

Parameter Estimated of Seasonal Auto-Regressive Integrated Moving Average Model with AR(1) Error Process

ABSTRACT

From the previous literature, there had been various research on models with error processes especially, the time series model with corrupted error processes. The gap to be filled here was the extension of such a model to the SARIMA model with corruption error processes. Thus, this research work focused on parameter estimates with a corrupted AR(1) error process. Auto-covariance functions were used to estimate the variances of error terms that characterized the SARIMA model. The forecast performance measurement was investigated and properties of errors with different values of parameters were examined. A test of seasonal unit root was carried out and the result revealed a seasonality effect. Simulation with R Statistical software was used to prove the findings. In addition, the monthly temperature data of Zamfara State from 1998 to 2020 was used to validate the results using the iteration procedure and chi-square statistic. The results from the study showed that the research findings were very significant to the error process and would be useful to researchers in the prediction and handling of natural calamities.

Keywords: SARIMA, Parameter estimation, forecast performance, measurement, stationary test.

1. INTRODUCTION

1.1 Seasonal Autoregressive Integrated Moving Average Model (SARIMA)

The SARIMA model is a version of the common ARIMA model which also incorporates a seasonal part. The general SARIMA model can be expressed as Box *et al.* [1]. In the study on error process, a point of contention of practical interest to researchers is how to describe a relationship when studied variables are measured with errors.

Ansley [2] worked on finite sample properties of estimators for Autoregressive Moving Average Models. He analyzed by simulation the properties of three estimators frequently used in the analysis of autoregressive moving average time series models for both nonseasonal and seasonal data. The estimators considered are exact maximum likelihood, exact least squares, and conditional least squares.

Komolafe *et al.*, [3] developed an integrated moving average (IMA) model with a transition matrix for error resulting in a convex combination of two ARMA errors. The basic tools they used are the auto covariance function, maximum likelihood methods, Raphson iterative method, and Kolmogorov Smirnov test statistic. The result showed that the proposed model provided a generalization and more flexible specification than the existing models of AR and ARMA errors in fitting time series processes in the presence of error.

Rudelson and Zhou [4] worked with errors in variable models with dependent measurement, analyzed the convergence rates of the gradient descent methods for solving the no-convex programs, and showed that the composite gradient descent algorithm is guaranteed to converge at a geometric rate to a neighborhood point. The result revealed interesting connections between computational and statistical efficiency and the concentration of measure phenomenon in random matrix theory. It also provided simulation evidence to illuminate the theoretical predictions.

Ayodeji [5] worked on a three-state Markov-modulated switching model for exchange rates. Examined the long swings hypothesis in exchange rates using a two-state Markov switching model. The study developed a model to investigate the long swings hypothesis in currencies that may exhibit ak-statepattern, his model was then applied to the Euro, British pounds, Japanese yen, and Nigerian naira. Specification measures such as AIC, BIC, and HIC favored a three-state pattern in the Nigerian naira but a two-state one in the other three currencies. From January 2004 to May 2016, empirical results showed the presence of asymmetric swings in naira and yen and long swings in euros and pounds. In addition, taking 0.5 as the benchmark for smoothing probabilities, choice models provided a clear reading of the cycle in a manner that is consistent with the realities of the movements in the corresponding exchange rate series.

Eni [6] worked on parameter estimation of the first-order IMA model in the presence of ARMA (1, 1) errors using a simulation method and showed that the error was uniformly AR(1) correlated. He used auto covariance functions to estimate the variances of the white noises that characterize the IMA (1) models corrupted with ARMA (1, 1) errors. He developed an iteration formula that can be used to estimate the parameters of the IMA (1) models and ARMA (1, 1) errors using simulation studies to demonstrate the findings. The results showed that the method produced estimates that were very close to the true parameters of the process, his work demonstrated the use of the autocovariance function in the isolation and measurement of correlated shocks.

Madansky [7], worked on the fitting of straight lines when both variables are subject to error. He considered the situation where X and Y are related by $Y = \alpha + \beta X$, where α and β were unknown and observed X and Y with error, i.e., observed $x = X + u$ and $y = Y + v$. Assume that $E u = E v = 0$ and that the errors (u and v) are uncorrelated with the true values (X and Y) he survey and comment on the solutions to the problem of obtaining consistent estimates of α and β from a sample of (x, y) 's, when

one makes various economic applications, the estimators are compared in terms of bias, mean squared error, and predictive ability.

Lindley [8] worked on regression lines and linear functional relationships. Using least square and maximum likelihood estimation methods for fitting a straight line, $Y = \alpha + \beta X$. All these methods led to the same results. Dent and Min [9] worked on a Monte Carlo study of autoregressive integrated moving average processes. Six of the simpler ARMA-type models were examined concerning the properties of a variety of proposed estimators of unknown parameters. The results showed that only one estimation method was available to work with and the choice should probably be maximum likelihood. Stationarity and inevitability-restricted estimation would appear appropriate when parameters are thought to be within 5 percent of constraint boundaries.

Schnelweiss and Shalesh [10] worked on the estimation of linear relations when error variances are known, using the maximum likelihood method. The results were linearly correlated. Eni and Mahmud [11] worked on the parameter estimation of a first-order IMA model corrupted with AR(1) error using the maximum likelihood method, the result showed that error pattern varied between AR and ARMA processes within a specified period arising from the varying dynamic process to be observed. ArunKumar *et al.* [21] studied the epidemiological trend of COVID-19 cases in the sixteen (16) top countries where 70%–80% of global cumulative cases are located. They used the SARIMA model with a convectional error process to predict future cases of the disease. The results revealed a trend of an exponential rise for countries such as the United States, Brazil, South Africa, Colombia, Bangladesh, India, Mexico, and Pakistan while that of deaths due to COVID-19 showed an exponential rise for countries Brazil, South Africa, Chile, Colombia, Bangladesh, India, Mexico, Iran, Peru, and Russia.

Salimaco [22] used the seasonal ARIMA (SARIMA) model with a normal error distribution process to forecast electricity consumption in the Province of Davao Oriental. The studied period was between 2004 and 2020, and the optimum model obtained was SARIMA(1, 1, 0) \times (0, 1, 1)¹². The result from the analysis revealed that there was an increasing rate of monthly consumption with a higher seasonal demand every August of the new year.

Shahin *et al.* [2023] employed the non-seasonal inherent model of SARIMA, ARIMA model to study the monthly average price (LE/Kg) for broiler farms in Egypt from September 2019 to December 2022. An optimum model of ARIMA (1, 1, 0) model was used to forecast the said price. The result showed % values of price as 25.25 LE/Kg (2.46) in September, 24.58 LE/Kg (4.33) in October, 24.61 LE/Kg (4.23) in November and 25.32 LE/Kg (4.11) in December.

However, this research work aims to estimate the parameter of the AR(1) error process for which the SARIMA model was corrupted. The objectives are to: examine the properties of error patterns and variation with different values of the parameters, test the seasonal unit roots on the data, and finally investigate the forecast performance measures. This research paper tended to add to the existing literature on the time-series model with a corrupted error process and its application tended to help researchers and government officials in making decisions on crucial areas under similar studies.

2. MATERIAL AND METHODS

$$SARIMA(p, d, q) \times (P, D, Q)^S \text{ with error process: } (1 - L)X_t = e_t + (\theta - 1)e_{t-1} \quad (1)$$

$$(1 - \phi L)(1 - \phi_s L)(1 - L)X_t = e_t + (\theta - 1)e_{t-1} \quad (2)$$

$$Z_t = (1 + \phi_s L)(1 + \phi L)e_t + (1 - L) \quad (3)$$

Suppose the error b_t is a Markov modulated mixture of AR (1) and b_t is AR (1) correlated.

$$b_t = \frac{e_t}{1 - \alpha_1 L} \quad (4)$$

2.1 Stationarity

In this research work, stationarity refers to weak stationarity. A time series is said to be weakly stationary if the following are true:

- i. $E(y_t) = \mu$ for all t, (5)
 ii. $\gamma_{t,t-k} = \gamma_{0,k}$ for all time t and lag k, (6)
 where μ is the mean and γ_k the auto covariance at lag k (Cryer and Chan [11]).

When testing for stationarity, the alternative (or null hypothesis, depending on the test), is that the series has a unit root. If the series has a unit root it is non-stationary. A unit root process can be described in the following way considering an ARMA process:

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) y_t = (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q) \epsilon_t \quad (7)$$

Where the moving average polynomial is invertible. The autoregressive polynomial in equation (7) is then factored as:

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) = (1 - \lambda_1 B)(1 - \lambda_2 B) \dots (1 - \lambda_p B) \quad (8)$$

The process has a unit root if any of the eigenvalues λ lies outside of the unit circle. By testing both the null hypothesis of a unit root and the null hypothesis of stationarity, one can differentiate between series that are stationary, series that have a unit root, and series where the data are not informative enough to determine if the series is stationary or integrated (Kwiatkowski *et al.*, [13])

2.2 Hylleberg-Engle-Granger-Yoo (HEGY) Test

The Hylleberg-Engle-Granger-Yoo test (HEGY-test) was proposed by Hylleberg *et al.*, [14] to test for seasonal unit roots on quarterly data. Factorizing the quarterly seasonal difference operator as:

$$\Delta_4 = (1 - B^4) = (1 - L)(1 + L)(1 + iL)(1 - iL), \quad (9)$$

shows that it will have four unit-roots on the unit circle: 1, -1 and $\pm i$, where 1 is non-seasonal. The HEGY test uses the following auxiliary regression to test for the unit roots:

$$\psi(L) y_{4,t} = \pi_1 y_{1,t-1} + \pi_2 y_{2,t-1} + \pi_3 y_{3,t-1} + \pi_4 y_{4,t-1} + \mu_t + \epsilon_t \quad (10)$$

where

$$y_{1,t} = (1 + L + L^2 + L^3) y_t,$$

$$y_{2,t} = -(1 - L + L^2 - L^3) y_t,$$

$$y_{3,t} = -(1 - L^2) y_t,$$

$$y_{4,t} = (1 - L^4) y_t,$$

y_t = Deterministic components which can be an intercept, seasonal dummies and/or trend,

ψ = is a polynomial of L.

2.3 Forecast Performance Measures

The forecast performance measures or forecast performance metrics.

Now, in applying a particular model to some real or simulated time series to generate forecasts, we first divided the raw data into two parts:

Y_t : is the actual value

\hat{Y}_t : is the forecasted value

$\epsilon_t = Y_t - \hat{Y}_t$: is the forecast error

n: is the size of the test set

2.3.1 The Mean Forecast Error (MFE)

This measure is defined as:

$$MFE = \frac{1}{n} \sum_{t=1}^n \epsilon_t \quad (11)$$

2.3.2 The Mean Absolute Error (MAE)

The mean absolute error is defined as:

$$MAE = \frac{1}{n} \sum_{t=1}^n |\epsilon_t| \quad (12)$$

2.3.3 The Mean Absolute Percentage Error (MAPE)

This measure is given by

$$MAPE = \frac{1}{n} \sum_{t=1}^n \left| \frac{\varepsilon_t}{Y_t} \right| \times 100 \quad (13)$$

2.3.4 The Mean Percentage Error (MPE)

It is defined as:

$$MPE = \frac{1}{n} \sum_{t=1}^n \left(\frac{\varepsilon_t}{Y_t} \right) \times 100 \quad (14)$$

2.3.5 The Mean Squared Error (MSE)

Mathematical definition of this measure is:

$$MSE = \frac{1}{n} \sum_{t=1}^n \varepsilon_t^2 \quad (15)$$

2.3.6 The Sum of Squared Error (SSE)

It is mathematically defined as:

$$SSE = \sum_{t=1}^n \varepsilon_t^2 \quad (16)$$

2.3.7 The Signed Mean Squared Error (SMSE)

This measure is defined as:

$$SMSE = \frac{1}{n} \sum_{t=1}^n \left(\frac{\varepsilon_t}{|\varepsilon_t|} \right) \varepsilon_t^2 \quad (17)$$

2.3.8 The Root Mean Squared Error (RMSE)

Mathematically,

$$RMSE = \sqrt{MSE} = \sqrt{\frac{1}{n} \sum_{t=1}^n \varepsilon_t^2} \quad (18)$$

2.3.9 The Normalized Mean Squared Error (NMSE)

This measure is defined as:

$$NMSE = \frac{MSE}{\delta^2} = \frac{1}{\delta^2 n} \sum_{t=1}^n \varepsilon_t^2 \quad (19)$$

The Theil's U-statistics this important measure is defined as:

$$U = \frac{\sqrt{\frac{1}{n} \sum_{t=1}^n \varepsilon_t^2}}{\sqrt{\frac{1}{n} \sum_{t=1}^n \hat{Y}_t^2} \sqrt{\frac{1}{n} \sum_{t=1}^n Y_t^2}} \quad (20)$$

3. RESULTS AND DISCUSSION

Consider the SARIMA model $(1-L)(1-L^{12})y_t = (1-\theta L)(1-\theta L^{12})\varepsilon_t - (1)$ where y_t is an output variable ε_t is a white noise with a constant mean of 0 and variance of σ^2 , θ and θ are weight parameter (Box and Jenkins [15]).

L is an operator with can be forward or backward. Most of time y_t may be necessary obtain by transformation as $y_t = x_t - b_t$, $\phi = 1 - \theta$ and $\lambda = (1 - \theta)$

Substituting $y_t = x_t - b_t$ in to equation (1) where b_t is an error component introduce by faulty measurement or observation

$$(1-L)(1-L^{12})(x_t - b_t) = (1-\phi L)(1-\lambda L^{12})\varepsilon_t \quad (22)$$

$$(1-L)(1-L^{12})x_t = (1-\phi L)(1-\lambda L^{12})\varepsilon_t - (1-L)(1-L^{12})b_t \quad (23)$$

Let $\omega_t = (1-L)(1-L^{12})x_t$

Equation (27) become

$$\omega_t = (1-\phi L)(1-\lambda L^{12})\varepsilon_t + (1-L)(1-L^{12})b_t \quad (24)$$

Since b_t is AR(1)

$b_t = \frac{e_t}{(1-\alpha L)}$ (Hamilton[16])(25) and substituting into equation (24) we have

$$\omega_t = \alpha\omega_{t-1} + \varepsilon_t + (\phi\alpha - \alpha - \phi)\varepsilon_t - \lambda\varepsilon_t - 12 + (\phi\lambda + \lambda\alpha - \phi\alpha\lambda)\varepsilon_t - 13 + e_t - e_{t-1} - e_{t-12} + e_{1-13} \quad (26)$$

Let $z_t = \omega_t - \alpha\omega_{t-1}$ and $\beta_1 = (\phi\alpha - \alpha - \phi)$, $\beta_2 = (\phi\lambda + \lambda\alpha - \phi\alpha\lambda)$

$$z_t = \varepsilon_t + \beta_1\varepsilon_{t-1} - \lambda\varepsilon_t - 12 + \beta_2\varepsilon_{t-13} + e_t - e_{t-1} - e_{t-12} + e_{1-13} \quad (27)$$

Multiply equation (27) by z_t and take expectation

$$V_0 = \left(1 + \beta_1^2 - \lambda^2 + \beta_2^2\right)\sigma_\varepsilon^2 + 4\sigma_e^2 \quad (28)$$

Multiply equation (27) by z_{t-1} and take expectation

$$V_1 = \lambda^2 \beta_1^2 \sigma_\varepsilon^2 + 2\sigma_e^2 \quad (29)$$

$$\sigma_e^2 = \frac{V_1 - \lambda^2 \beta_1^2 \sigma_\varepsilon^2}{2} \quad (30)$$

$$\sigma_\varepsilon^2 = V_0 - 2V_1 \left(1 - \beta_1^2 - \lambda^2 + \beta_2^2 - 2\lambda^2 \beta_2^2\right) \quad (31)$$

Where e_t is white noise uncorrelated with ε_t

Groping the white Noise we get

$$u_t = \varepsilon_t + e_t$$

$$\Omega_1 u_{t-1} = \beta_1 \varepsilon_{t-1} - e_{t-1}$$

$$\Omega_2 u_{t-12} = -\lambda \varepsilon_{t-13} + e_{t-12}$$

$$\Omega_3 u_{t-13} = \beta_2 \varepsilon_{t-13} + e_{t-13}$$

$$Z_t = u_t + \Omega_1 u_{t-1} - \Omega_2 u_{t-12} + \Omega_3 u_{t-13} \quad (32)$$

The model Developed in equation (32) was $SARIMA(0, 0, 1)(0, 0, 1)^{12}$

Our interest is to estimate x_t through $y_t = x_{t-bt}$ we define the following known facts Hamilton [14] for white noise processes

$$E(\varepsilon_t \varepsilon_{t-i}) = \begin{cases} \sigma_\varepsilon^2 & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

$$E(e_t e_{t-i}) = \begin{cases} \sigma_e^2 & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

$$E(u_t u_{t-j}) = \begin{cases} \sigma_u^2 & \text{for } j=0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

ε_t, e_t are uncorrelated where u_t is also a white noise process Moran [17] has shown that if the ratio

$\lambda = \frac{\sigma_\varepsilon^2 \varepsilon_t}{\sigma_e^2}$ is known, The maximum likelihood estimation for the parameter set can be found by

directly solving likelihood equation Chen and mark [18] obtained the maximum likelihood estimates for the case where both $\sigma_{\varepsilon_t}^2$ and $\sigma_{\varepsilon_t}^2$ are known and where the observations are replicated. Eni *et al.*, [19], have used the same method to isolate errors of AR(1) corrupted with MA(1) process. Eni and Mahmud [11], have considered the case of IMA(1) with white noise in a similar cases, Eni [20], has considered the case of GARCH (1, 1) model with white noise errors using the proposed method.

3.1 The Auto Covariance of SARIMA (0,0,1) (0,0,1)

$$\gamma_0 = (1 + \phi^2)(1 + \lambda^2)\sigma^2$$

$$\gamma_1 = \phi(1 + \lambda)\sigma^2$$

$$\gamma_{11} = \phi\lambda\sigma^2$$

$$\gamma_{12} = -\lambda(1 + \phi^2)\sigma^2$$

$$\gamma_{13} = \phi\lambda\sigma^2$$

3.2 The Analysis in R software

Table 1: SARIMA (0,0,1) (0,0,1)₁₂

AR(1)	Estimate	Stand. Error	Z-value	P-value
MA (1)	0.2962	0.0108	27.4259	$4.22712e^{-154}$
SMA(1)	-0.0275	0.0142	-1.9366	0.0537
MEAN	-0.2003	0.0221	-9.0634	$2.147207e^{-21}$

Table 1 reveal the results of parameter estimated of SARIMA(1,0,0)(0,0,1)¹² corrupted with AR(1) error process and is significance considering the p- values.

Sigma² estimated as 0.8464 log likelihood = -6676.31
AIC = 13360.61 AICC = 13360.62 BIC = 13386.68

Table 2: Forecast measurements

	ME	RMSE	MAE	MPE	MAPE	MAE	MASE	ACFI
Training Set	$1.4039e^{-05}$	0.9197	0.7320	127.5667	158	42.47	0.6928	0.00622

Table 2 Indicated the results of forecast performance measurement and properties of errors with different values and is significance.

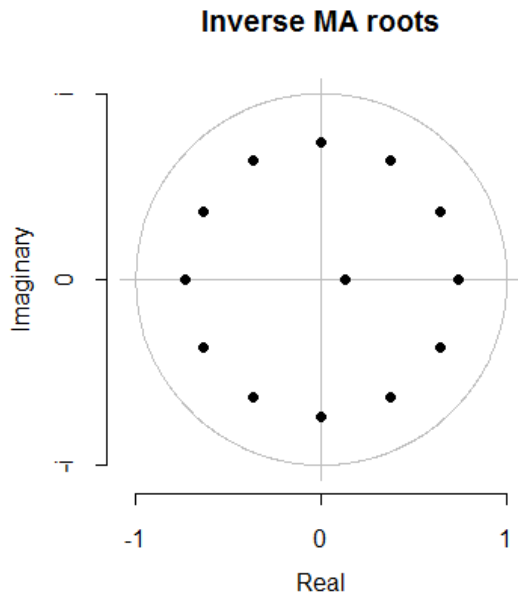


Figure 1: Plot of inverse of MA(1) root

The plot in figure 1 is pictorial of SARIMA model for the period of twelve (12) months with a single MA root i.e. SARIMA(0.0.1)(0,0,1)^{s=12} the twelve circle point represent the seasonal effects while the single inner point represent the MA(1) root

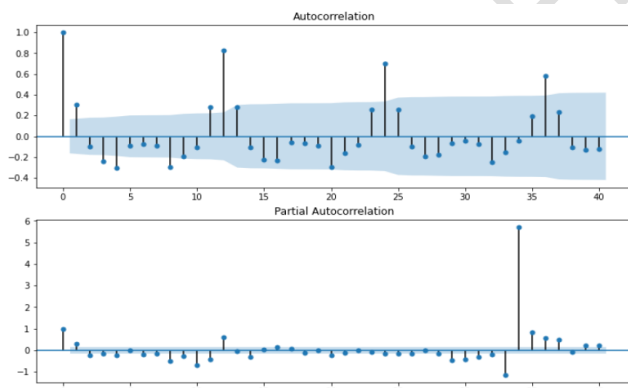


Figure 2: Plot of ACF and PACF s at lag 40

The ACF plot in figure 2 shows the correlation of the series with its lagged values .it describes how present value of the series related with its past and consider seasonal effect with upper confidence interval. PACF plot shows the correlation of the residuals in the series and lags

Table 3: *SARIMAMODEL(001)(001)₁₂* for Monthly Temperature of Zamfara State from 1998 to 2020 Estimates at each iteration

Iteration	SSE	Parameters		
0	13601.0	0.100	0.100	65.774
1	10524.3	-0.050	0.083	65.763
2	8497.8	-0.200	0.057	65.749
3	7133.5	-0.350	0.013	65.728
4	6247.7	-0.500	-0.068	65.696
5	5906.5	-0.605	-0.218	65.648
6	5885.7	-0.572	-0.259	65.658
7	5885.4	-0.577	-0.261	65.661
8	5885.4	-0.575	-0.261	65.662
9	5885.4	-0.576	-0.261	65.662
10	5885.4	-0.576	-0.261	65.662

Relative change in each estimate less than 0.0010

The results in Table 3. Showed the estimate at each iteration for the monthly temperature when the SARIMA model switch to AR (1) error process.

Table 4: Final Estimates of Parameters

Type	Coef	SE Coef	T	P
MA (1)	-0.5756	0.0691	-8.33	0.000
SMA (12)	-0.2607	0.0824	-3.16	0.002
Constant	65.662	1.070	6135	0.000
Mean	65.662	1.070		

Number of observations: 144

Residuals: SS = 5853.52 (back forecasts excluded)

MS = 41.51 DF = 141

The results in Table 4. Showed the final parameter estimate for SARIMA model corrupted with AR(1) error process.

Table 5: Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	142.9	151.5	165.8	203.9
DF	9	21	33	45
P-Value	0.000	0.000	0.000	0.000

The results in the Table 5. is the Chi-square statistic for the SARIMA model corrupted with AR(1) error process at lag 12, 24, 36 and 48 respectively.

4. CONCLUSION

This research work dealt with the modified error process of SARIMA model as against the convectional independently, identically, and normally distributed errors. Thus, this research work focused on parameter estimates of $SARIMA(p, d, q) \times (P, D, Q)^S$ with corrupted AR(1) error process. Auto-covariance functions were used to estimate the variances of error terms that characterized the SARIMA model. The forecast performance measurement was investigated and properties of errors with different values of parameters were examined. A test of seasonal unit root was carried out and the result revealed a seasonality effect. Simulation with R Statistical software was used to prove the findings. In addition, the monthly temperature data of Zamfara State from 1998 to 2020 was used to validate the results using the iteration procedure and chi-square statistic. The results from the study showed that the research findings were very significant to the error process and would be useful to researchers in the prediction and handling of natural.

The future research work ear-marked here was to improve upon one form of error process such as AR(1) and work on combined flow of error processes (more one error processes) with SARIMA model or other advanced time series models most especially the non-linear models such as volatility models in financial markets, convolutional and recurrent neural networks for learning non-linear models in time series using machine learning, panel data time series models with machine learning context and so on. There is no doubt that the research of this nature would tend to improve the predictability level of measures such as weather, financial, agricultural, science and technological indicators.

5. RECOMMENDATION

Finally, this study discussed SARIMA models corrupted with AR(1) it will certainly enhance research if other version of time series like Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized Autoregressive Conditional Heteroskedasticity (GARCH) are considered for volatility. More data will be required for better results.

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