Integer Solutions of the Diophantine Equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$

Abstract: In this paper, we mainly find all solutions of the diophantine equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$ in positive integer variables (x, y, z, l) .

Key words: Diophantine Equation ; Integer Solution

1 Introduction

The so-called indefinite equation is a type of equation, characterized by fewer equations than the number of variables, and its solution is subject to certain limitations (such as rational numbers, integers, or positive integers, etc.) [1][2].

In the early 3rd century, the ancient Greek mathematician Diophantus, who had systematically studied this type of equation, was known as the ancestor of the indefinite equation, hence the indefinite equation is also called the Diophantus equation. Ko Zhao was one of the founders and pioneers of modern number theory in China. He devoted himself to the study of indefinite equations and made important contributions in many aspects of this field, especially in the study of the famous Catalan conjecture, which obtained an important result known as KOch's theorem.

Indefinite equations are an important branch of number theory and one of the most active mathematical fields in history. They have also been fully demonstrated in mathematical competitions around the world. For solving indefinite equations, people must be require to creatively solve them through the ideas, methods, and techniques of elementary number theory. Here we mainly discuss the integer solutions for a equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$[1].

2 Positive integer solutions of the equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}$

For the symmetry of m, n, k in the indeterminate equation

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{l}, \tag{2.1}$$

, we assume $x \geq y \geq z$ in this section. Obviously, z and l are not equal to 1, thus, $x \geq y \geq z \geq 2$, therefore,

$$\frac{1}{l}=(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq (\frac{1}{2})^3=\frac{1}{8},$$

so

$$l \leq 8.$$

Cosidering $l \geq 2$, if the equation (2.1) has positive integer solution, there must be $2 \leq l \leq 8$. We will discuss these seven cases sequencely.

Case 1. $l=2$

At this point, the equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{2} \tag{2.2}$$

(1) If $z \geq 5$, and $x \geq y \geq 5$, thus

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq (\frac{4}{5})^3 > \frac{1}{2}.$$

so, the equation (2.2) has no integer solution in this case.

(2) If $z = 4$, the equation (2.2) is

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{3}{4}=\frac{1}{2} \tag{2.3}$$

namely,

$$(x-3)(y-3)=6.$$

For the assumption $x \geq y$, we get

$$\begin{cases} x = 9 \\ y = 4 \end{cases} \qquad \begin{cases} x = 6 \\ y = 5 \end{cases}$$

There are two integer solutions of the equation (2.1) , that is

$$\begin{cases} x = 9 \\ y = 4 \\ k = 4 \\ l = 2 \end{cases} \qquad \begin{cases} x = 6 \\ y = 5 \\ k = 4 \\ l = 2 \end{cases}$$

(3) If $z = 3$, the equation (2.2) is

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{2}{3}=\frac{1}{2} \tag{2.4}$$

namely,

$$(x-4)(y-4)=12 \tag{2.5}$$

For the assumption $x \geq y$, we get

$$\begin{cases} x = 16 \\ y = 5 \end{cases} \qquad \begin{cases} x = 10 \\ y = 6 \end{cases} \qquad \begin{cases} x = 8 \\ y = 7 \end{cases}$$

There are three integer solutions of the equation (2.1) , that is

$$\begin{cases} x = 16 \\ y = 5 \\ k = 3 \\ l = 2 \end{cases} \qquad \begin{cases} x = 10 \\ y = 6 \\ k = 3 \\ l = 2 \end{cases} \qquad \begin{cases} x = 8 \\ y = 7 \\ k = 3 \\ l = 2 \end{cases}$$

(4) If $z = 2$, the equation (2.2) is

$$(1-\frac{1}{x})(1-\frac{1}{y})=1 \tag{2.6}$$

There is not hold for the positive integer number x, y, at the above equation, so the equation(2.1) has no positive integer solution for $z = 2$.

Case 2. $l = 3$

At this point, the equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{3} \tag{2.7}$$

(1) If $k \geq 4$, and $x \geq y \geq 4$, thus

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq (\frac{3}{4})^3 > \frac{1}{3}$$

so, the equation (2.7) has no integer solution in this case.

(2) If $z = 3$, the equation (2.7) can be transformed into

$$(x-2)(y-2)=2 \tag{2.8}$$

that is $l = 3, k = 3$,and

$$\begin{cases} x &= 4 \\ y &= 3 \end{cases}$$

(3) If $z = 2$, the equation (2.7) can be changed to

$$(x-3)(y-3)=6 \tag{2.9}$$

we obtain that $z = 2, l = 3$,

$$\begin{cases} x-3 &= 6 \\ y-3 &= 1 \end{cases} \qquad \begin{cases} x-3 &= 3 \\ n-3 &= 2 \end{cases}$$

namely,

$$\begin{cases} x &= 9 \\ y &= 4 \\ z &= 2 \\ l &= 3 \end{cases} \qquad \begin{cases} x &= 6 \\ y &= 5 \\ z &= 2 \\ l &= 3 \end{cases}$$

Case 3. $l = 4$

The equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z})=\frac{1}{4} \tag{2.10}$$

(1) If $k \geq 3$, then $x \geq y \geq z \geq 3$, we obtain

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) \geq (\frac{2}{3})^3 > \frac{1}{4}$$

The equation (2.10) has no solutions.

(2) If $z = 2$, the equation (2.10) can be transformed into

$$(x-2)(y-2)=2$$

then $l = 4, z = 2,$, and

$$\begin{cases} x &= 4 \\ y &= 3 \\ z &= 2 \\ l &= 4 \end{cases}$$

Case 4. $l = 5$

The equation (2.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{5} \tag{2.11}$$

(1) If $z \geq 3$, then $x \geq y \geq z \geq 3$, we obtain

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) \geq (\frac{2}{3})^3 > \frac{1}{5}$$

The equation (2.11) has no solutions.

(2) If $z = 2$, the equation (2.11) can be simplified to

$$(3x - 5)(3y - 5) = 10. \tag{2.12}$$

then $l = 5, z = 2,$, and

$$\begin{cases} x &= 5 \\ y &= 2 \\ z &= 2 \\ l &= 5 \end{cases}$$

Case 5. $l = 6$

The equation (2.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{6} \tag{2.13}$$

Samely, z must be less than 3, that is $z = 2$, the equation (2.13) can be simplified to

$$(2x - 3)(2y - 3) = 3 \tag{2.14}$$

Then $l = 6, k = 2$,and

$$\begin{cases} x &= 3 \\ y &= 2 \\ z &= 2 \\ l &= 6 \end{cases}$$

Case 6. $l = 7$

Samely, the equation (2.1) is

$$(1 - \frac{1}{x})(1 - \frac{1}{y})(1 - \frac{1}{z}) = \frac{1}{7} \tag{2.15}$$

, and $z = 2$, then the equation (2.15) can be simplified to

$$(5x - 7)(5y - 7) = 14 \tag{2.16}$$

we can see that no positive integer solutions for the above equation.

Case 7. $l = 8$

Samely, the equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = \frac{1}{8} \tag{2.17}$$

and $z = 2$, then the equation (2.17) can be simplified to

$$(3x-4)(3y-4) = 4 \tag{2.18}$$

we can obtain a group of positive integer solution, namely, $l = 8, z = 2$,

$$\begin{cases} x &= 2 \\ y &= 2 \\ z &= 2 \\ l &= 8 \end{cases}$$

To summarize, the equation (2.17) has positive integer solutions ($Suppose x \geq y \geq z$), namely,

$$\begin{cases} x_1 &= 16 \\ y_1 &= 5 \\ z_1 &= 3 \\ l_1 &= 2 \end{cases} \quad \begin{cases} x_2 &= 10 \\ y_2 &= 6 \\ z_2 &= 3 \\ l_2 &= 2 \end{cases} \quad \begin{cases} x_3 &= 8 \\ y_3 &= 7 \\ z_3 &= 3 \\ l_3 &= 2 \end{cases} \quad \begin{cases} x_4 &= 9 \\ y_4 &= 4 \\ z_4 &= 4 \\ l_4 &= 2 \end{cases}$$

$$\begin{cases} x_5 &= 6 \\ y_5 &= 5 \\ z_5 &= 4 \\ l_5 &= 2 \end{cases} \quad \begin{cases} x_6 &= 9 \\ y_6 &= 4 \\ z_6 &= 2 \\ l_6 &= 3 \end{cases} \quad \begin{cases} x_7 &= 6 \\ y_7 &= 5 \\ z_7 &= 2 \\ l_7 &= 3 \end{cases} \quad \begin{cases} x_8 &= 4 \\ y_8 &= 3 \\ z_8 &= 3 \\ l_8 &= 3 \end{cases}$$

$$\begin{cases} x_9 &= 4 \\ y_9 &= 3 \\ z_9 &= 2 \\ l_9 &= 4 \end{cases} \quad \begin{cases} x_{10} &= 5 \\ y_{10} &= 2 \\ z_{10} &= 2 \\ l_{10} &= 5 \end{cases} \quad \begin{cases} x_{11} &= 3 \\ y_{11} &= 2 \\ z_{11} &= 2 \\ l_{11} &= 6 \end{cases} \quad \begin{cases} x_{12} &= 2 \\ y_{12} &= 2 \\ z_{12} &= 2 \\ l_{12} &= 8 \end{cases}$$

3 Negative integer solutions of the equation $(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = \frac{1}{l}$

In this section, we try to discuss the case where x, y and z can be negative integers. Hence the item

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > 0,$$

whether x, y and z are positive or negative numbers, thus l is a postive integer.

Case 1. If x, y and z are negative integer number,

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > 1.$$

we can see that the equation (2.1) has no solution.

Case 2. If there are two negative integer in x, y and z, there is no harm in suppose that x, y are negative, then

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > 1-\frac{1}{z} \geq 1-\frac{1}{2} = \frac{1}{2}$$

that is l=1 only.

We transforming the equation (2.1) into

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = 1$$

that is

$$(x+z-1)(y+z-1) = z(z-1)$$

Suppose $z = t$, than $t \geq 2$. If d is a fator of $t(t-1)$, we obtain

$$\begin{cases} x + t-1 = d \\ y + t-1 = \frac{t(t-1)}{d} \end{cases}$$

so

$$\begin{cases} x = 1 - t + d \\ y = 1 - t + \frac{t(t-1)}{d} \end{cases}$$

where $d < t-1, \frac{t}{d} < 1$, the condition must be $d < 0$. So let's $d = -d$, we obtain the solution of the equation (2.1) is

$$\begin{cases} x = 1-t-d \\ y = 1-t-\frac{t(t-1)}{d} \\ z = t \\ l = 1 \end{cases}$$

Case 3. If there is only a negative integer in x, y and z, suppose that x is negative, for $y \geq z \geq 2$, we have

$$\frac{1}{l} = (1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

so $l < 4$. We will discuss the cases for $l = 1, 2, 3$ at the following.

(1) If $l = 1$, the equation (2.1) is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = 1$$

Following the discussion above, we get

$$\begin{cases} x = 1-t+d \\ y = 1-t+\frac{t(t-1)}{d} \\ z = t \\ l = 1 \end{cases}$$

where d is a factor of $t(t-1)$, $t \geq 2$.

(2) If $l = 2$, the equation (2.1) is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = \frac{1}{2} \tag{3.19}$$

1^o. $z \geq 4$, the above equation is equal to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > (1-\frac{1}{y})(1-\frac{1}{z}) \geq (\frac{3}{4})^2 > \frac{1}{2}$$

we can obtain that the equation (2.1) has no solution if $z \geq 4$.

2^o. $z = 2$, we have

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{1}{2} = \frac{1}{2}$$

namely, $x + y = 1$.

Let $y = t \geq 2$, than

$$\begin{cases} x = 1-t \\ y = t \\ z = 2 \\ l = 2 \end{cases} \qquad (t \geq 2)$$

3^o. $z = 3$ we have

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{2}{3} = \frac{1}{2}$$

namely

$$(x-4)(y-4) = 12$$

.

considering $x < 0$, so

$$\begin{cases} x = -2 \\ y = 2 \\ z = 3 \\ l = 2 \end{cases} \qquad \begin{cases} x = -8 \\ y = 3 \\ z = 3 \\ l = 2 \end{cases}$$

(3) If $l = 3$, the equation (2.1) can be simplified to

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) = \frac{1}{3}$$

Suppose $y \geq z \geq 3$, so

$$(1-\frac{1}{x})(1-\frac{1}{y})(1-\frac{1}{z}) > (\frac{2}{3})^2 > \frac{1}{3}.$$

We can see that the equation (2.1) has no solution.

So, $z = 2$, the equation (2.1) is

$$(1-\frac{1}{x})(1-\frac{1}{y}) \times \frac{1}{2} = \frac{1}{3}.$$

It can be simplified to

$$(x-3)(y-3) = 6$$

so

$$\begin{cases} x = -3 \\ y = 2 \\ z = 2 \\ l = 3 \end{cases}$$

Based on the discussion above, we have the negative integer solutions of the equation (2.1) are

$$\begin{cases} x &= 1-t-d \\ y &= 1-t-\frac{t(t-1)}{d} \\ z &= t \\ l &= 1 \end{cases} \qquad (t \geq 2, d \text{ is a positive factor of } t(t-1))$$

$$\begin{cases} x &= 1-t+d \\ y &= 1-t+\frac{t(t-1)}{d} \\ z &= t \\ l &= 1 \end{cases} \qquad (t \geq 2, d \text{ is a factor of } t(t-1))$$

$$\begin{cases} x &= 1-t \\ y &= t \\ z &= 2 \\ l &= 2 \end{cases} \qquad (t \geq 2,)$$

$$\begin{cases} x &= -2 \\ y &= 2 \\ z &= 3 \\ l &= 2 \end{cases} \quad \begin{cases} x &= -8 \\ y &= 3 \\ z &= 3 \\ l &= 2 \end{cases} \quad \begin{cases} x &= -3 \\ y &= 2 \\ z &= 2 \\ l &= 3 \end{cases}$$

References

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