Original Research Article

Gaussian Generalized Guglielmo Numbers

Abstract:

In this study, we define Gaussian generalized Guglielmo numbers in detail, and focus on four specific cases: Gaussian triangular numbers, Gaussian triangular-Lucas numbers, Gaussian oblong numbers, and Gaussian pentagonal numbers.

In addition, we present some identities and matrices related to these sequences, as well as recurrence relations, Binet's formulas, generating functions, Simpson's formulas, and summation formulas.

Keywords: Gaussian triangular numbers, Gaussian oblong numbers, Gaussian pentegonal numbers, triangular numbers, oblong numbers, pentegonal numbers.

1. Introduction

In this section, firstly, we give some prelimanary result on Guglielmo numbers.

The generalized Guglielmo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$. Hence, recurrence (1.1) is true for all integer n. Soykan has conducted a study on this particular sequence, for more details, see [25]

Third order reccurance relations has been studied by many authors, for more detail see [2,3,5,7, 8,13,18,20,21,23,24,27,28,29].

Next, we present Binet's formula of generalized Guglielmo numbers.

THEOREM 1.1. *[25 , Theorem 1] Binet formula of generalized Guglielmo numbers can be presented as follows:*

$$W_n = A_1 + A_2 n + A_3 n^2$$

where A_1, A_2 and A_3 are given as

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0), \end{aligned}$$

i.e.,

$$W_n = W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2. \tag{1.2}$$

Now we define four particular cases of the sequence $\{W_n\}$ as follows: the triangular sequence $\{T_n\}_{n\geq 0}$, the triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, the oblong sequence $\{O_n\}_{n\geq 0}$ and the pentagonal sequence $\{p_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations,

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 3, \tag{1.3}$$

$$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3, H_1 = 3, H_2 = 3, \tag{1.4}$$

$$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0, O_1 = 2, O_2 = 6, \tag{1.5}$$

$$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 5. \tag{1.6}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned} T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)}, \\ H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)}, \\ O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)}, \\ p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.3)-(1.6) hold for all integer n.

Note that, Gaussian numbers, generally known as Gaussian integers, are a subset of the complex numbers. A complex number is expressed in the form $a + bi$ where a and b are arbitrary real numbers, and i is the imaginary unit such that $i^2 = -1$. Gaussian integers are a specific type of complex number. In other word, z is a Gaussian integers such that $z = a + bi$ where a and b are arbitrary integers.

Next, we give some information about Gaussian sequences from literature.

First, we give some Gaussian numbers with second order reccurance relations.

- Horadam [12] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).

- Pethe and Horadam [19] introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1},$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [11] studied Gaussian Pell and Pell Lucas numbers by written , respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + ij_{n-1} \end{aligned}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [15] introduced and studied Gaussian Mersenne numbers defined by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [17] introduced and studied Gaussian balancing and Gaussian Lucas Balancing numbers given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [9] studied Gaussian Oresme numbers and defined them as

$$GS_n = S_n + iS_{n-1}$$

where oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some gaussian numbers with third order reccurance relations.

- Soykan, Taşdemir, Okumuş and Göcen [26] presented Gaussian generalized Tribonacci numbers given by

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

- Taşcı [16] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GR_n &= R_n + iR_{n-1} \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [4] defined Gaussian third-order Jacobsthal numbers as

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

2. Gaussian Generalized Guglielmo Numbers

In this section, we define Gaussian generalized Guglielmo numbers and present some proporties such as Binet's formula and generating function.

Gaussian generalized Guglielmo numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 3GW_{n-1} - 3GW_{n-2} + GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i(3W_0 - 3W_1 + W_2),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 3GW_{-(n-1)} - 3GW_{-(n-2)} + GW_{-(n-3)} \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) hold for all integer n. Note that for all integers n, we get

$$GW_n = W_n + iW_{n-1} \tag{2.3}$$

The first few generalized Gaussian Guglielmo numbers with positive subscript and negative subscript are presented in the following table.

Table 1. The first few generalized Gaussian Guglielmo numbers.

n	GW_n	GW_{-n}
0	$W_0 + i(3W_0 - 3W_1 + W_2)$	$W_0 + i(3W_0 - 3W_1 + W_2)$
1	$W_1 + iW_0$	$3W_0 - 3W_1 + W_2 + i(6W_0 - 8W_1 + 3W_2)$
2	$W_2 + iW_1$	$6W_0 - 8W_1 + 3W_2 + i(10W_0 - 15W_1 + 6W_2)$
3	$W_0 - 3W_1 + 3W_2 + iW_2$	$10W_0 - 15W_1 + 6W_2 + i(15W_0 - 24W_1 + 10W_2)$
4	$3W_0 - 8W_1 + 6W_2 + i(W_0 - 3W_1 + 3W_2)$	$15W_0 - 24W_1 + 10W_2 + i(21W_0 - 35W_1 + 15W_2)$
5	$6W_0 - 15W_1 + 10W_2 + i(3W_0 - 8W_1 + 6W_2)$	$21W_0 - 35W_1 + 15W_2 + i(28W_0 - 48W_1 + 21W_2)$
6	$10W_0 - 24W_1 + 15W_2 + i(6W_0 - 15W_1 + 10W_2)$	$28W_0 - 48W_1 + 21W_2 + i(36W_0 - 63W_1 + 28W_2)$
7	$15W_0 - 35W_1 + 21W_2 + i(10W_0 - 24W_1 + 15W_2)$	$36W_0 - 63W_1 + 28W_2 + i(45W_0 - 80W_1 + 36W_2)$

Gaussian triangular numbers, $GW_n : GW_n(0, 1, 3 + i) = GT_n$, are defined by

$$GT_n = 3GT_{n-1} - 3GT_{n-2} + GT_{n-3} \tag{2.4}$$

with the initial conditions

$$GT_0 = 0, GT_1 = 1, GT_2 = 3 + i.$$

Gaussian triangular-Lucas numbers, $GW_n(3 + 3i, 3 + 3i, 3 + 3i) = GH_n$, are defined by

$$GH_n = 3GH_{n-1} - 3GH_{n-2} + GH_{n-3} \tag{2.5}$$

with the initial conditions

$$GH_0 = 3 + 3i, GH_1 = 3 + 3i, GH_2 = 3 + 3i.$$

Gaussian oblong numbers , $GW_n(0, 2, 6 + 2i) = GO_n$, are defined by

$$GO_n = 3GO_{n-1} - 3GO_{n-2} + GO_{n-3} \tag{2.6}$$

with the initial conditions

$$GO_0 = 0, GO_1 = 2, GO_2 = 6 + 2i.$$

and Gaussian pentegonal numbers, $GW_n(2i, 1, 5 + i) = Gp_n$, are defined by

$$Gp_n = 3Gp_{n-1} - 3Gp_{n-2} + Gp_{n-3} \tag{2.7}$$

with the initial conditions

$$Gp_0 = 2i, Gp_1 = 1, Gp_2 = 5 + i.$$

Note that for all integers n, we have

$$\begin{aligned} GT_n &= T_n + iT_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \\ GO_n &= O_n + iO_{n-1}, \\ Gp_n &= p_n + ip_{n-1}. \end{aligned}$$

The first few values of Gaussian triangular numbers, Gaussian triangular-Lucas numbers, Gaussian oblong numbers and Gaussian pentegonal numbers with positive and negative subscript are given in the Table 2.

Table 2. Special cases of Gaussian generalized Guglielmo numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8
GT_n	0	1	$3+i$	$6+3i$	$10+6i$	$15+10i$	$21+15i$	$28+21i$	$36+28i$
GT_{-n}		i	$1+3i$	$3+6i$	$6+10i$	$10+15i$	$15+21i$	$21+28i$	$28+36i$
GH_n	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$
GH_{-n}		$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$
GO_n	0	2	$6+2i$	$12+6i$	$20+12i$	$30+20i$	$42+30i$	$56+42i$	$72+56i$
GO_{-n}		$2i$	$2+6i$	$6+12i$	$12+20i$	$20+30i$	$30+42i$	$42+56i$	$56+72i$
Gp_n	$2i$	1	$5+i$	$12+5i$	$22+12i$	$35+22i$	$51+35i$	$70+51i$	$92+70i$
Gp_{-n}		$2+7i$	$7+15i$	$15+26i$	$26+40i$	$40+57i$	$57+77i$	$77+100i$	$100+126i$

Next, we present The Binet's formula for the Gaussian generalized Guglielmo numbers

Theorem 2.1. *The Binet's formula for the Gaussian generalized Guglielmo numbers is*

$GW_n = (W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2) + i(W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)(n - 1) + \frac{1}{2}(W_2 - 2W_1 + W_0)(n-1)^2)$.

Proof. The proof follows from (1.2) and (2.3). □

The previous Theorem gives the following results, as special cases.

Corollary 2.2. *For all integers n, we have following identities,*

- **(a):** $GT_n = \frac{1}{2}n\,(n+1) + i(\frac{1}{2}n\,(n-1))$.
- **(b):** $GH_n = 3 + 3i$.
- **(c):** $GO_n = n\,(n+1) + in(n-1)$.
- **(d):** $Gp_n = \frac{1}{2}n\,(3n-1) + i(\frac{1}{2}\,(n-1)\,(3n-4))$.

The next Theorem presents the generating function of Gaussian generalized Guglielmo numbers.

Theorem 2.3. *Let $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$ denote the generating function of Gaussian generalized Guglielmo numbers. Then,*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + 3GW_0)x^2}{1 - 3x + 3x^2 - x^3}. \tag{2.8}$$

Proof. Using the definition of Gaussian Guglielmo numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain

$$\begin{aligned}
(1-3x+3x^2-x^3)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 3x\sum_{n=0}^{\infty} GW_n x^n + 3x^2\sum_{n=0}^{\infty} GW_n x^n - x^3\sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=0}^{\infty} GW_n x^{n+1} + 3\sum_{n=0}^{\infty} GW_n x^{n+2} - \sum_{n=0}^{\infty} GW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=1}^{\infty} GW_{n-1} x^{n} + 3\sum_{n=2}^{\infty} GW_{n-2} x^{n} - \sum_{n=3}^{\infty} GW_{n-3} x^{n}, \\
&= (GW_0 + GW_1x + GW_2x^2) - 3(GW_0x + GW_1x^2) + 3GW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(GW_n - 3GW_{n-1} + 3GW_{n-2} - GW_{n-3})x^n, \\
&= GW_0 + GW_1x + GW_2x^2 - 3GW_0x - 3GW_1x^2 + 3GW_0x^2, \\
&= GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + 3GW_0)x^2,
\end{aligned}$$

and rearranging above equation, we get (2.8). □

Theorem (2.3) gives following results as special cases,

$$f_{GT_n}(x) = \frac{x+ix^2}{1-3x+3x^2-x^3} \quad , \quad f_{GH_n}(x) = \frac{(3+3i)x^2-(6+6i)x+3+3i}{1-3x+3x^2-x^3},$$
$$f_{GO_n}(x) = \frac{2ix^2+2x}{1-3x+3x^2-x^3} \quad , \quad f_{Gp_n}(x) = \frac{(2+7i)x^2+(1-6i)x+2i}{1-3x+3x^2-x^3}.$$

3. Some Identities About Recurrence Relations of Gaussian Generalized Guglielmo Numbers

In this section, we present some identities on Gaussian triangular,Gaussian triangular-Lucas, Gaussian oblong, Gaussian pentegonal numbers.

Theorem 3.1. *The following equations hold for all integer n*

$$GT_n = \frac{1}{2}GO_{n+3} - \frac{3}{2}GO_{n+2} + \frac{3}{2}GO_{n+1}, \tag{3.1}$$

$$GO_n = 2GT_{n+3} - 6GT_{n+2} + 6GT_{n+1}, \tag{3.2}$$

$$GT_n = \frac{-2}{27}Gp_{n+2} + \frac{10}{27}Gp_{n+1} + \frac{1}{27}Gp_n, \tag{3.3}$$

$$Gp_n = 2GT_{n+2} - 6GT_{n+1} + 7GT_n, \tag{3.4}$$

$$GO_n = \frac{-4}{27}Gp_{n+2} + \frac{20}{27}Gp_{n+1} + \frac{2}{27}Gp_n, \tag{3.5}$$

$$Gp_n = GO_{n+2} - 3GO_{n+1} + \frac{7}{2}GO_n. \tag{3.6}$$

Proof. To proof identity (3.1), we can write $GT_n = aGO_{n+3} + bGO_{n+2} + cGO_{n+1}$ and solve the system of equations we get,

$$\begin{aligned} GT_0 &= aGO_3 + bGO_2 + cGO_1, \\ GT_1 &= aGO_4 + bGO_3 + cGO_2, \\ GT_2 &= aGO_5 + bGO_4 + cGO_3. \end{aligned}$$

Then, we obtain $a = \frac{1}{2}, b = -\frac{3}{2}, c = \frac{3}{2}$. The other identities can be found similarly. □

LEMMA 3.2. *([10])We assume that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n}x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1}x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The generating functions of the even and odd-indexed generalized Guglielmo sequences are provided by the following theorem.

THEOREM 3.3. *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are provided by*

$$f_{GW_{2n}}(x) = \frac{GW_0 + (GW_2 - 3GW_0)x + (6GW_0 - 8GW_1 + 3GW_2)x^2}{1 - 3x + 3x^2 - x^3}. \tag{3.7}$$

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (GW_0 - 6GW_1 + 3GW_2)x + (3GW_0 - 3GW_1 + GW_2)x^2}{1 - 3x + 3x^2 - x^3} \tag{3.8}$$

Proof. We only proof (3.7). From Theorem (2.3) we can obtain following identities:

$$\begin{aligned} f_{GW_n}(\sqrt{x}) &= \frac{GW_0 - \sqrt{x}\,(GW_1 - 3GW_0) + x\,(3GW_0 - 3GW_1 + GW_2)}{3x + 3\sqrt{x} + x^{\frac{3}{2}} + 1}. \\ f_{GW_n}(-\sqrt{x}) &= -\frac{GW_0 + \sqrt{x}\,(GW_1 - 3GW_0) + x\,(3GW_0 - 3GW_1 + GW_2)}{3\sqrt{x} - 3x + x^{\frac{3}{2}} - 1}. \end{aligned}$$

Thereby, using lemma (3.2) identity (3.7) can be proved . The other identity can be found similarly. □

From Theorem (3.3), we get the following corollary.

COROLLARY 3.4.

(a):

$$f_{GT_{2n}}(x) = \frac{(1+3i)\,x^2 + (3+i)\,x}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GT_{2n+1}}(x) = \frac{ix^2 + (3+3i)\,x + 1}{1 - 3x + 3x^2 - x^3}.$$

(b):

$$f_{GH_{2n}}(x) = \frac{(3+3i)\,x^2 - (6+6i)\,x + 3 + 3i}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GH_{2n+1}}(x) = \frac{(3+3i)\,x^2 - (6+6i)\,x + 3 + 3i}{1 - 3x + 3x^2 - x^3}.$$

(c):

$$f_{GO_{2n}}(x) = \frac{(2+6i)\,x^2 + (6+2i)\,x}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GO_{2n+1}}(x) = \frac{2ix^2 + (6+6i)\,x + 2}{1 - 3x + 3x^2 - x^3}.$$

(d):

$$f_{Gp_{2n}}(x)=\frac{(7+15i)\,x^{2}+(5-5i)\,x+2i}{1-3x+3x^{2}-x^{3}} \text{ and } f_{Gp_{2n+1}}(x)=\frac{(2+7i)\,x^{2}+(9+5i)\,x+1}{1-3x+3x^{2}-x^{3}}.$$

From Corollary (3.4) we can obtain the following corollary which presents the identities on Gaussian Guglielmo sequences

Corollary 3.5.

(a): $(3+i)GH_{2n-2}+(1+3i)GH_{2n-4}=(3+3i)GT_{2n}-(6+6i)GT_{2n-2}+(3+3i)GT_{2n-4}$.

(b): $2iGT_{2n-4}+(6+6i)GT_{2n-2}+2GT_{2n}=(3+i)GO_{2n-1}+(1+3i)GO_{2n-3}$.

(c): $(7+15i)GT_{2n-4}+(5-5i)GT_{2n-2}+2iGT_{2n}=(3+i)Gp_{2n-2}+(1+3i)Gp_{2n-4}$.

(d): $(3+3i)GO_{2n-4}-(6+6i)GO_{2n-2}+(3+3i)GO_{2n}=(2+6i)GH_{2n-4}+(6+2i)GH_{2n-2}$.

(e): $(7+15i)GH_{2n-4}+(5-5i)GH_{2n-2}+2iGH_{2n}=(3+3i)Gp_{2n-4}-(6+6i)Gp_{2n-2}+(3+3i)Gp_{2n}$.

(f): $(7+15i)GO_{2n-4}+(5-5i)GO_{2n-2}+2iGO_{2n}=(2+6i)Gp_{2n-4}+(6+2i)Gp_{2n-2}$.

(g): $iGH_{2n-3}+(3+3i)GH_{2n-1}+GH_{2n+1}=(3+3i)GT_{2n-3}-(6+6i)GT_{2n-1}+(3+3i)GT_{2n+1}$.

(h): $iGH_{2n-4}+(3+3i)GH_{2n-2}+GH_{2n}=(3+3i)GT_{2n-3}-(6+6i)GT_{2n-1}+(3+3i)GT_{2n+1}$.

(i): $iGO_{2n-4}+(3+3i)GO_{2n-2}+GO_{2n}=(2+6i)GT_{2n-3}+(6+2i)GT_{2n-1}$.

(j): $iGp_{2n-3}+(3+3i)Gp_{2n-1}+Gp_{2n+1}=(2+7i)GT_{2n-3}+(9+5i)GT_{2n-1}+GT_{2n+1}$.

(k): $(3+3i)GO_{2n-3}-(6+6i)GO_{2n-1}+(3+3i)GO_{2n+1}=2iGH_{2n-3}+(6+6i)GH_{2n-1}+2GH_{2n+1}$.

(l): $2iGp_{2n-3}+(6+6i)Gp_{2n-1}+2Gp_{2n+1}=(2+7i)GO_{2n-3}+(9+5i)GO_{2n-1}+GO_{2n+1}$.

Proof. From Corollary (3.4) we obtain

$$((3+i)x+(1+3i)x^{2})f_{GH_{2n}}=((3+3i)-(6+6i)x+(3+3i)x^{2})f_{GT_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((3+i)x+(1+3i)x^{2})\sum_{n=0}^{\infty}GH_{2n}x^{n}\\
&= (3+i)x\sum_{n=0}^{\infty}GH_{2n}x^{n}+(1+3i)x^{2}\sum_{n=0}^{\infty}GH_{2n}x^{n}\\
&= (3+i)\sum_{n=0}^{\infty}GH_{2n}x^{n+1}+(1+3i)\sum_{n=0}^{\infty}GH_{2n}x^{n+2}\\
&= (3+i)\sum_{n=1}^{\infty}GH_{2n-2}x^{n}+(1+3i)\sum_{n=2}^{\infty}GH_{2n-4}x^{n}\\
&= (3+i)(3+3i)x+\sum_{n=2}^{\infty}((3+i)GH_{2n-2}+(1+3i)GH_{2n-4})x^{n}
\end{aligned}$$

whereas the RHS (right hand side) is equal to

$$\begin{aligned}
RHS &= ((3+3i)-(6+6i)x+(3+3i)x^2)\sum_{n=0}^{\infty}GT_{2n}x^n \\
&= (3+3i)\sum_{n=0}^{\infty}GT_{2n}x^n-(6+6i)x\sum_{n=0}^{\infty}GT_{2n}x^n+(3+3i)x^2\sum_{n=0}^{\infty}GT_{2n}x^n \\
&= (3+3i)\sum_{n=0}^{\infty}GT_{2n}x^n-(6+6i)\sum_{n=0}^{\infty}GT_{2n}x^{n+1}+(3+3i)\sum_{n=0}^{\infty}GT_{2n}x^{n+2} \\
&= (3+3i)\sum_{n=0}^{\infty}GT_{2n}x^n-(6+6i)\sum_{n=1}^{\infty}GT_{2n-2}x^n+(3+3i)\sum_{n=2}^{\infty}GT_{2n-4}x^n \\
&= (3+i)(3+3i)x+\sum_{n=2}^{\infty}((3+3i)GT_{2n}-(6+6i)GT_{2n-2}+(3+3i)GT_{2n-4})x^n
\end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can present other identity similarly. □

We can get an identitiy related to Gaussian Guglielmo numbers and triangular numbers given below.

Theorem 3.6. *For all integers m,n the following identity holds:*

$$GW_{m+n}=T_{m-1}GW_{n+2}+(T_{m-3}-3T_{m-2})GW_{n+1}+T_{m-2}GW_n.$$

Proof. First, we assume that $m,n\geqslant 0$. the theorem (3.6) can be proved by mathematical induction on m. If $m=0$ we get

$$GW_n=T_{-1}GW_{n+2}+(T_{-3}-3T_{-2})GW_{n+1}+T_{-2}GW_n$$

which is true since $T_{-1}=0, T_{-2}=1, T_{-3}=3$. We assume that the identity given holds for $m\leq k$. For $m=k+1$, we get

$$\begin{aligned}
GW_{(k+1)+n} &= 3GW_{n+k}-3GW_{n+k-1}+GW_{n+k-2} \\
&= 3(T_{k-1}GW_{n+2}+(T_{k-3}-3T_{k-2})GW_{n+1}+T_{k-2}GW_n) \\
&\quad -3(T_{k-2}GW_{n+2}+(T_{k-4}-3T_{k-3})GW_{n+1}+T_{k-3}GW_n) \\
&\quad +(T_{k-3}GW_{n+2}+(T_{k-5}-3T_{k-4})GW_{n+1}+T_{k-4}GW_n) \\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})GW_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5}) \\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))GW_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})GW_n \\
&= T_kGW_{n+2}+(T_{k-2}-3T_{k-1})GW_{n+1}+T_{k-1}GW_n \\
&= T_{(k+1)-1}GW_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})GW_{n+1}+T_{(k+1)-2}GW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (3.6). The case $m,n<0$ can be proved similarly. □

For $n \geq 0$, $m \geq 0$ and taking $GW_n = GT_n$ or $GW_n = GH_n$ or $GO_n = GW_n$ or $GW_n = Gp_n$, respectively, we get,

$$\begin{aligned}
GT_{m+n} &= T_{m-1}GT_{n+2} + (T_{m-3} - 3T_{m-2})GT_{n+1} + T_{m-2}GT_n, \\
GH_{m+n} &= T_{m-1}GH_{n+2} + (T_{m-3} - 3T_{m-2})GH_{n+1} + T_{m-2}GH_n, \\
GO_{m+n} &= T_{m-1}GO_{n+2} + (T_{m-3} - 3T_{m-2})GO_{n+1} + T_{m-2}GO_n, \\
Gp_{m+n} &= T_{m-1}Gp_{n+2} + (T_{m-3} - 3T_{m-2})Gp_{n+1} + T_{m-2}Gp_n.
\end{aligned}$$

4. Simpson's Formula

In this section, we present Simpson's formula of generalized Gaussian Guglielmo numbers. This is a special cases of [22, Theorem 4.1]. We give the proof by calculating determinant and using Binet's formula of Gaussian generalized Guglielmo numbers.

THEOREM 4.1 (Simpson's formula of generalized Gaussian Guglielmo numbers). *For all integers n, we can write following equality*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = -(GW_0 - 2GW_1 + GW_2)^3.$$

Proof. Using Theorem (2.1) we can obtain

$$\begin{aligned}
\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} &= i\left((1-i)\,W_0 - (2-2i)\,W_1 + (1-i)\,W_2\right)^3 \\
&= -i^3(1-i)^3(W_0 - 2W_1 + W_2)^3 \\
&= (-i - i^4)^3(W_0 - 2W_1 + W_2)^3 \\
&= -(1+i)^3(W_0 - 2W_1 + W_2)^3 \\
&= -(W_0 - 2W_1 + W_2 + i(W_0 - 2W_1 + W_2))^3 \\
&= -(GW_0 - 2GW_1 + GW_2)^3. \;\square
\end{aligned}$$

From the Theorem (4.1) we get the following Corollary.

COROLLARY 4.2. *For all integers n, we get the following identities.*

(a): $$\begin{vmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{vmatrix} = 2(1-i).$$

(b): $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 0.$

(c): $\begin{vmatrix} GO_{n+2} & GO_{n+1} & GO_n \\ GO_{n+1} & GO_n & GO_{n-1} \\ GO_n & GO_{n-1} & GO_{n-2} \end{vmatrix} = 16(1-i).$

(d): $\begin{vmatrix} Gp_{n+2} & Gp_{n+1} & Gp_n \\ Gp_{n+1} & Gp_n & Gp_{n-1} \\ Gp_n & Gp_{n-1} & Gp_{n-2} \end{vmatrix} = 54(1-i).$

5. SUM FORMULAS

In this section, we identify some sum formulas of generalized Gaussian Guglielmo numbers.

THEOREM 5.1. *For all integers $n \geq 0$, we have sum formulas given below*

(a): $\sum\limits_{k=0}^{n} GW_k = \frac{1}{6}(n+1)(n(n-1)W_2 - n(2n-5)W_1 + (n^2-4n+6)W_0) + \frac{1}{6}(n+1)((n^2-4n+6)W_2 - (2n^2-11n+18)W_1 + (n^2-7n+18)W_0)i.$

(b): $\sum\limits_{k=0}^{n} GW_{2k} = \frac{1}{6}(n+1)(\left(4n^2-n\right)W_2 - 8\left(n^2-n\right)W_1 + \left(4n^2-7n+6\right)W_0) + \frac{1}{6}(n+1)((4n^2-7n+6)W_2 - 2(4n^2-10n+9)W_1 + (4n^2-13n+18)W_0)i.$

(c): $\sum\limits_{k=0}^{n} GW_{2k+1} = \frac{1}{6}(n+1)(\left(4n^2+5n\right)W_2 - 2\left(4n^2+2n-3\right)W_1 + \left(4n^2-n\right)W_0) + \frac{1}{6}(n+1)((4n^2-n)W_2 - 8(n^2-n)W_1 + (4n^2-7n+6)W_0)i.$

Proof. From (2.3) we can write the following sum formulas.

$$\begin{aligned} \sum_{k=0}^{n} GW_k &= \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1}, \\ \sum_{k=0}^{n} GW_{2k} &= \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1}, \\ \sum_{k=0}^{n} GW_{2k+1} &= \sum_{k=0}^{n} W_{2k+1} + i\sum_{k=0}^{n} W_{2k}. \end{aligned}$$

Using Soykan [25, Theorem 34] we can write

(a): $\sum\limits_{k=0}^{n} W_k = \frac{1}{6}(n+1)(n(n-1)W_2 - n(2n-5)W_1 + (n^2-4n+6)W_0).$

(b): $\sum\limits_{k=0}^{n} W_{k-1} = \frac{1}{6}(n+1)\left(\left(n^2-4n+6\right)W_2 - \left(2n^2-11n+18\right)W_1 + \left(n^2-7n+18\right)W_0\right).$

(c): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{6}(n+1)\left(\left(4n^2-n\right)W_2 - 8\left(n^2-n\right)W_1 + \left(4n^2-7n+6\right)W_0\right).$

(d): $\sum\limits_{k=0}^{n} W_{2k-1} = \frac{1}{6}(n+1)\left(\left(4n^2-7n+6\right)W_2 - 2\left(4n^2-10n+9\right)W_1 + \left(4n^2-13n+18\right)W_0\right).$

(e): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{6}(n+1)\left(\left(4n^2+5n\right)W_2 - 2\left(4n^2+2n-3\right)W_1 + \left(4n^2-n\right)W_0\right).$

So that, the proof is done easily.□

The previous theorem gives the following Corollary.

Corollary 5.2.

(a): $\sum_{k=0}^{n} GT_k = \frac{1}{6}n(n+2)(n+1) + \frac{1}{6}n(n-1)(n+1)i.$

(b): $\sum_{k=0}^{n} GH_k = 3(n+1) + 3(n+1)i.$

(c): $\sum_{k=0}^{n} GO_k = \frac{1}{3}n(n+2)(n+1) + \frac{1}{3}n(n-1)(n+1)i.$

(d): $\sum_{k=0}^{n} Gp_k = \frac{1}{2}n^2(n+1) + \frac{1}{2}(n+1)\left(-3n+n^2+4\right)i.$

Next, we give sum formulas which are given by even subscripts.

Corollary 5.3.

(a): $\sum_{k=0}^{n} GT_{2k} = \frac{1}{6}n(4n+5)(n+1) + \frac{1}{6}n(n+1)(4n-1)i.$

(b): $\sum_{k=0}^{n} GH_{2k} = 3(n+1) + 3(n+1)i.$

(c): $\sum_{k=0}^{n} GO_{2k} = \frac{1}{6}\left(8n^2+10n\right)(n+1) + \frac{1}{6}\left(8n^2-2n\right)(n+1)i.$

(d): $\sum_{k=0}^{n} Gp_{2k} = \frac{1}{2}n(4n+1)(n+1) + \frac{1}{2}(n+1)\left(-5n+4n^2+4\right)i.$

Next, we give sum formulas which are given by odd subscripts.

Corollary 5.4.

(a): $\sum_{k=0}^{n} GT_{2k+1} = \frac{1}{6}(n+1)\left(4n^2+11n+6\right) + \frac{1}{6}n(4n+5)(n+1)i.$

(b): $\sum_{k=0}^{n} GH_{2k+1} = 3(n+1) + 3(n+1)i.$

(c): $\sum_{k=0}^{n} GO_{2k+1} = \frac{1}{6}(n+1)\left(8n^2+22n+12\right) + \frac{1}{6}\left(8n^2+10n\right)(n+1)i.$

(d): $\sum_{k=0}^{n} Gp_{2k+1} = \frac{1}{6}(n+1)\left(12n^2+21n+6\right) + \frac{1}{6}\left(12n^2+3n\right)(n+1)i.$

5.1. Sums of Squares.

THEOREM 5.5. *For all integers m and j, W_0, W_1, W_2 are the initial values of (1.1), we have the following sum formulas for generalized Gaussian Guglielmo numbers*

$$\sum_{k=0}^{n} GW_k^2 = \frac{n+1}{120}\Psi$$

where Ψ, ζ, γ and ϑ are $\Psi = \zeta - \gamma + i\vartheta$,

$\zeta = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$,

$\gamma = W_2^2(6n^4 - 36n^3 + 86n^2 - 116n + 120) + 4W_1^2(6n^4 - 51n^3 + 161n^2 - 251n + 270) + W_0^2(6n^4 - 66n^3 + 296n^2 - 716n + 1080) + 2W_0W_2(6n^4 - 51n^3 + 171n^2 - 306n + 360) - 2W_1W_2(12n^4 - 87n^3 + 237n^2 - 342n + 360) - 2W_0W_1(12n^4 - 117n^3 + 447n^2 - 882n + 1080)$,

$\vartheta = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$.

$$\sum_{k=0}^{n} GW_kGW_{k-1} = \frac{1}{240}(n+1)\left((\lambda_1 - \lambda_2) + i(\Gamma_1 + 2\Gamma_2)\right)$$

where λ_1, λ_1, Γ_1 and Γ_2 are

$\lambda_1 = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$,

$\lambda_2 = 8W_1^2(6n^4 - 66n^3 + 276n^2 - 576n + 720) + W_2^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + W_0^2(12n^4 - 162n^3 + 882n^2 - 2532n + 4320) + 4W_0W_2(6n^4 - 66n^3 + 286n^2 - 646n + 900) - 2W_1W_2(24n^4 - 234n^3 + 874n^2 - 1684n + 2040) - 2W_0W_1(24n^4 - 294n^3 + 1414n^2 - 3484n + 5040)$,

$\Gamma_1 = W_2^2(12n^4 - 72n^3 + 132n^2 - 72n) + 8W_1^2(6n^4 - 51n^3 + 141n^2 - 141n) + W_0^2(12n^4 - 132n^3 + 552n^2 - 1152n + 1440) + 4W_0W_2(6n^4 - 51n^3 + 151n^2 - 196n + 180) - 2W_1W_2(24n^4 - 174n^3 + 394n^2 - 304n) - 2W_0W_1(24n^4 - 234n^3 + 814n^2 - 1264n + 960)$,

$\Gamma_2 = W_2^2(6n^4 - 36n^3 + 86n^2 - 116n + 120) + 4W_1^2(6n^4 - 51n^3 + 161n^2 - 251n + 270) + W_0^2(6n^4 - 66n^3 + 296n^2 - 716n + 1080) + 2W_0W_2(6n^4 - 51n^3 + 171n^2 - 306n + 360) - 2W_1W_2(12n^4 - 87n^3 + 237n^2 - 342n + 360) - 2W_0W_1(12n^4 - 117n^3 + 447n^2 - 882n + 1080)$.

Proof: From (2.3) we can write the following sum formulas.

$$\sum_{k=0}^{n} GW_k^2 = \sum_{k=0}^{n} W_k^2 - \sum_{k=0}^{n} W_{k-1}^2 + 2i\sum_{k=0}^{n} W_kW_{k-1}.$$

$$\sum_{k=0}^{n} GW_kGW_{k-1} = \sum_{k=0}^{n}(W_kW_{k-1} - W_{k-1}W_{k-2}) + i\sum_{k=0}^{n}(W_kW_{k-2} + W_{k-1}^2).$$

Using Soykan [25, Theorem 41], we write following equalities.

$$\sum_{k=0}^{n} W_k^2 = \frac{n+1}{120}\Delta_1.$$
$$\sum_{k=0}^{n} W_{k-1}^2 = \frac{n+1}{120}\Delta_2.$$
$$\sum_{k=0}^{n} W_k W_{k-1} = \frac{n+1}{240}\Omega_1.$$
$$\sum_{k=0}^{n} W_{k-1} W_{k-2} = \frac{n+1}{240}\Omega_2.$$
$$\sum_{k=0}^{n} W_k W_{k-2} = \frac{n+1}{240}\Omega_3.$$

where Δ_1, Δ_2, Ω_1, Ω_2 and Ω_3 are

$\Delta_1 = 4W_1^2(6n^4-21n^3+11n^2+19n)-W_2^2(-6n^4+6n^3+4n^2-4n)+W_0^2(6n^4-36n^3+86n^2-116n+120)+2W_0W_2(6n^4-21n^3+21n^2-6n)+2W_1W_2(-12n^4+27n^3+3n^2-18n)-2W_0W_1((12n^4-57n^3+87n^2-42n)$,

$\Delta_2 = W_2^2(6n^4-36n^3+86n^2-116n+120)+4W_1^2(6n^4-51n^3+161n^2-251n+270)+W_0^2(6n^4-66n^3+296n^2-716n+1080)+2W_0W_2(6n^4-51n^3+171n^2-306n+360)-2W_1W_2(12n^4-87n^3+237n^2-342n+360)-2W_0W_1(12n^4-117n^3+447n^2-882n+1080)$,

$\Omega_1 = W_2^2(12n^4-42n^3+42n^2-12n)+8W_1^2(6n^4-36n^3+66n^2-36n)+W_0^2(12n^4-102n^3+342n^2-612n+720)+4W_0W_2(6n^4-36n^3+76n^2-76n+60)-2W_1W_2(24n^4-114n^3+154n^2-64n)-2W_0W_1(24n^4-174n^3+454n^2-544n+360)$,

$\Omega_2 = 8W_1^2(6n^4-66n^3+276n^2-576n+720)+W_2^2(12n^4-102n^3+342n^2-612n+720)+W_0^2(12n^4-162n^3+882n^2-2532n+4320)+4W_0W_2(6n^4-66n^3+286n^2-646n+900)-2W_1W_2(24n^4-234n^3+874n^2-1684n+2040)-2W_0W_1(24n^4-294n^3+1414n^2-3484n+5040)$,

$\Omega_3 = W_2^2(12n^4-72n^3+132n^2-72n)+8W_1^2(6n^4-51n^3+141n^2-141n)+W_0^2(12n^4-132n^3+552n^2-1152n+1440)+4W_0W_2(6n^4-51n^3+151n^2-196n+180)-2W_1W_2(24n^4-174n^3+394n^2-304n)-2W_0W_1(24n^4-234n^3+814n^2-1264n+960)$. □

The previous theorem provides following corollary.

Corollary 5.6.

(a): $\sum\limits_{k=0}^{n} GT_k^2 = \frac{1}{4}n^2(n+1)^2 + i(\frac{1}{20}(n+1)n(n-1)(2n+1)(n+2))$.

(b): $\sum\limits_{k=0}^{n} GH_k^2 = 18i(n+1)$.

(c): $\sum\limits_{k=0}^{n} GO_k^2 = n^2(n+1)^2 + i(\frac{1}{5}(n+1)n(n-1)(2n+1)(n+2))$.

(d): $\sum\limits_{k=0}^{n} Gp_k^2 = \frac{1}{4}(3n-4)(n+1)\left(-n+3n^2+4\right) + i(\frac{1}{60}(n+1)n(n-1)\left(-45n+54n^2-26\right))$.

(e): $\sum\limits_{k=0}^{n} GT_k GT_{k-1} = \frac{1}{4}n^2(n+1)(n-1) + i(\frac{1}{30}n(n-1)(n+1)\left(3n^2-7\right))$.

(f): $\sum_{k=0}^{n} GH_k GH_{k-1} = 18i\,(n+1)\,.$
(g): $\sum_{k=0}^{n} GO_k GO_{k-1} = n^2\,(n-1)\,(n+1) + i(\frac{2}{15}n\,(n-1)\,(n+1)\,\left(3n^2-7\right)).$
(h): $\sum_{k=0}^{n} Gp_k Gp_{k-1} = \frac{1}{4}\,(3n-7)\,(n+1)\,\left(-4n+3n^2+8\right) + i(\frac{1}{30}\,(n+1)\,(27n^4-117n^3+167n^2-107n+120).$

6. Matrix Formulation of GW_n

Consider the triangular sequence $\{T_n\}$ defined by the third-order recurrence relation following

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$$

with the initial conditions

$$T_0 = 0,\ T_1 = 1,\ T_2 = 3.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then we give the following Lemma.

Lemma 6.1. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \tag{6.1}$$

Proof. The identity (6.1) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_2 \\ GW_1 \\ GW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus, the following identity is true.:

$$\begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}$$
$$= \begin{pmatrix} 3GW_{k+2} - 3GW_{k+1} + GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

For the proof see [24].

We define

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \tag{6.2}$$

$$E_{GW} = \begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}. \tag{6.3}$$

Now, we have the following theorem with N_{GW} and E_{GW}

THEOREM 6.2. *Using N_{GW} and E_{GW}, we get*

$$A^n N_{GW} = E_{GW}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{GW} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= GW_2T_{n+1} + GW_1(T_{n-1} - 3T_n) + GW_0T_n, \\ a_{12} &= GW_1T_{n+1} + GW_0(T_{n-1} - 3T_n) + GW_{-1}T_n, \\ a_{13} &= GW_0T_{n+1} + GW_{-1}(T_{n-1} - 3T_n) + GW_{-2}T_n, \\ a_{21} &= GW_2T_n + GW_1(T_{n-2} - 3T_{n-1}) + GW_0T_{n-1}, \\ a_{22} &= GW_1T_n + GW_0(T_{n-2} - 3T_{n-1}) + GW_{-1}T_{n-1}, \\ a_{23} &= GW_0T_n + GW_{-1}(T_{n-2} - 3T_{n-1}) + GW_{-2}T_{n-1}, \\ a_{31} &= GW_2T_{n-1} + GW_1(T_{n-3} - 3T_{n-2}) + GW_0T_{n-2}, \\ a_{32} &= GW_1T_{n-1} + GW_0(T_{n-3} - 3T_{n-2}) + GW_{-1}T_{n-2}, \\ a_{33} &= GW_0T_{n-1} + GW_{-1}(T_{n-3} - 3T_{n-2}) + GW_{-2}T_{n-2}, \end{aligned}$$

Using the Theorem (3.6) the proof is done. □

By taking, $GW_n = GT_n$ with GT_0, GT_1, GT_2 in (6.2) and (6.3), $GW_n = GH_n$ with GH_0, GH_1, GH_2 in (6.2) and (6.3), $GW_n = GO_n$ with GO_0, GO_1, GO_2 in (6.2) and (6.3), $GW_n = Gp_n$ with Gp_0, Gp_1, Gp_2 in (6.2) and (6.3) respectively, we get:

$$N_{GT} = \begin{pmatrix} 3+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1+3i \end{pmatrix}, \quad E_{GT} = \begin{pmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{pmatrix}$$

$$N_{GH} = \begin{pmatrix} 3+3i & 3+3i & 3+3i \\ 3+3i & 3+3i & 3+3i \\ 3+3i & 3+3i & 3+3i \end{pmatrix}, \quad E_{GH} = \begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix}$$

$$N_{GO}=\begin{pmatrix}6+2i & 2 & 0\\ 2 & 0 & 2i\\ 0 & 2i & 2+6i\end{pmatrix}\quad , \quad E_{GO}=\begin{pmatrix}GO_{n+2} & GO_{n+1} & GO_{n}\\ GO_{n+1} & GO_{n} & GO_{n-1}\\ GO_{n} & GO_{n-1} & GO_{n-2}\end{pmatrix}$$

$$N_{Gp}=\begin{pmatrix}5+i & 1 & 2i\\ 1 & 2i & 2+7i\\ 2i & 2+7i & 7+15i\end{pmatrix}\quad , \quad E_{Gp}=\begin{pmatrix}Gp_{n+2} & Gp_{n+1} & Gp_{n}\\ Gp_{n+1} & Gp_{n} & Gp_{n-1}\\ Gp_{n} & Gp_{n-1} & Gp_{n-2}\end{pmatrix}.$$

From Theorem (6.2), we can write the following corollary.

Corollary 6.3. *The following identities are holds*

(a): $A^n N_{GT} = E_{GT}$.

(b): $A^n N_{GH} = E_{GH}$.

(b): $A^n N_{GO} = E_{GO}$.

(c): $A^n N_{Gp} = E_{Gp}$.

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