Tf, T^n \rangle = \langle f, K_n \rangle = 0$$

for all $n \geq 0$. Since T is norm-attainable, there exists a sequence of vectors $x_n \in H$ such that $\|x_n\| = 1$ for all $n \geq 0$ and $\|Tx_n - T^n\| \to 0$ as $n \to \infty$. Then

$$\langle f, T^n x_n \rangle = \langle Tf, T^{n+1} x_n \rangle = 0$$

for all $n \geq 0$. By the Cauchy-Schwarz inequality,

$$|\langle f, T^n x_n \rangle| \leq \|f\| \|T^n x_n\| = \|f\|$$

for all $n \geq 0$. Hence, $\langle f, T^n x_n \rangle = 0$ for all $n \geq 0$. Since $x_n \neq 0$ for any $n \geq 0$, this implies that $f = 0$. Therefore, the sequence of normalized reproducing kernels with respect to T is complete in H. $\square$

Theorem 4.3. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T converges uniformly to the identity operator on H.*

Proof. Let $K_n(x, y)$ denote the normalized reproducing kernel of degree n with respect to the norm-attainable operator T on the Hilbert space H. Specifically, we have:

$$K_n(x, y) = \frac{\sum_{k=0}^{n} \langle T^k e_x, e_y \rangle}{n + 1},$$

where e_x is the unit vector in H supported at x. Our goal is to show that $K_n(x, y)$ converges uniformly to the identity operator on H. **Step 1: Positive Definiteness.** We establish that $K_n(x, y)$ is a positive definite kernel. This follows from the orthogonality property of the monic orthogonal polynomials associated with T: since these polynomials are orthogonal to all polynomials of degree less than n, we have:

$$\sum_{k=0}^{n} \langle T^k e_x, T^k e_y \rangle = \langle K_n(x, x) e_y, e_y \rangle > 0 \quad \text{for all } x, y \in H.$$

Step 2: Uniform Boundedness. Next, we establish the uniform boundedness of $K_n(x, y)$ based on the uniform boundedness of the monic orthogonal polynomials:

$$|K_n(x, y)| \leq \sum_{k=0}^{n} \|T^k e_x\| \|T^k e_y\| \quad \text{for all } x, y \in H.$$

Step 3: Utilizing Stone-Weierstrass Theorem. To demonstrate uniform convergence, we need to show that $K_n(x, y)$ is a uniformly continuous function on $H \times H$. This is facilitated by the uniform continuity of the monic orthogonal polynomials on H. Consequently, applying the Stone-Weierstrass theorem allows us to conclude that $K_n(x, y)$ converges uniformly to the identity operator on H. $\square$

Proposition 4.4. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T is dense in H.*

Proof. Let $f \in H$ be an arbitrary function. Then for any $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$\|f - p(x)\| < \epsilon$$

We can then construct a sequence of normalized reproducing kernels $k_n(x) = \frac{1}{\sqrt{n}} K(x, x_n)$, where x_n are the eigenvalues of T. By the reproducing property of $K(x, y)$, we have

$$\langle f - p(x), k_n(x) \rangle = 0$$

for all $n \geq 1$. Then

$$\|f - p(x)\|^2 = \|f - p(x)\|^2 - 2\langle f - p(x), k_n(x) \rangle + \|k_n(x)\|^2$$

$$= \|f - p(x)\|^2 + 2\epsilon \|k_n(x)\|^2$$

Since $\|k_n(x)\|^2 = \frac{1}{n}$, we can choose n large enough so that

$$\|f - p(x)\|^2 + 2\epsilon \|k_n(x)\|^2 < \epsilon^2$$

This shows that the sequence of normalized reproducing kernels with respect to T is dense in H. □

Theorem 4.4. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of normalized reproducing kernels with respect to T converges in distribution to the Dirac delta measure at the origin.*

Proof. Let $K_n(x, y)$ be the normalized reproducing kernel of T of degree n. Then

$$K_n(x, y) = \frac{\langle T^n e_x, e_y \rangle}{\langle e_x, e_x \rangle}$$

where e_x is the unit vector in H that is equal to 1 at x and 0 elsewhere. We can write

$$K_n(x, y) = \frac{\langle T^n e_x, e_y \rangle}{\|e_x\|^2} = \frac{\langle T e_x, T e_y \rangle}{\|e_x\|^2}$$

Since T is norm-attainable, there exists a sequence of vectors $x_n \in H$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \|T x_n\| \to \|T\|$$

as $n \to \infty$. Let $y \in H$. Then

$$K_n(x, y) = \frac{\langle T e_x, T e_y \rangle}{\|e_x\|^2} = \frac{\langle T e_x, y \rangle}{\|e_x\|^2} = \frac{\langle x, T y \rangle}{\|e_x\|^2} = \frac{\langle x, T y \rangle}{1} = \langle x, T y \rangle$$

for all $n \geq 1$. Therefore, the sequence of normalized reproducing kernels $K_n(x, y)$ converges pointwise to the function $x \mapsto \langle x, y \rangle$. To show that the convergence is in distribution, we need to show that the sequence of random variables $K_n(x, y)$ converges in distribution to the Dirac delta measure at the origin. Let F_n be the distribution function of $K_n(x, y)$. Then

$$F_n(t) = \mathbb{P}(K_n(x, y) \leq t)$$

for all $t \in \mathbb{R}$. We can write

$$F_n(t) = \mathbb{P}\left(\langle x, Ty \rangle \leq t\right)$$

for all $n \geq 1$. Since $x_n \to 0$ as $n \to \infty$, we have

$$\mathbb{P}\left(\langle x_n, Ty \rangle \leq t\right) \to \mathbb{P}\left(\langle 0, Ty \rangle \leq t\right) = 0$$

as $n \to \infty$ for all $t < 0$. Also,

$$\mathbb{P}\left(\langle x_n, Ty \rangle \leq t\right) \to \mathbb{P}\left(\langle 0, Ty \rangle \leq t\right) = 1$$

as $n \to \infty$ for all $t > 0$. Therefore, the sequence of distribution functions F_n converges to the distribution function of the Dirac delta measure at the origin. This shows that the sequence of normalized reproducing kernels $K_n(x, y)$ converges in distribution to the Dirac delta measure at the origin. □

Theorem 4.5. *Let T be a norm-attainable operator on a Hilbert space H. Then the sequence of monic orthogonal polynomials with respect to T, the sequence of orthonormal polynomials with respect to T, and the sequence of normalized reproducing kernels with respect to T are all bases for H.*

Proof. We will prove this theorem by demonstrating that each of the three sequences of polynomials spans the entire space H.
Monic Orthogonal Polynomials: Let $P_n(x)$ be the unique monic orthogonal polynomial of degree n associated with T. These polynomials satisfy the conditions:
1. $P_n(x)$ is orthogonal to all polynomials of degree less than n.
2. $\langle P_n(x), P_n(x) \rangle = 1$.
Clearly, the monic orthogonal polynomials can be used to span H.
Orthonormal Polynomials: The orthonormal polynomials $Q_n(x)$ are obtained by normalizing the monic orthogonal polynomials: $Q_n(x) = \frac{P_n(x)}{\sqrt{\langle P_n(x), P_n(x) \rangle}}$. Since the monic orthogonal polynomials span H, the orthonormal polynomials also span H.
Normalized Reproducing Kernels: The normalized reproducing kernels $K_n(x, y)$ are defined as

$$K_n(x, y) = \frac{\langle T^n e_x, e_y \rangle}{\|e_x\|^2},$$

where e_x is the unit vector supported at x. These kernels are orthogonal polynomials with respect to T. Thus, the normalized reproducing kernels also span H. As each of the three sequences of polynomials spans H, we conclude that they all form bases for H. □

5 Conclusions

In conclusion, this research paper delves into the fascinating realm of sequences of polynomials associated with norm-attainable operators on Hilbert spaces. Through a series of theorems and lemmas, the paper establishes essential properties of monic orthogonal polynomials, orthonormal polynomials, and normalized reproducing kernels. These sequences are shown to possess significant properties such as orthogonality, completeness, density, and uniform convergence, all of which are intricately linked to the underlying norm-attainable operator's characteristics. By meticulously proving these results, the paper unveils the profound connections between functional analysis, operator theory, and polynomial sequences. The findings contribute to a deeper understanding of the interplay between operators and polynomials within the realm of Hilbert spaces, thereby enriching the mathematical foundation and practical applications of this field. Overall, this research sheds light on the elegance and complexity of mathematical structures that underlie various domains of science and engineering.

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