Original Research Article

ALPHA POWER CUBIC TRANSMUTED DAGUM DISTRIBUTION WITH APPLICATIONS TO REAL LIFE DATA

Abstract:

In this article, a new model is proposed called alpha power cubic transmuted distribution. Some properties of the new distribution such as survival and hazard functions and other useful measures were obtained. The model parameters were estimated using the method of maximum likelihood. The usefulness of the proposed distribution has been shown using three real datasets. This indicates that the proposed compound distribution would be useful in the area of distribution theory in Statistics, and in modeling real life data sets relating to biology, demography, geology, economics, environmental sciences, engineering, finance, medicine, hydrology, insurance and many other fields.

Keywords: Alpha power transformation, Dagum distribution, Cubic transmuted Dagum distribution, Statistical properties, parameter estimation, Applications.

1 Introduction

The Dagum distribution was proposed by [1]. His proposals enable the development of statistical distributions used to fit empirical income and wealth data, that could accommodate both heavy tails in empirical income and wealth distributions, and also permit interior mode. Dagum distribution has both Type-I and Type-II specification, where Type-I is the three parameter specifications and Type-II deal with four parameter specification. Dagum in 1977 motivated his model from empirical observation that the income elasticity of the cumulative distribution function (cdf) of income into a decreasing and bounded function of F.

Recently, very useful families of probability distributions have been proposed in the literature and it has been shown that they are useful for adding skewness and flexibility to other models. A brief summary of the recently proposed families include the a new generalized Weibull-G family by [2], Logistic-X family by [3], a new Weibull-G family by [4], a Lindley-G family by [5], a Gompertz-G family by [6], an odd Lindley-G family by [7], Alpha Power transformation by [8], odd Lomax generator of distributions (Odd Lomax-G family) by [9] and Poisson-X family by [10] as so on.

According to [11], the cubic transmuted Dagum distribution based on the cubic transmuted family was found to be better than the transmuted Dagum distribution and the conventional Dagum distribution when applied to three real life data sets.

Therefore our interest in this article is to present another extension of the Dagum distribution using the alpha power transformation proposed by [8] and hope that it will yield a better model for analyzing real life datasets.

According to [11], the cumulative distribution function (*cdf*) and probability density function (*pdf*) of the cubic transmuted Dagum distribution are given by:

$$F(x)=(1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta} \tag{1}$$

and

$$f(x)=\gamma\theta\beta x^{-\theta-1}\left(1+\gamma x^{-\theta}\right)^{-\beta-1}\left[1-\lambda+6\lambda\left(1+\gamma x^{-\theta}\right)^{-\beta}-6\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}\right] \tag{2}$$

respectively. Where, $x>0,\gamma>0,\theta>0,\beta>0,-1\leq\lambda\leq1$ and γ is a scale parameter while θ and β are the shape parameters and λ is the transmuted parameter of the cubic transmuted Dagum distribution (CTDaD).

This article is organized in sections as follows: the new model and graphical representation is given in section 2. Section 3 derived some properties of the new distribution. The estimation formula for the unknown parameters of the distribution using maximum likelihood estimation is provided in section 4. An application of the new model to three data sets on survival time is done in section 5 and a summary and conclusion is presented in section 6.

2. Apha Power Cubic Transmuted Dagum distribution (APCTDaD)

2.1 Definition

According to [8], the cumulative distribution function (cdf) and the probability density function (pdf) of the Alpha Power transformed family of distributions are defined as:

$$F(x)=\frac{\alpha^{G(x)}-1}{\alpha-1} \tag{3}$$

and

$$f(x)=\frac{\log(\alpha)}{(\alpha-1)}g(x)\alpha^{G(x)} \tag{4}$$

"respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\alpha>0$ and $\alpha\neq1$ is the power or shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Substituting equation (1) and (2) in (3) and (4) above and simplifying, we obtain the *cdf* and *pdf* of the APCTDaD for a random variable *X* as follows:

$$F(x)=\frac{\alpha^{\left((1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta}\right)}-1}{\alpha-1} \tag{5}$$

and

$$f(x)=\frac{\log(\alpha)\gamma\theta\beta x^{-\theta-1}\left(1+\gamma x^{-\theta}\right)^{-\beta-1}\left[1-\lambda+6\lambda\left(1+\gamma x^{-\theta}\right)^{-\beta}-6\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}\right]}{(\alpha-1)\alpha^{-(1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta}}} \tag{6}$$

respectively. Where, $x,\alpha,\theta,\gamma,\beta>0,-1\leq\lambda\leq1$ and $\alpha\neq1$ is a power parameter and γ is a scale parameter while θ and β are the shape parameters and λ is the cubic transmuted parameter of the cubic transmuted Dagum distribution (CTDD).

2.3 Graphical Presentation of Pdf and Cdf of APCTDaD

The pdf and cdf of the APCTDaD using some parameter values are displayed in **figures 1** and **2** as follows.

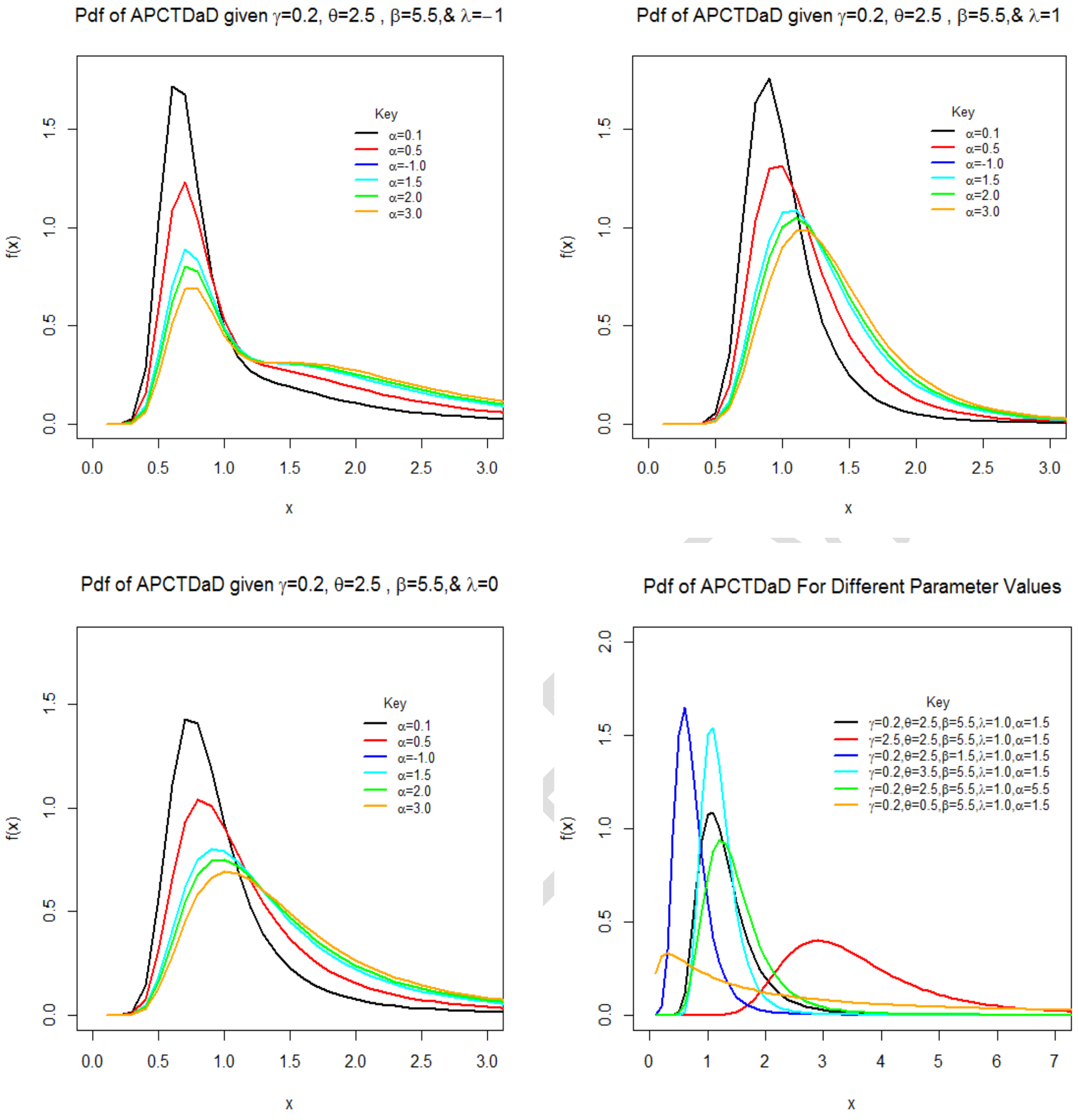


Fig. 1: PDF of the APCTDaD.

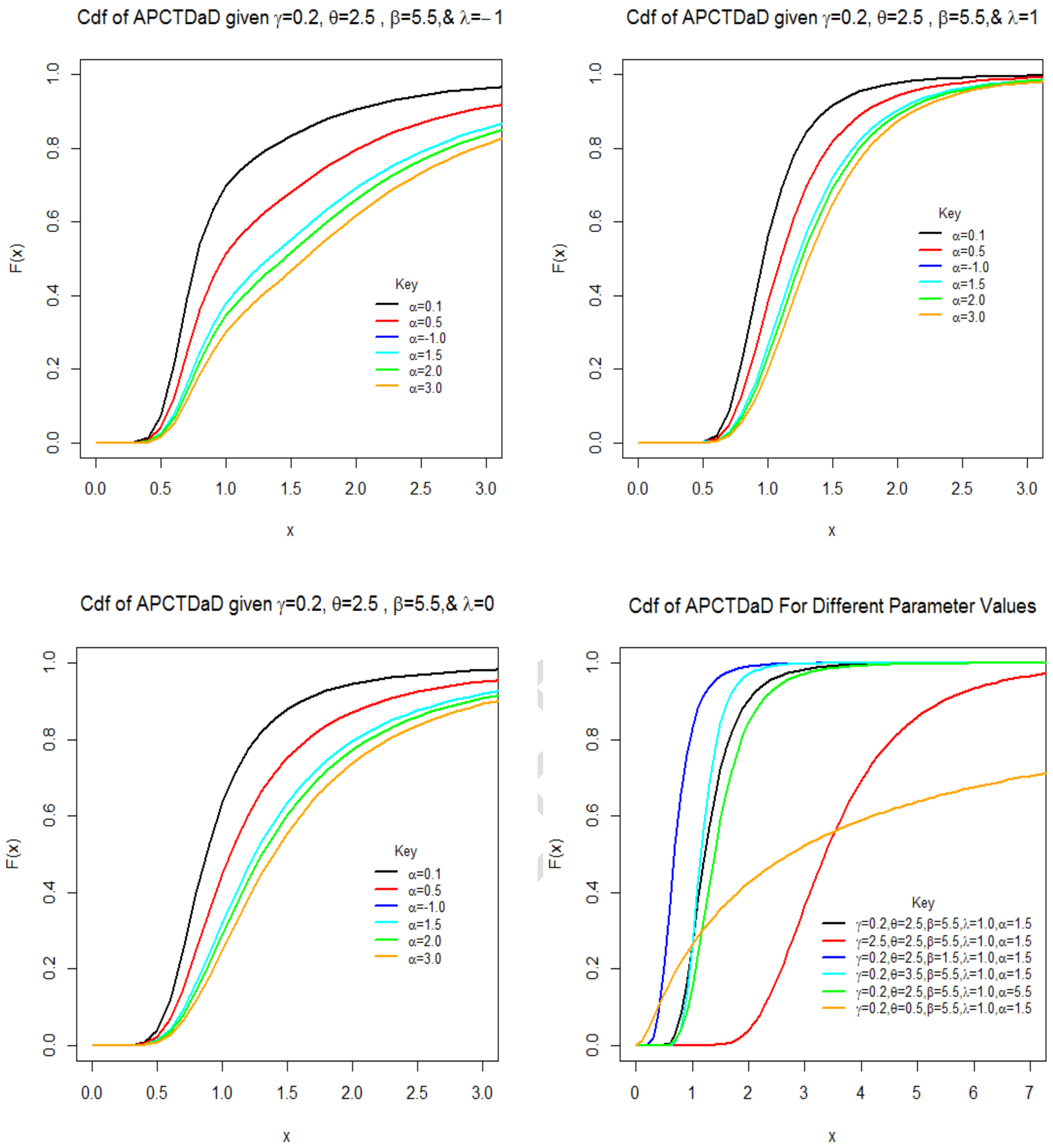


Fig. 2: CDF of the APCTDaD.

Considering the plot of the pdf of the ATCTDaD in figure 1 above, it is clear that the distribution is skewed and flexible and that its shape is flexible depending on the values of the parameters. The plot of cdf in figure 2 also shows that the distribution is a valid probability distribution.

3. Mathematical and Statistical Properties of APCTDaD

This section defined and discusses some properties of the APCTDaD. They are as presented below:

3.1 Reliability analysis of the APCTDaD.

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S\left(x\right)=1-F\left(x\right) \tag{7}$$

Applying the cdf of the APCTDaD in (7), the survival function for the APCTDaD is obtained as:

$$S\left(x\right)=1-\frac{\alpha^{\left((1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta}\right)}-1}{\alpha-1} \tag{8}$$

The following is a plot for the survival function of the APCTDaD using different parameter values as shown in Figure **3** below;

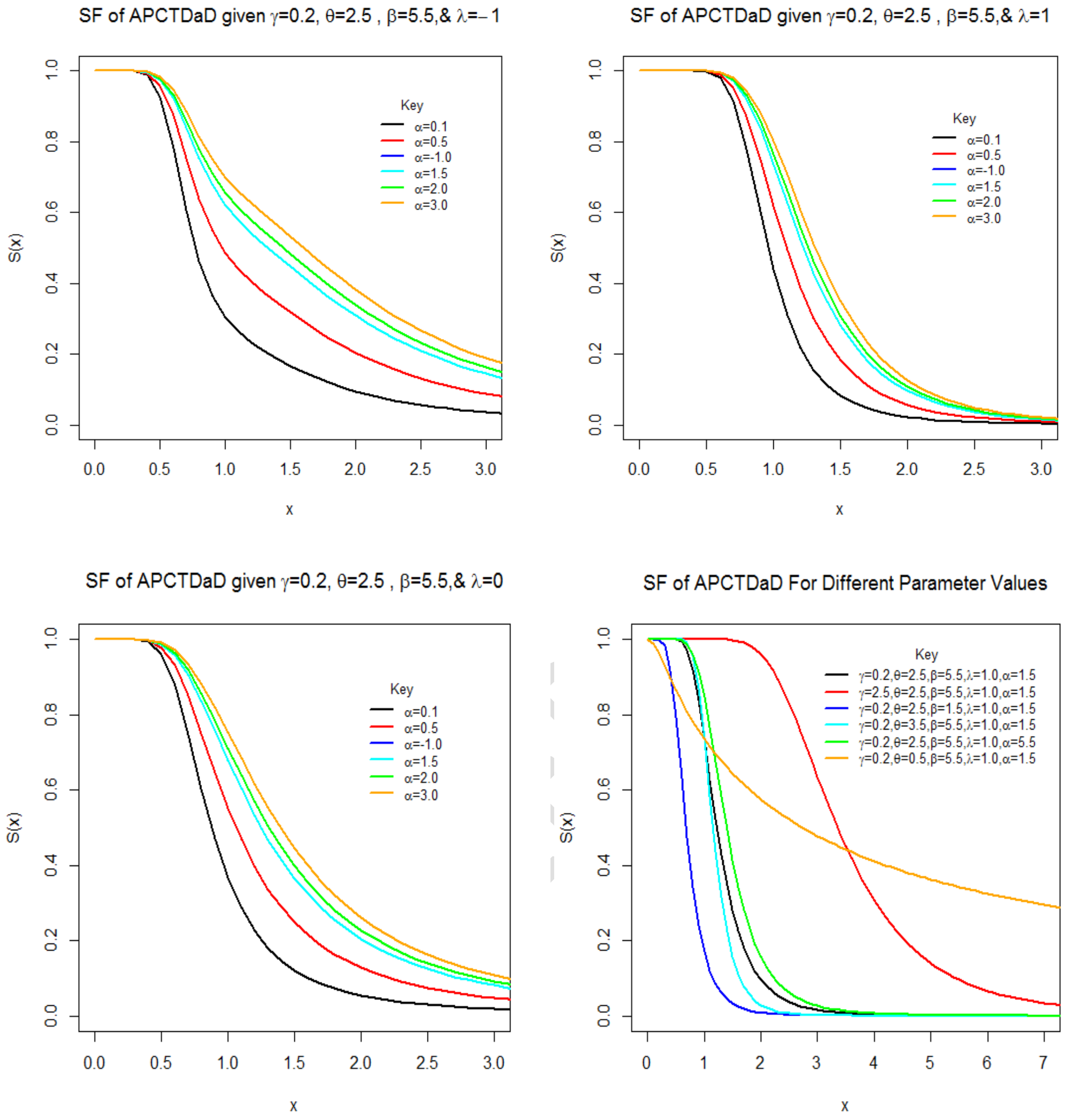


Figure 3.: Survival function of APCTDaD.

Based on the plots of the survival function in figure 3 above, it can be seen that the function is decreasing, that is probability of survival decreases over time or survival rate decreases with increase in age or time.

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined generally as:

$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \tag{9}$$

Therefore, our definition of the hazard rate of the APCTDaD is given by:

$$h\left(x\right)=\frac{\dfrac{\log\left(\alpha\right)\gamma\theta\beta x^{-\theta-1}\left(1+\gamma x^{-\theta}\right)^{-\beta-1}\left[1-\lambda+6\lambda\left(1+\gamma x^{-\theta}\right)^{-\beta}-6\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}\right]}{\left(\alpha-1\right)\alpha^{-(1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta}}}}{1-\dfrac{\alpha^{\left((1-\lambda)\left(1+\gamma x^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x^{-\theta}\right)^{-3\beta}\right)}-1}{\alpha-1}} \qquad (10)$$

The following is a plot of the hazard function for some chosen parameter values

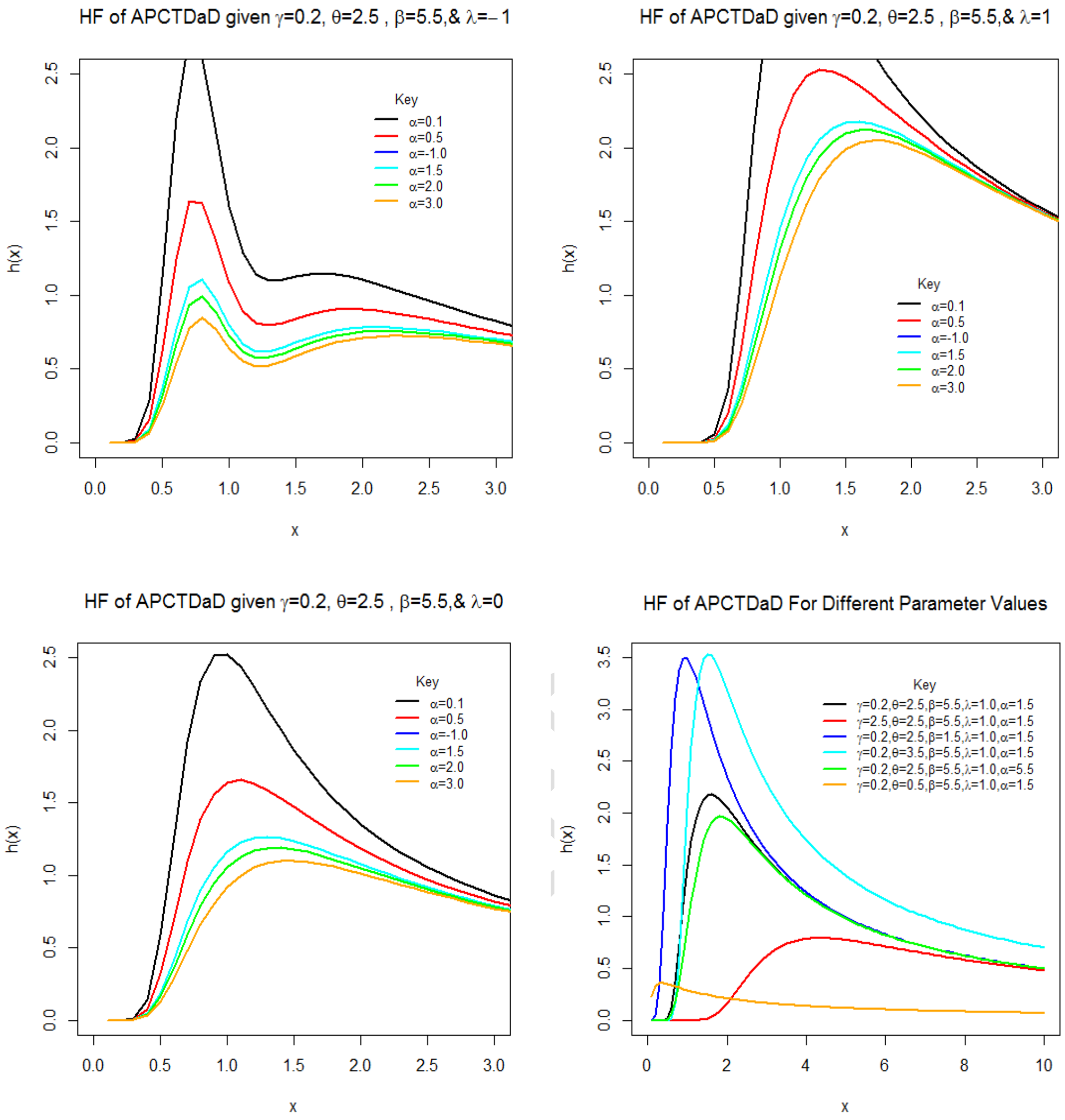


Figure 4: Hazard function of APCTDaD.

Also, the plots of the hazard rate of the distribution are increasing in shape which is an indication that the proposed model would be appropriate for analyzing random variables or events whose failure rate increases and decreases exponentially throughout the entire process.

4. Estimation of unknown Parameters of the APCTDaD

Let $X_1, X_2, ..., X_n$ be a sample of size "n" independently and identically distributed random variables from the APCTDaD with unknown parameters $\alpha, \theta, \gamma, \lambda$ and θ defined previously. The likelihood function is given by:

$$L\left(X \mid \alpha,\beta,\theta,\gamma,\lambda\right)=\frac{\left(\log\left(\alpha\right)\gamma\theta\beta\right)^{n}\prod_{i=1}^{n}\left(x_i^{-\theta-1}\left(1+\gamma x_i^{-\theta}\right)^{-\beta-1}\left[1-\lambda+6\lambda\left(1+\gamma x_i^{-\theta}\right)^{-\beta}-6\lambda\left(1+\gamma x_i^{-\theta}\right)^{-2\beta}\right]\right)}{\left(\alpha-1\right)^{n}\prod_{i=1}^{n}\left(\alpha^{-\left(1-\lambda\right)\left(1+\gamma x_i^{-\theta}\right)^{-\beta}+3\lambda\left(1+\gamma x_i^{-\theta}\right)^{-2\beta}-2\lambda\left(1+\gamma x_i^{-\theta}\right)^{-3\beta}}\right)}$$

(11)

Taking the partial derivative of the natural logarithm of the likelihood function equated to zero with respect to the parameters and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α, β, λ, θ *and* γ. Note that it is difficult to solve the equations analytically and therefore the Newton-Raphson's iteration method is applied using computer packages such as Maple or R with available data.

5. Applications

In this section, three applications of the proposed model to real life datasets are provided to illustrate the flexibility of the APCTDaD introduced in Section 2. The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models. For all the datasets, the fits of APCTDaD is compared with those of the cubic transmuted Dagum distribution (CTDaD), transmuted Dagum distribution (TDaD) and the conventional Dagum distribution (DaD).

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs, ℓ, Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in [12]. Meanwhile, the smaller these statistics are, the better the fit of the distribution is. The required computations are carried out using the R package "AdequacyModel" which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

5.1. Application 1: Based on Dataset I

This data represents the survival times of a group of patients suffering from head and neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT) ([13], [14], [15]).

The following table and figures present a good exploration of the dataset with some explanations.

Table 1: Descriptive Statistics for dataset I

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
44	12.20	67.21	128.5	219.0	223.48	1776.00	93286.4	3.38382	13.5596

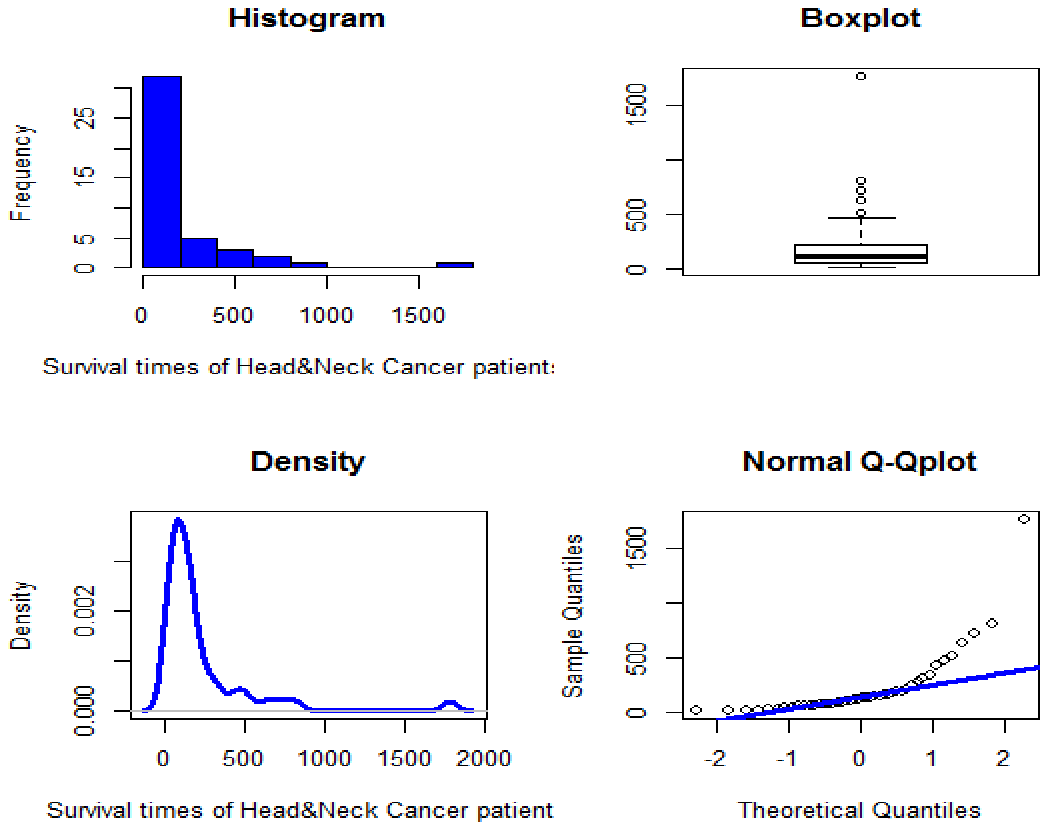


Figure 5: A graphical summary of the dataset I

Considering the dataset in table 1 and figure 5, it can be seen that data set I is skewed to the right or positively skewed and hence suitable for skewed models like the proposed APCTDaD.

Table 2: Maximum Likelihood Parameter Estimates for dataset I

Distribution	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
APCTDaD	2.6115052	0.8723578	9.9574230	0.9234020	8.1805877
CTDaD	9.0478552	0.8359463	4.4704835	0.8879190	-
TDaD	9.3003812	0.8012132	6.9245910	0.9839051	-
DaD	8.4264614	0.9986907	8.9957360	-	-

Table 3: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset I

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
APCTDaD	277.9149	563.8297	564.8554	570.9665	566.4764	1st
CTDaD	277.8219	563.6439	564.6695	570.7806	566.2905	2nd
TDaD	277.8373	565.6745	567.2535	574.5955	568.9828	3rd
DaD	279.5193	565.0386	565.6386	570.3912	567.0236	4th

Table 4: The A^*, W^*, K-S statistic and P-values for dataset I

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
APCTDaD	0.1708722	0.02519533	0.063566	0.9893	1st
CTDaD	0.1479777	0.02205189	0.076078	0.944	2nd
TDaD	0.1489243	0.02100251	0.08537	0.8785	3rd
DaD	0.3116772	0.04943709	0.10655	0.6606	4th

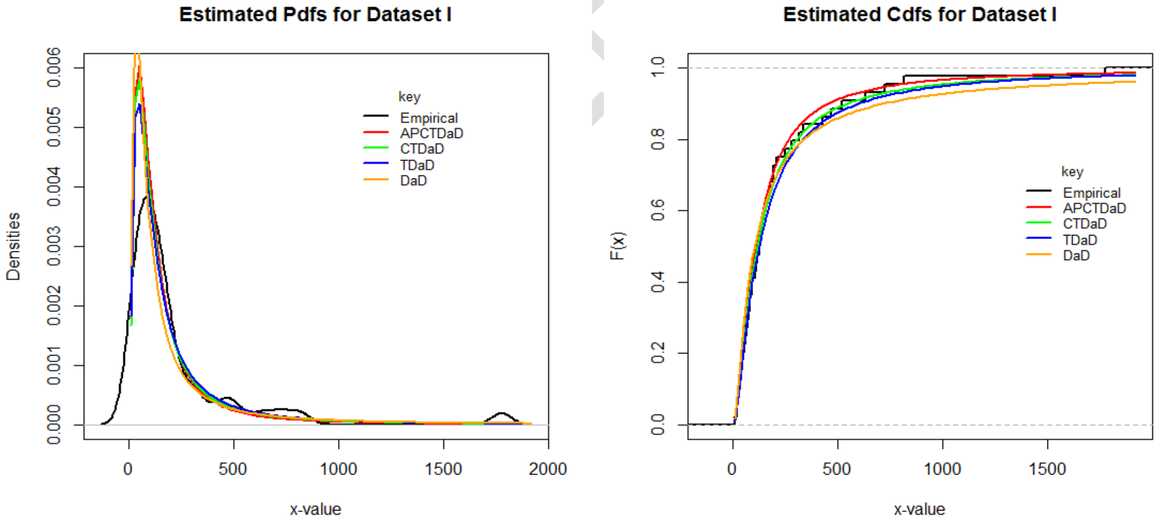


Figure 6: Plots of the estimated densities and cdfs of the fitted distributions to dataset I.

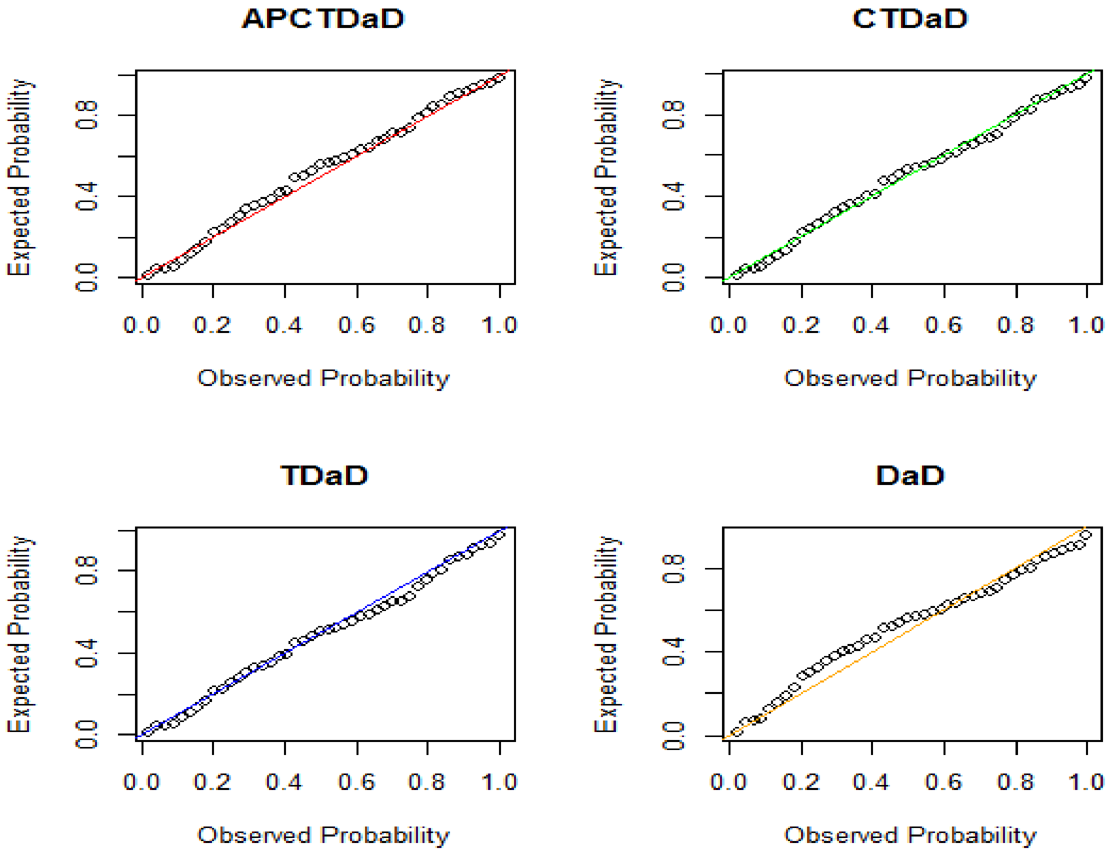


Figure 7: Probability plots for the fit of the APCTDaD, CTDaD, TDaD & DaD based on dataset I.

5.2. Application 2: Based on Dataset II

This data represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by [16], [17], [18] and [19]. It is given and summarized as follows:

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

The following table and figures present a good exploration of the dataset with some explanations.

Table 5: Summary Statistics for the data set II

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

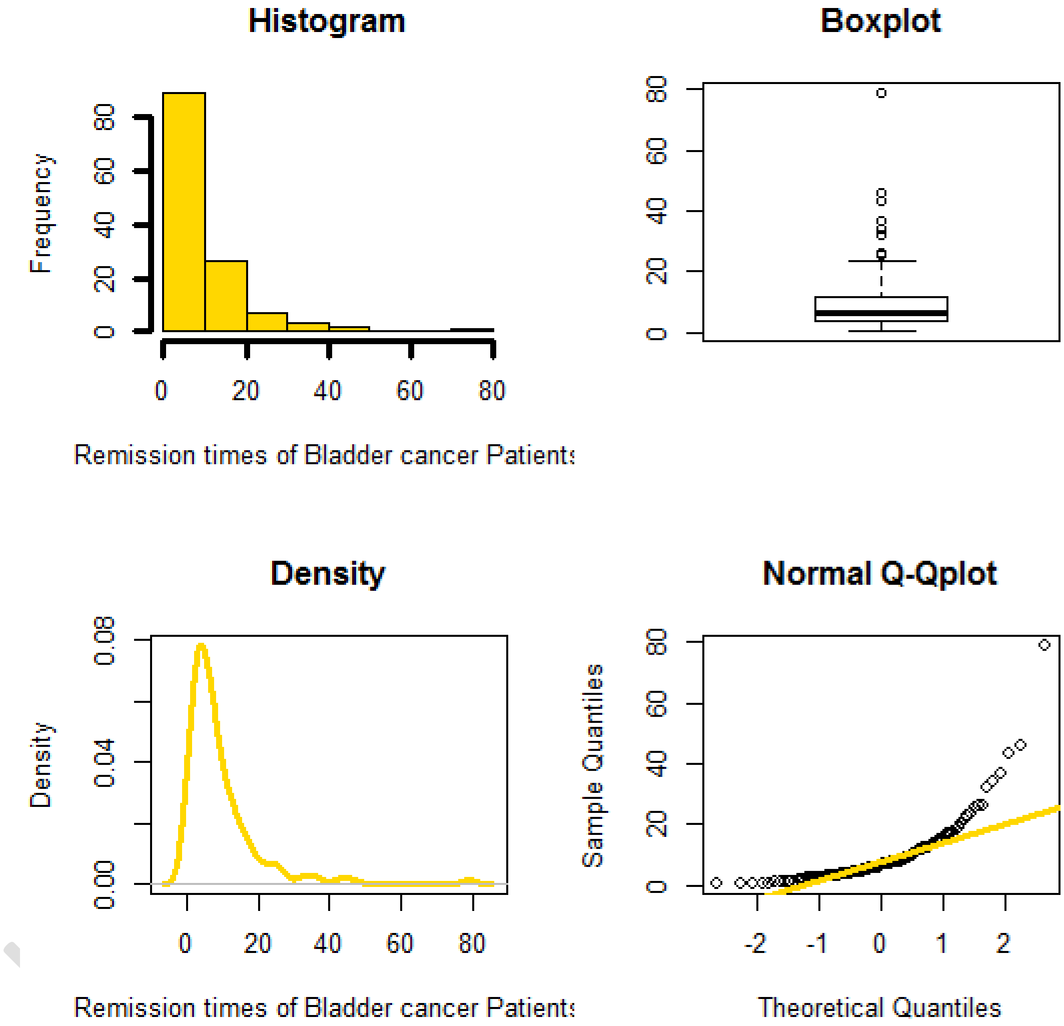


Figure 8: A graphical summary of the dataset II.

Also, looking at the summary statistics and graphical presentation of dataset II in table 5 and figure 8 respectively, it still clear that data set II is skewed to the right or positively skewed and hence good for skewed models like the proposed APCTDaD.

Table 6: Maximum Likelihood Parameter Estimates for dataset II

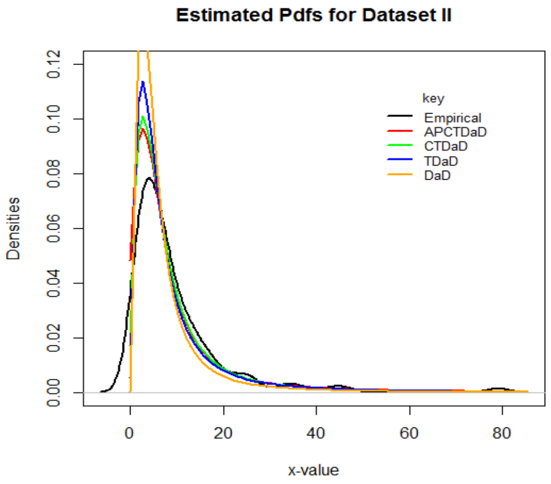
Distribution	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
APCTDaD	9.2007268	1.3511256	0.6522685	0.5313419	7.7493862
CTDaD	9.8212178	1.2807240	1.0175680	0.6221916	-
TDaD	6.1177948	1.5123881	1.1742541	-0.6486219	-
DaD	8.235246	1.712200	1.574800	-	-

Table 7: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset II

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
APCTDaD	411.3755	830.7511	831.0763	842.1592	835.3863	1st
CTDaD	411.5118	833.0236	833.5154	847.2838	838.8176	2nd
TDaD	413.2724	834.5448	834.87	845.9529	839.18	3rd
DaD	420.3174	846.6349	846.8284	855.191	850.1113	4th

Table 8: The A^*, W^*, K-S statistic and P-values for dataset II

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
APCTDaD	0.2838934	0.03962499	0.040366	0.9852	1st
CTDaD	0.2021853	0.02865787	0.040712	0.9838	2nd
TDaD	0.6238288	0.09281483	0.069858	0.5599	3rd
DaD	1.088383	0.1649073	0.14645	0.008252	4th


Estimated Pdfs for Dataset II

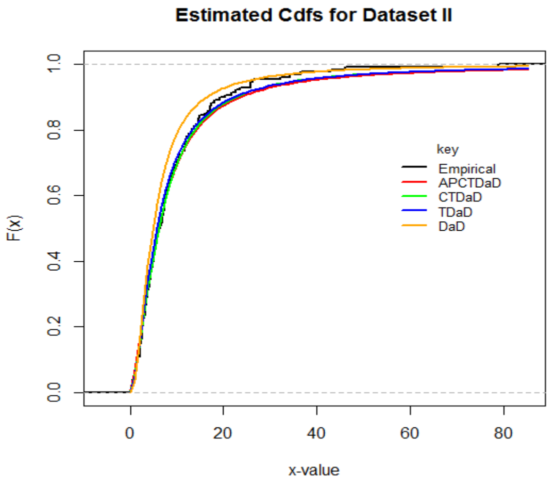

Estimated Cdfs for Dataset II

Figure 9: Plots of the estimated densities and cdfs of the fitted distributions to dataset II.

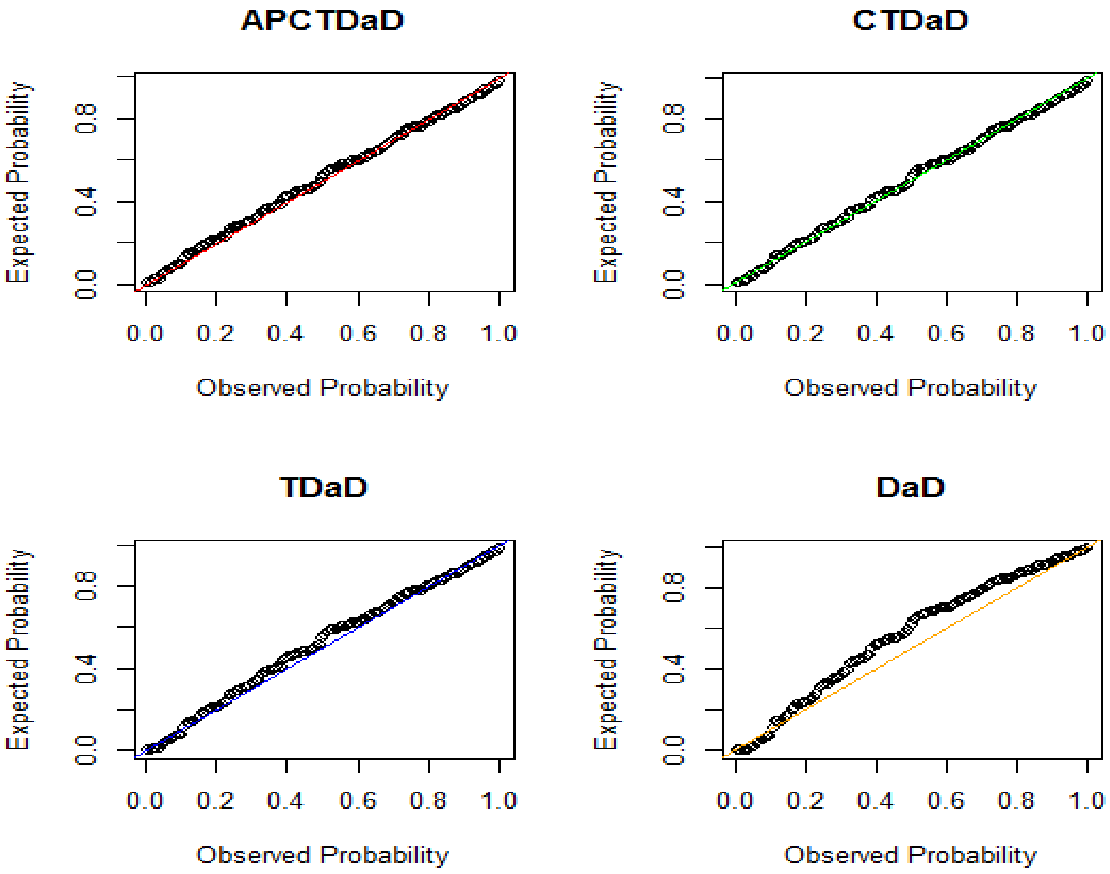


Figure 10: Probability plots for the fit of the APCTDaD, CTDaD, TDaD & DaD based on dataset II.

5.3. Application 3: Based on Dataset III

The dataset represents the survival times (in days) of 72 guinea pigs infected with virulenttubercle bacilli reported by [20]. They are the Regiment 4.3, Study M.: 10, 33, 44,56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116,120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176,183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327,342, 347, 361, 402, 432, 458, 555.

The following table and figures present a good exploration of the dataset with some explanations.

Table 9: Descriptive Statistics for dataset III

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Dataset A	72	10.0	108.0	149.5	224.0	176.8	555.0	10705.1	1.34128	1.98852

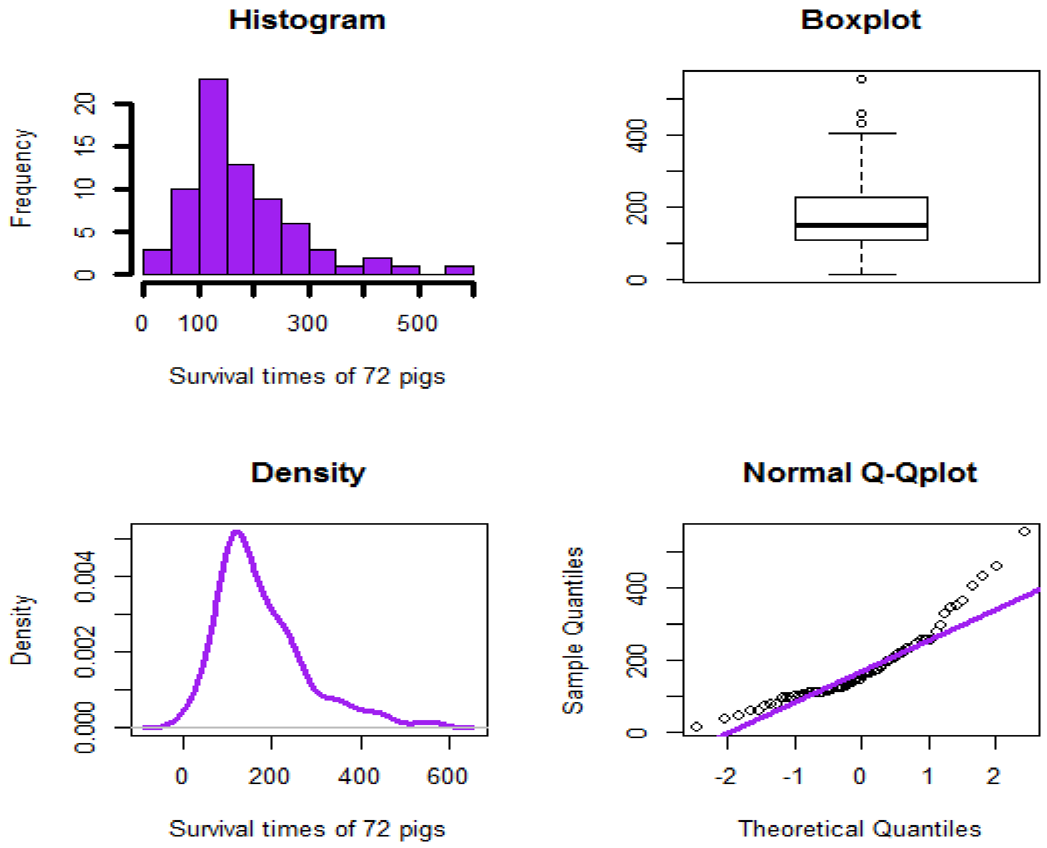


Figure 11: A graphical summary of the dataset III

A summary of dataset III in table 9 and figure 11 has shown that it is a skewed data and therefore requires a skewed and flexible probability distribution for its proper analysis.

Table 10: Maximum Likelihood Parameter Estimates for dataset III

Distribution	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
APCTDaD	8.0772768	1.0429266	9.4344820	0.9586407	8.2604805
CTDaD	8.6642460	0.9446059	9.2356805	0.9792477	-
TDaD	8.5805922	0.8044233	8.6707749	0.9692952	-
DaD	9.0129861	0.9666121	9.1183652	-	-

Table 11: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset III

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
APCTDaD	435.1453	880.2905	881.1996	891.6738	884.8222	1st
CTDaD	438.1489	884.2979	884.8949	893.4045	887.9233	2nd
TDaD	444.912	897.824	898.421	906.9307	901.4494	3rd
DaD	454.7097	915.4195	915.7724	922.2495	918.1385	4th

Table 12: The A^*, W^*, K-S statistic and P-values for dataset III

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
APCTDaD	1.059438	0.1451422	0.16932	0.03223	1st
CTDaD	1.442234	0.2061458	0.16971	0.03161	2nd
TDaD	1.193157	0.1675743	0.19743	0.007299	3rd
DaD	1.775791	0.259787	0.26276	9.62e-05	4th

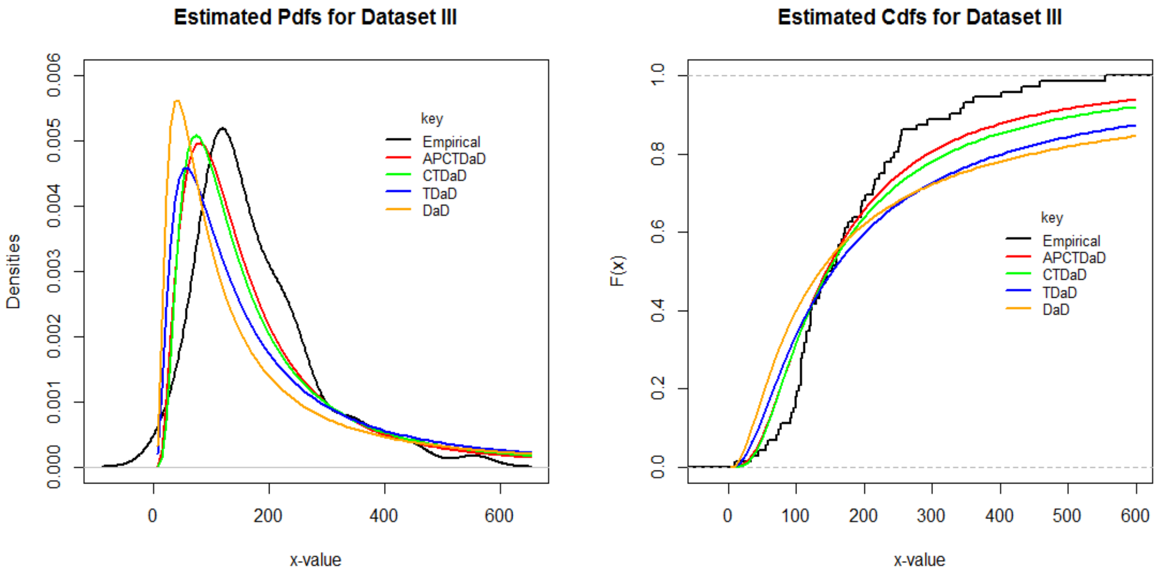


Figure 12: Plots of the estimated densities and cdfs of the fitted distributions to dataset III.

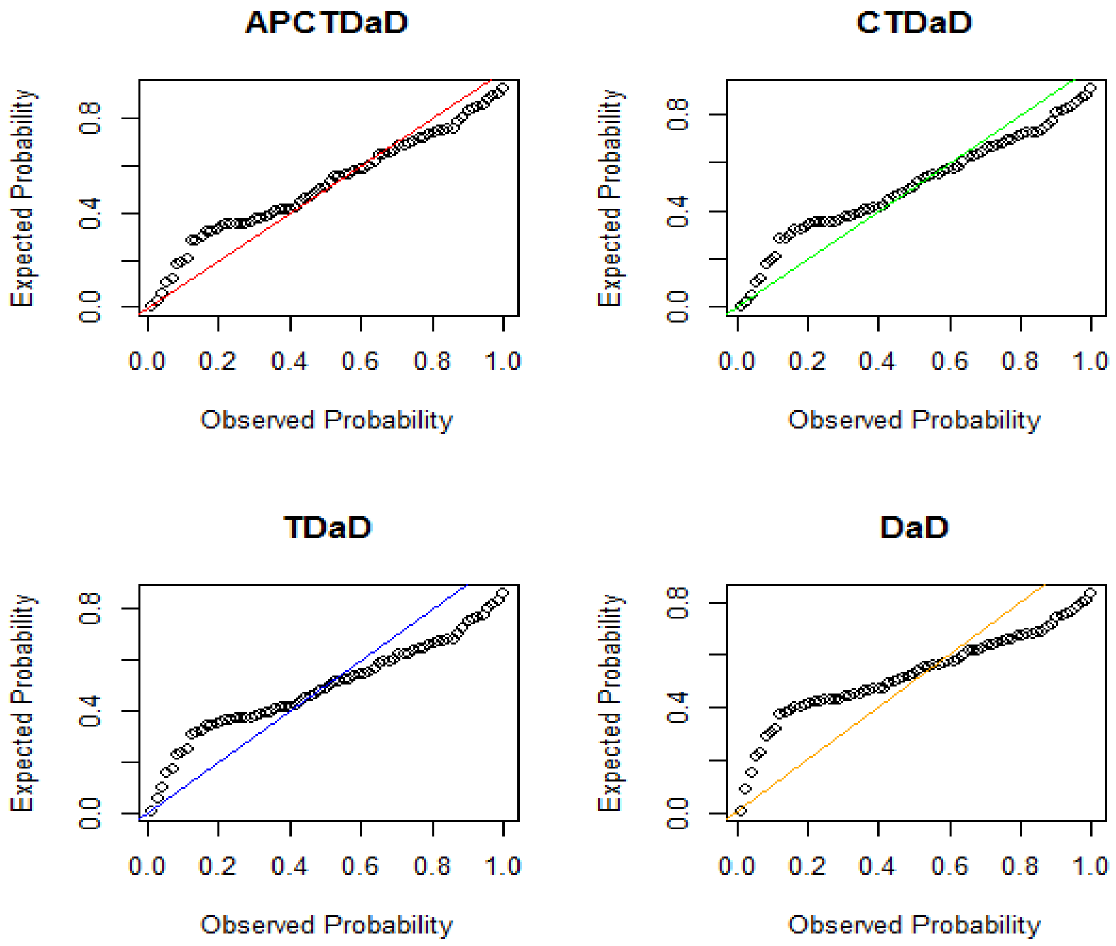


Figure 13: Probability plots for the fit of the APCTDaD, CTDaD, TDaD & DaD based on dataset III.

Tables 2, 6 and 10 list the values of the MLEs of the model parameters for all the three datasets, whereas the values of AIC, CAIC, BIC and HQIC are listed in Tables 3, 7 and 11 for datasets I, II and III respectively. Also, the values of A*, W* and K-S for datasets I, II and III are provided in Tables 4, 8 and 12 respectively.

The plots of the fitted APCTDaD density and cumulative distribution functions with those of competing distributions for datasets I, II and III are displayed in Figures 6, 9 and 12 respectively. The PP-plots of the fitted distributions are also given in Figures 7, 10 and 13 for datasets I, II and III respectively. From the results based on all the measures above, it is observed that the alpha power cubic transmuted Dagum distribution (APCTDaD) performs much better for all the real life datasets compared to the other three fitted distributions (cubic transmuted Dagum distribution (CTDaD), transmuted Dagum distribution (TDaD) and the conventional Dagum distribution (DaD)). These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in the figures 7, 10 and 13 above. Considering all these results, it is proven that the proposed distribution (alpha power cubic transmuted Dagum distribution (APCTDaD)) is a more flexible distribution than the other existing distributions fitted in this study.

6. Summary and Conclusion

This research considered the alpha power transformation approach to define and study a Dagum distribution producing a new distribution called “alpha power cubic transmuted Dagum distribution (APCTDaD)”. The research derived and studied the reliability functions of the proposed distribution which include survival and hazard functions with their graphical presentations and discussions on their usefulness and applications. It is concluded that the proposed distribution is better compared to other existing distributions based on our application of the model to a real life datasets.

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