Review Article

Applications of Mathematics in Agricultural Engineering

Abstract

This paper investigates the important role of mathematics in the solution of complicated issues in the field of agricultural engineering and technology. It shows examples of mathematical modelling and analytical techniques that are used in agriculture, such as Crop Growth, Irrigation Management, Soil Moisture Modeling , Environmental management, Pest and Disease Management, Fertilizer Applications, Watershed Management etc.

Keywords: Mathematical Models; Agricultural Engineering;

2010 Mathematics Subject Classification: 97M50;93A30;81T80;97M40;97M10;97M99

1 Introduction

Agricultural engineering is very much important in ensuring the agricultural sector's long-term viability.Quality and productivity are essential components of global food security. At the core of this subject ,which is a combination of different subjects , is the seamless integration of mathematical and engineering principles to address the complex difficulties of agriculture. This paper tries to examines how various mathematical models enable us to optimise crop production, manage resources efficiently, design innovative machinery, and navigate the complex web of factors influencing agriculture in a time vitnessing abrupt change in environmental and technological dynamics.

2 Mathematical Modeling in Crop Growth

Mathematical modelling has become a vital tool in modern agriculture research, enhancing our understanding of crop growth. These models are extremely useful for forecasting and optimising agricultural yields, resource management, and environmentally friendly farming practises. Different Mathematical models give insights into how different variables impact crop growth over time by measuring the sophisticated relationship between environmental parameters such as humidity,temperature, composition of soil , and sunlight, as well as genetic identities of the crop itself.This multidisciplinary approach has been very important in solving global problems such as food security, climate variations, and resource efficiency. To account for the complicated dynamics of crop growth, a class of mathematical models, ranging from simple equations to very complicated mechanistic simulations, have been created and modified over the decades. These mathematical models have been effectively used to a variety of crop species, including major grains like wheat and rice, to improve crop management tactics, optimise planting schedules, and generate robust cultivars.

The so called logistic growth model, for example, which is extensively used in several sectors including agricultural research, represents the increase of a population of organisms over time, such as crops. The logistic growth equation is as follows [1,2]:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{2.1}$$

where:

N : Population size (e.g., crop yield)
r : Intrinsic growth rate
K : Carrying capacity (maximum sustainable yield)

As we grapple with the pressing concerns of feeding a growing global population while minimising environmental impact, continual refining and implementation of mathematical models in crop growth is vital for sustainable agricultural practises.[3,4]

3 Mathematics in Irrigation Management

Agriculture depends heavily on irrigation, and maths may assist reduce water waste. One example of an optimisation challenge is the scheduling of irrigation. Algorithms for dynamic programming can figure out the best irrigation plan to increase crop output while using the least amount of water. Since it provides the fundamental framework for maximising water supplies and raising agricultural production, mathematics is crucial to irrigation management. Mathematical models that take into account elements like soil type, crop requirements, climatic patterns, and irrigation infrastructure are largely reliant on for the effective allocation of water resources, a crucial feature of irrigation. Farmers and water resource managers may use these mathematical models to determine when, where, and how much water to apply, in order to ensure that crops get the hydration they require while minimising the amount of water utilised.

For instance, the Penman-Monteith equation may be used to estimate crop evapotranspiration rates precisely, which helps with irrigation planning. Furthermore[5], to find the best irrigation plans under different limitations and ensure sustainable water usage, sophisticated mathematical procedures like linear programming and dynamic optimisation are used. Irrigation management may greatly help to global food security and water conservation by utilising mathematics, highlighting its crucial role in agricultural practises[6].

4 Soil Moisture Modeling

Hydrological and environmental studies must include soil moisture modelling because it offers crucial insights into the changing dynamics of water content in the soil profile. Many mathematical models have been created to mimic changes in soil moisture over time[7]. Understanding water availability in agriculture, managing water resources, and protecting the environment all depend on these models. The Richards Equation, the Thornthwaite-Mather model, and the Soil Conservation Service Curve Number (SCS-CN) approach are three widely used soil moisture models.

The Richards Equation[8] is a widely employed model for simulating soil moisture dynamics in unsaturated soils. It is described as:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K(\theta) \nabla \Psi) \tag{4.1}$$

where:

$$\begin{aligned} \theta &: \text{Volumetric water content} \\ t &: \text{Time} \\ K(\theta) &: \text{Hydraulic conductivity as a function of } \theta \\ \Psi &: \text{Soil water potential} \end{aligned}$$

The Thornthwaite-Mather model[9,10,11] estimates potential evapotranspiration based on temperature and can be used to assess soil moisture. It is given by:

$$PET = 16 \left(\frac{T}{5} \right)^{1.5} \tag{4.2}$$

where PET is potential evapotranspiration, and T is the mean monthly temperature in degrees Celsius.

The Soil Conservation Service Curve Number (SCS-CN) method[12] is commonly used in hydrology for estimating runoff and soil moisture conditions. The model equation is:

$$Q = \frac{(P - I_a)^2}{P - I_a + S} \tag{4.3}$$

where Q is runoff, P is precipitation, I_a is initial abstraction, and S is potential maximum retention.

When properly calibrated and validated, these mathematical models are useful resources for regulating soil moisture dynamics and enabling well-informed decisions across a variety of fields.

5 Environmental Control in Greenhouses

The production of crops year-round in a controlled environment is made possible by greenhouse farming, which is essential to contemporary agriculture. To get the crop yields and quality you want, greenhouses must have effective environmental management. This control entails regulating variables including CO2 concentration, humidity, and temperature to generate the perfect development environment.

Regulating temperature is an important component of environmental control. The following equation may be used to represent the energy balance within a greenhouse[13]:

$$\frac{dT}{dt} = \frac{1}{C} \left(P_{\text{in}} - P_{\text{out}} \right) + \frac{T_{\text{out}} - T}{R} \tag{5.1}$$

where T is the greenhouse air temperature, C is the heat capacity of the air, P_{in} and P_{out} represent heat gains and losses, respectively, and T_{out} is the outdoor temperature.

Humidity control is another critical factor. The greenhouse humidity can be managed using the following equation:

$$\frac{dh}{dt} = \frac{1}{V} \left(M_{\text{in}} - M_{\text{out}} \right) \tag{5.2}$$

where h is the greenhouse air humidity, V is the greenhouse volume, and M_{in} and M_{out} are the moisture gains and losses.

Additionally, controlling CO_2 concentration for photosynthesis optimization is essential. The CO_2 balance in the greenhouse can be described as:

$$\frac{dC_{\text{CO2}}}{dt} = \frac{1}{V}\left(F_{\text{in}} - F_{\text{out}}\right) \tag{5.3}$$

where C_{CO2} is the CO_2 concentration, F_{in} and F_{out} represent CO_2 flow rates into and out of the greenhouse, respectively.

Efficient control strategies based on these equations are essential for optimizing crop growth and resource utilization in greenhouse agriculture.

6 Mathematical Modeling in Pest and Disease Management

In managing pests and diseases, mathematical modelling is essential for formulating efficient control plans and determining how they affect agricultural systems. To comprehend and forecast the dynamics of pests and illnesses, many mathematical models have been presented. Using compartmental models, such as the Susceptible-Infectious-Recovered (SIR) model, which has been modified for pests and illnesses, is a popular strategy. For instance, the SIR model may be altered as follows in the context of agricultural diseases:

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I, \quad \frac{dR}{dt} = \gamma I \tag{6.1}$$

Here, S represents the susceptible crop population, I is the infected crop population, and R is the recovered crop population. Parameters β and γ govern disease transmission and recovery rates, respectively.

Another example is the logistic growth model[14] applied to pest populations, which can be represented as:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{6.2}$$

In this equation, N represents the pest population, r is the intrinsic growth rate, and K is the carrying capacity of the environment. This model helps in predicting pest population dynamics and determining optimal control strategies.

Additionally, spatial models, such as the reaction-diffusion model, are valuable in analyzing the spread of pests and diseases across agricultural landscapes. One example is the Fisher-KPP equation:

$$\frac{\partial u}{\partial t} = d\Delta u + ru(1-u) \tag{6.3}$$

Here, u represents the pest or disease density, d is the diffusion coefficient, and r represents the reaction rate. This model can inform the deployment of spatially targeted control measures.

7 Optimization of Fertilizer Application

Optimising fertiliser application in agriculture is critical for increasing crop output while minimising environmental consequences. This problem has received a lot of attention in recent years, with many research concentrating on establishing exact techniques for fertiliser usage. Various aspects such as soil type, crop type, climate conditions, and economic limitations are taken into account throughout the optimisation process. Mathematical modelling and sophisticated optimisation algorithms are a well-established way to attaining this optimisation.

Some examples of optimization models used for fertilizer application are as follows:

The nutrient absorption model[15], which describes how crops receive nutrients from the soil, is a typical model for fertiliser optimization's. This model is often expressed as:

$$N_{\text{uptake}} = f(N_{\text{soil}}, N_{\text{fertilizer}}) \tag{7.1}$$

Where N_{uptake} is the nutrient uptake by the crop, N_{soil} is the nutrient content in the soil, and $N_{\text{fertilizer}}$ represents the amount of fertilizer applied.

Crop growth models simulate the growth and development of crops over time, incorporating fertilizer application as a variable. A widely used model is the Monod equation[16], which relates crop growth rate (μ) to nutrient availability:

$$\mu = \frac{\mu_{\text{max}} \cdot N_{\text{fertilizer}}}{K_s + N_{\text{fertilizer}}} \tag{7.2}$$

Here, μ_{max} is the maximum specific growth rate, K_s is the half-saturation constant, and $N_{\text{fertilizer}}$ is the nutrient concentration from fertilizer.
Economic factors are also involved in optimising fertiliser application. An economic optimisation model[17] seeks to maximise profit while accounting for fertiliser costs and crop output. This can be written as:

$$\text{Maximize Profit} = \text{Crop Yield} \cdot (\text{Crop Price} - \text{Fertilizer Cost}) \tag{7.3}$$

8 Mathematical Modeling of Soil Erosion

Soil erosion is a major environmental concern with far-reaching implications for agriculture, ecology, and land management. Mathematical modelling is essential for understanding and forecasting soil erosion processes, as well as for developing efficient erosion control techniques. To estimate erosion rates and analyse their influence on soil quality, many models have been presented. The Universal Soil Loss Equation (USLE)[[18], for example, is frequently used to predict soil erosion based on rainfall erosivity (R), soil erodibility (K), slope length and steepness (LS), cover and management (C), and support practises (P). The USLE is written as follows:

$$A = R \cdot K \cdot LS \cdot C \cdot P \tag{8.1}$$

The Revised Universal Soil Loss Equation (RUSLE)[19,20,21] expands on USLE by taking into account the impact of land management practises and conservation measures. It may be stated as follows:

$$A = R \cdot K \cdot LS \cdot C \cdot P \cdot C_{\text{factor}} \cdot P_{\text{factor}} \tag{8.2}$$

Furthermore, physically-based models such as the Soil and Water Assessment Tool (SWAT)[22] mimic hydrological processes such as erosion by combining soil parameters, land use, climate, and terrain. SWAT's erosion component may be expressed using the following equations:

$$\text{Erosion} = \text{R} \cdot \text{K} \cdot \text{LS} \cdot \text{C} \cdot \text{P} \cdot \text{O}$$

(8.3)

9 Hydrological Modeling for Watershed Management

Finally, mathematical models such as USLE, RUSLE, and SWAT can be used to estimate soil erosion risk and guide soil conservation measures. These models assist researchers and land managers in better understanding the complex processes involved in soil erosion and making educated decisions to reduce its negative environmental effects.

SWAT[23,24] is a widely-used model for simulating the hydrology of large, complex watersheds. Its key equations include the water balance equation :

$$\Delta S = P - Q_s - Q_q - E - \Delta S_{surf} \tag{9.1}$$

HSPF, on the other hand, is a complete model that incorporates various hydrological processes, including runoff, groundwater flow, and water quality. A very important equation in HSPF is the equation corresponds to kinematic wave for overland flow :

$$\frac{\partial V}{\partial t} = \frac{\partial (SI)}{\partial x} \tag{9.2}$$

VIC is a distributed hydrological model suitable for mountainous regions. It employs equations such as the VIC soil moisture accounting equation :

$$\frac{dS_w}{dt} = P - R - E - \frac{S_w}{S_{max}} \cdot (R_{in} - E) \tag{9.3}$$

These models are useful for measuring the effects of land use changes and climatic variability on water resources, allowing for more informed decision-making for long-term watershed management.

10 Mathematics in Precision Livestock Farming

With the advances in mathematics, data science, and sensor technology, Precision livestock farming (PLF) has evolved as a game-changing strategy in modern agriculture. Some specific mathematical models demonstrate the relevance of mathematics in PLF are as follows:

Understanding and forecasting animal behaviour is essential for effective farm management. The Hidden Markov Model (HMM)[25,26,27], which can represent the transitions between distinct behavioural states of cattle, is a popular model in PLF. The following are the model equations:

$$P(S_{t+1} = s_j | S_t = s_i) = \pi_{ij}, \quad i, j \in \{1, 2, \ldots, N\} \tag{10.1}$$

Where S_t represents the hidden behavioral state at time t, and π_{ij} is the transition probability from state s_i to state s_j. Optimising feed efficiency is a primary goal of PLF. Linear programming is commonly used to solve feed ration optimisation concerns[28,29]. The objective function and limitations are as follows:

$$\begin{aligned}
\text{Minimize } & Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{Subject to } & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \geq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}$$

Where x_i represents the amount of a particular feed component, c_i is the cost of that component, and a_{ij} are coefficients describing the nutritional content of the feed components.

Early diagnosis of animal illnesses is critical for outbreak prevention. In PLF, machine learning methods such as Support Vector Machines (SVM)[30,31] are used to predict illness. The SVM model is expressed as follows:

$$f(x) = \sum_{i=1}^{N} \alpha_i y_i K(x_i, x) + b \tag{10.2}$$

Where x is the input data vector, N is the number of support vectors, α_i are the support vector coefficients, y_i is the class label, and $K(x_i, x)$ is the kernel function.

PLF relies heavily on mathematical techniques, which allows for the modelling of optimisation of feed efficiency , behaviour of animal , and prediction of diseases , eventually leading to more effective and sustainable practises in livestock management.

11 Conclusion

In this study, we explore the applications of mathematics in different fields of agricultural engineering, such as crop growth, irrigation management, soil moisture modelling, environmental management, pest and disease management, fertiliser applications, watershed management, etc. The study is not exhaustive but gives an insight into why mathematics is essential in agricultural engineering. It highlights the potential of different mathematical techniques for addressing complex challenges in agriculture, leading to innovative solutions for a sustainable global food system.

References

1. Verhulst, P. F. (1838). "Notice sur la loi que la population poursuit dans son accroissement." Correspondance mathématique et physique, 10, 113-121.
2. Thornley, J. H., & France, J. (2007). "Mathematical models in agriculture: A quantitative approach to problems in agriculture and related sciences." CABI.
3. Monteith, J. L. (1977). "Climate and the efficiency of crop production in Britain [and discussion]." Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences, 281(980), 277-294.
4. Ritchie, J. T. (1972). "Model for predicting evaporation from a row crop with incomplete cover." Water Resources Research, 8(5), 1204-1213.
5. Allen, R. G., Pereira, L. S., Raes, D., & Smith, M. (1998). Crop evapotranspiration: Guidelines for computing crop water requirements. FAO Irrigation and Drainage Paper 56
6. Doorenbos, J., & Pruitt, W. O. (1977). Guidelines for predicting crop water requirements. FAO Irrigation and Drainage Paper 24.
7. Etienne Muller, Henri Décamps,Modeling soil moisture–reflectance, Remote Sensing of Environment,Volume 76, Issue 2,2001,Pages 173-180
8. Abriola, L. M., J. Lang, and K. Rathfelder, Michigan soil vapor ex- traction remediation model: A computer program to model soil vapor extractions and bioventing of organic chemicals in unsatur- ated geologic material, Rep. R-97/099, EPA/600, U.S. Environ. Prot. Agency, Washington, D.C., 1997
9. Thornthwaite, C. W., An approach toward a rational classification of climate, Geogr. Rev., 38(1), 55-94, 1948.
10. Thornthwaite, C. W., and J. R. Mather, The water balance, Publ. Climatol. Lab. Climatol. Drexel Inst. Technol., 8(1), 1-104, 1955.
11. Thornthwaite, C. W., and J. R. Mather, Instructions and tables for computing potential evapotranspiration and the water balance, Publ. Climatol. Lab. Climatol. Drexel Inst. Technol., 10(3), 185-311, 1957
12. Ponce, V. M., and R. H. Hawkins (1995), Runoff curve number: Has it reached maturity? J. Hydrol. Eng., 1, 11–19.

UNDER PEER REVIEW

13. T. Takakura, C. Kubota, S. Sase, M. Hayashi, M. Ishii, K. Takayama, H. Nishina, K. Kurata, G.A. Giacomelli, Measurement of evapotranspiration rate in a single-span greenhouse using the energy-balance equation,Biosystems Engineering,Volume 102, Issue 3, Pages 298-304,2009
14. Matis, J. H., and M. J. Al-Muhammed. "Theory and application of the logistic probability density function as a population growth model." Damascus University Journal for Basics Science 26, no. 1, 9-19,2010.
15. Bastianelli, Denis, Daniel Sauvant, and Alain Rérat. "Mathematical modeling of digestion and nutrient absorption in pigs." Journal of animal science 74.8,1873-1887,1996.
16. Monod J The growth of bacterial cultures. Annu Rev Microbiol 3:371–394, 1949
17. Bara, Najat, Frédéric Gautier, and Vincent Giard. "An economic evaluation of operational decisions–an application in scheduling evaluation in fertilizer plants." Production Planning Control 32.9, 699-714,2021.

18 Wischmeier WH, Smith DD. Predicting rainfall erosion losses – a guide for conservation planning. U.S. Department of Agriculture, Agriculture Handbook 537. 1978:20-152.

19 Renard KG, Freimund JR , Using monthly precipitation data to estimate the R factor in the revised USLE. J Hydrol 157:287–306,1994

20 Renard KG, Foster GR, Weesies GA, McCool DK , Predicting soil erosion by water. A guide to conservation planning with the revised universal soil loss equation (RUSLE). Agric. Handbook 703. US Govt Print Office, Washington, DC,1996

21 Renard KG, Foster GR, Weesies GA, McCool DK, Yoder DC Predicting soil erosion by water: a guide to conservation planning with the Revised Universal Soil Loss Equation (RUSLE). Agriculture Handbook No. 703, USDA-ARS,1997

22 Arnold, J. G., R. Srinivasan, R. S. Muttiah, and J. R. Williams. . Large-area hydrologic modeling and assessment: Part I. Model,1998

23 Neitsch, S. L., J. G. Arnold, J. R. Kiniry, R. Srinivasan, and J. R. Williams. 2004. Soil and Water Assessment Tool Input / Output File Documentation. Ver. 2005. Temple, Tex.: USDA-ARS Grassland Soil and Water Research Laboratory.

24 Neitsch, S. L., J. G. Arnold, J. R. Kiniry, and J. R. Williams. 2005. Soil and Water Assessment Tool Theoretical Documentation. Ver. Temple, Tex.: USDA-ARS Grassland Soil and Water Research Laboratory, and Texas AM University, Blackland Research and Extension Center development. J. American Water Resour. Assoc. 34(1): 73-89,2005.

25 Guo, Ying, et al. "Using accelerometer, high sample rate GPS and magnetometer data to develop a cattle movement and behaviour model." Ecological Modelling 220.17: 2068-2075,2009.

26 Sean R Eddy,Hidden Markov models,Current Opinion in Structural Biology, Volume 6, Issue 3,Pages 361-365,1996

27 Williams, M. L., W. P. James, and M. T. Rose. "Fixed-time data segmentation and behavior classification of pasture-based cattle: Enhancing performance using a hidden Markov model." Computers and electronics in agriculture 142: 585-596,2017.

28 F. Waugh, —The Minimum Cost Diary Feed,‖ J. Farm Econ., vol. 33, no. 3, pp. 299–310, 1951.

29 G. B.Dantzig, Linear Programming and Extensions. NJ.: Princeton, 1963

30 Jiang, Feng, et al. "Image recognition of four rice leaf diseases based on deep learning and support vector machine." Computers and Electronics in Agriculture 179 ,105824,2020.

31 Pinki, Farhana Tazmim, Nipa Khatun, and SM Mohidul Islam. "Content based paddy leaf disease recognition and remedy prediction using support vector machine." 2017 20th international conference of computer and information technology (ICCIT). IEEE, 2017.