Original Research Article

ESTIMATION OF A SHAPE PARAMETER OF A GOMPERTZ-LINDLEY DISTRIBUTION USING BAYESIAN AND MAXIMUM LIKELIHOOD METHODS

Abstract

The Gompertz-Lindley distribution is an extension of the Lindley distribution with three parameters. It was found to be more flexible for modeling real life events. The distribution contains two shape parameters and a scale parameter. Despite the necessity of parameter estimation theory in modeling, it has not been shown that a method of estimation method is better for any of these three parameters of the Gompertz-Lindley distribution. This paper identifies the best estimation method for the shape parameter of the Gompertz-Lindley distribution by deriving Bayesian estimators for the shape parameter of the distribution using two non-informative prior distributions (Uniform and Jeffery) and an informative prior (gamma) under squared error loss function (SELF), quadratic loss function (QLF) and precautionary loss function (PLF). These estimators were evaluated and the results compared with the maximum likelihood estimation method using Monte Carlo simulations with the mean square error (MSE) as a criterion for choosing the best estimator.

Keywords*: Gomperz-Lindley distribution; Bayesian analysis; Prior distributions; Loss functions; Maximum Likelihood Estimation and Mean square error.*

1. Introduction

A number of classical probability distributions have been used over the years for modeling real life datasets and one of such distributions is the Lindley distribution. The Lindley distribution is a probability distribution that was investigated in context of fiducial statistics as a counter example of Bayesian theory [1]. Its fundamental properties, estimation and applications have been discussed using different data sets [2], [3], [4], [5] and [6].

Despite the useful properties and various applications of the Lindley distribution, its applicability may be restricted to non-monotone hazard rate data according to [7]. This has lead to the introduction of other extensions of the Lindley distribution such as the transmuted Lindley distribution by [8], the exponentiated Power Lindley distribution by [9], Generalized Lindley distribution by [10], Transmuted Generalized Lindley distribution by [11], Extended Power Lindley distribution by [12], Transmuted Two-Parameter Lindley distribution by [13] and a three-parameter Lindley distribution by [14], power Lindley distribution by [15], the transmuted Lindley-geometric distribution by [16], the beta-Lindley distribution by [17], Kumaraswamy-Lindley distribution by [18] and Gompertz-Lindley distribution by [19].

Besides extended Lindley distributions, numerous compound probability distributions have been proposed for modeling real life situations and these compound distributions are found to be skewed, flexible and more better in statistical modeling compared to the classical distributions ([20]- [30]).

According to [19], the probability density function (pdf), the cumulative distribution function (cdf), survival function, hazard function and quantile function (qf) of the Gompertz-Lindley distribution (GomLinD) are respectively defined as:

$$f(x)=\frac{\alpha\theta^{2}(1+x)}{\theta+1}e^{-\theta x}\left[\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]^{-\beta-1} e^{\frac{\alpha}{\beta}\left\{1-\left[\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]^{-\beta}\right\}} \tag{1.1}$$

$$F(x)=1-\exp\left\{\frac{\alpha}{\beta}\left\{1-\left[\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]^{-\beta}\right\}\right\} \tag{1.2}$$

$$S(x)=\exp\left\{\frac{\alpha}{\beta}\left\{1-\left[\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]^{-\beta}\right\}\right\} \tag{1.3}$$

$$h(x)=\frac{\alpha\theta^{2}(1+x)e^{\beta\theta x}}{(\theta x+\theta+1)} \tag{1.4}$$

and

$$Q(u)=X_{q}=-1-\frac{1}{\theta}-\frac{1}{\theta}W\left(-\frac{\left(1-\frac{\beta}{\alpha}\ln(1-u)\right)^{-\frac{1}{\beta}}}{e^{\theta+1}}\right) \tag{1.5}$$

where $\theta>0$ is a scale parameter while $\alpha>0$ and $\beta>0$ are the extra shape parameters of the Gompertz-Lindley distribution (GomLinD).

These functions are represented graphically using some arbitrary parameter values in the figure below:

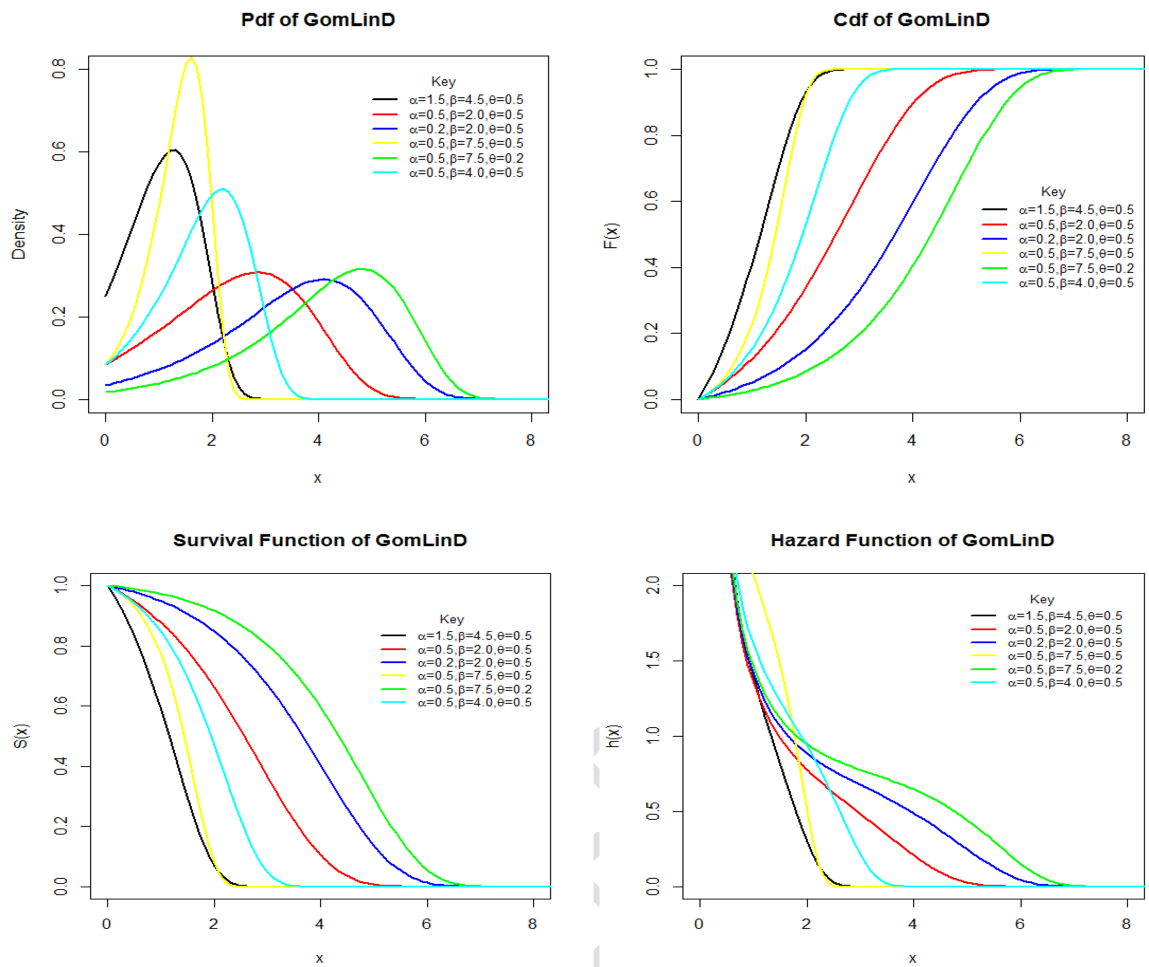

Fig. **1**: Plots of the PDF, CDF, Survival and Hazard Function of the GomLinD for Selected Parameter Values.

The Gompertz-Lindley distribution (GomLinD) has three parameters (two shapes parameters and a scale parameter). It was found to be skewed and flexible with a decreasing hazard rate and different shapes and also performed better than the Generalized Lindley distribution (GenLinD), a three-parameter Lindley distribution (ATPLinD), Transmuted two-parameter Lindley distribution (TTPLinD), Transmuted Lindley distribution (TLinD) and the conventional Lindley distribution (LinD) based on an application of the models to a lifetime dataset [19].

Estimation of parameters of a distribution differs by method from one parameter of the distribution to another and therefore this study aims at estimating the shape parameter of the GomLinD using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation.

We have two basic methods of parameter estimation and these are the classical and the non classical methods. The classical method of estimation involves a situation where the parameters are considered to be constant but unknown whereas the parameters are considered to be unknown and random just like variables under non classical approach. The most widely used method in classical theory is the method of maximum likelihood estimation while the Bayesian estimation method is used in the non classical theory. However, in most real life problems described by life time distributions, the parameters cannot be considered as constants in all the life testing period ([31]-[33]). Based on the reason above, it is true that the classical approach can no longer handle problems of parameter estimation in life time models and hence there is need for Bayesian estimation in life time models.

The aim of this article is to estimate the shape parameter of the GomLinD using Bayesian approach assuming three prior distributions and three loss functions. The remaining parts of this paper are presented as follows: in Section 2, maximum likelihood estimator (MLE) for the shape parameter is obtained. In Section 3, Bayesian estimators based on the prior beliefs (distributions) and loss functions are derived. The proposed estimators are evaluated using their mean squared errors (MSEs) in Section 4 and the summary and conclusion is presented in Section 5.

2. Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from a population X of size 'n' independently and identically distributed random variables with probability density function $f(x)$, . The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Given that the values, $\underline{x} = \left(x_1, x_2, ..., x_n\right)$ are obtained independently from the GomLinD with unknown parameters, α , β and θ .

The likelihood function is given by:

$$L\left(\underline{x} \mid \alpha, \beta, \theta\right) = P\left(x_1, x_2, ..., x_n \mid \alpha, \beta, \theta\right) = \prod_{i=1}^{n} P\left(\underline{x} \mid \alpha, \beta, \theta\right) \tag{2.1}$$

The likelihood function, $L\left(\underline{x} \mid \alpha, \beta, \theta\right)$ based on the pdf of GomLinD is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L\left(\underline{x} \mid \alpha, \beta, \theta\right) = \frac{\left(\alpha\theta^2\right)^n \prod_{i=1}^{n}\left\{(1+x_i)e^{-\theta x_i}\right\} \prod_{i=1}^{n}\left\{\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right\}^{-\beta-1} e^{\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}}}{\left(\theta+1\right)^n} \tag{2.2}$$

For the shape parameter of the GomLinD, α , the likelihood function is given by;

$$L\left(\underline{x} \mid \alpha\right) \propto \alpha^n e^{\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}}$$

$$L\left(\underline{x} \mid \alpha\right) = \mathrm{K}\alpha^n e^{\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \tag{2.3}$$

Where $K = \dfrac{\left(\theta^{2}\right)^{n}\prod_{i=1}^{n}\left\{\left(1+x_i\right)e^{-\theta x_i}\right\}\prod_{i=1}^{n}\left\{\left[1+\dfrac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right\}^{-\beta-1}}{\left(\theta+1\right)^{n}}$ is a constant which is independent of the shape parameter, α .

Let the log-likelihood function, $l=\log L\left(\underline{x}\mid\alpha\right)$, therefore

$$\log L\left(\underline{x}\mid\alpha\right)=n\log\alpha+\tfrac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\} \tag{2.4}$$

Differentiating ***l*** partially with respect to α gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}=0 \tag{2.5}$$

And solving for $\hat{\alpha}$ gives;

$$\hat{\alpha}=n\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{-1} \tag{2.6}$$

where $\hat{\alpha}$ is the maximum likelihood estimator of the shape parameter, α. More information about the maximum likelihood estimation of the shape parameter of the GomLinD can be obtained from [19].

3. Bayesian Estimation

This paper has made use of two non-informative priors (uniform and Jeffrey) and an informative prior (gamma) to estimate the shape parameter of a GomLinD. These assumed priors distributions or beliefs have been used over the years by several authors including [34]-[42]. Our article also considered three loss functions which are squared error, quadratic and precautionary loss functions and these loss functions have been studied by other authors [43]-[51] etc. The stated prior distributions and loss functions are defined as follows:

a. The uniform prior is defined as:

$$p\left(\alpha\right)\propto 1;0<\alpha<\infty \tag{3.1}$$

b. Also, the Jeffrey's prior is defined as:

$$p\left(\alpha\right)\propto\frac{1}{\alpha};0<\alpha<\infty \tag{3.2}$$

c. Also, the gamma prior is defined as:

$$P(\alpha)=\frac{a^b}{\Gamma(b)}\alpha^{b-1}e^{-a\alpha} \tag{3.3}$$

i. Squared Error Loss Function (*SELF*)

The squared error loss function relating to the shape parameter α is defined as:

$$L(\alpha,\alpha_{SELF})=(\alpha-\alpha_{SELF})^2 \tag{3.4}$$

where α_{SELF} is the estimator of the parameter α under *SELF*.

ii. Quadratic Loss Function (QLF)

The quadratic loss function is defined from [52] as

$$L(\alpha,\alpha_{QLF})=\left(\frac{\alpha-\alpha_{QLF}}{\alpha}\right)^2 \tag{3.5}$$

where α_{QLF} is the estimator of the parameter α under *QLF*.

iii. Precautionary Loss Function (*PLF*)

The precautionary loss function (*PLF*) introduced by [53] is an asymmetric loss function and is defined as

$$L(\alpha_{PLF},\alpha)=\frac{(\alpha_{PLF}-\alpha)^2}{\alpha_{PLF}} \tag{3.6}$$

where α_{PLF} is the estimator of the shape parameter α under *PLF*.

The posterior distribution of a parameter is the distribution of the parameter after observing the available data and it is obtained by using Bayes' theorem in relation to the shape parameter, α, likelihood function and prior distribution as follows:

$$P(\alpha\,|\,\underline{x})=\frac{P(\alpha,\underline{x})}{P(\underline{x})}=\frac{P(\underline{x}\,|\,\alpha)P(\alpha)}{P(\underline{x})}=\frac{P(\underline{x}\,|\,\alpha)P(\alpha)}{\int P(\underline{x}\,|\,\alpha)P(\alpha)d\alpha}=\frac{L(\underline{x}\,|\,\alpha)P(\alpha)}{\int L(\underline{x}\,|\,\alpha)P(\alpha)d\alpha} \tag{3.7}$$

where $P(\underline{x})$ is the marginal distribution of X and $P(\underline{x})=\sum_{x}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)$ when the prior distribution of α is discrete and $P(\underline{x})=\int_{-\infty}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)d\alpha$ when the prior distribution of α is continuous. Also note that $p(\alpha)$ and $L(\underline{x}\,|\,\alpha)$ are the prior distribution and the Likelihood function respectively.

3.1 Bayesian Analysis under Uniform Prior with Three Loss Functions

The posterior distribution of the shape parameter α assuming a uniform prior distribution is obtained from (3.7) using integration by substitution method as:

$$P\left(\alpha \mid \underline{x}\right)=\frac{\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{n+1}}{\Gamma\left(n+1\right)\alpha^{-n}\exp\left\{-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right\}} \tag{3.8}$$

Bayes estimators under uniform prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF}=E\left(\alpha\right)=E\left(\alpha \mid \underline{x}\right)=\int_{0}^{\infty}\alpha P\left(\alpha \mid \underline{x}\right)d\alpha=\left(n+1\right)\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{-1} \tag{3.9}$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1} \mid \underline{x}\right)}{E\left(\alpha^{-2} \mid \underline{x}\right)}=\frac{\int_{0}^{\infty}\alpha^{-1}P\left(\alpha \mid \underline{x}\right)d\alpha}{\int_{0}^{\infty}\alpha^{-2}P\left(\alpha \mid \underline{x}\right)d\alpha}=\left(n-1\right)\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{-1} \tag{3.10}$$

and

$$\alpha_{PLF}=\left\{E\left(\alpha^{2} \mid \underline{x}\right)\right\}^{\frac{1}{2}}=\left\{\int_{0}^{\infty}\alpha^{2}P\left(\alpha \mid \underline{x}\right)d\alpha\right\}^{\frac{1}{2}}=\frac{-\beta\left[\left(n+1\right)\left(n+2\right)\right]^{0.5}}{\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \tag{3.11}$$

3.2 Bayesian Analysis under Jeffrey's Prior with Three Loss Functions

The posterior distribution of the shape parameter α for a given data assuming a Jeffrey's prior distribution is obtained from (3.7) using integration by substitution method as:

$$P\left(\alpha \mid \underline{x}\right)=\frac{\alpha^{n-1}\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{n}}{\Gamma\left(n\right)\exp\left\{-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right\}} \tag{3.12}$$

Bayes estimators under Jeffrey's prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF}=E\left(\alpha\right)=E\left(\alpha \mid \underline{x}\right)=\int_{0}^{\infty}\alpha P\left(\alpha \mid \underline{x}\right)d\alpha=n\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{-1} \tag{3.13}$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1} \mid \underline{x}\right)}{E\left(\alpha^{-2} \mid \underline{x}\right)}=\frac{\int_{0}^{\infty}\alpha^{-1}P\left(\alpha \mid \underline{x}\right)d\alpha}{\int_{0}^{\infty}\alpha^{-2}P\left(\alpha \mid \underline{x}\right)d\alpha}=\left(n-2\right)\left(-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{-1} \tag{3.14}$$

and

$$\alpha_{PLF} = \left\{E\left(\alpha^2 \mid \underline{x}\right)\right\}^{\frac{1}{2}} = \left\{\int_0^\infty \alpha^2 P\left(\alpha \mid \underline{x}\right) d\alpha\right\}^{\frac{1}{2}} = \frac{-\beta\left[n\left(n+1\right)\right]^{0.5}}{\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \quad (3.15)$$

3.3 Bayesian Analysis under Gamma Prior with Three Loss Functions

The posterior distribution of the shape parameter α for a given data assuming a gamma prior distribution is obtained from (3.7) using integration by substitution method as

$$P\left(\alpha \mid \underline{x}\right) = \frac{\alpha^{(n+b-1)}\left(a-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)^{n+b}}{\Gamma\left(n+b\right)\exp\left\{-\alpha\left(a-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}\right)\right\}} \quad (3.16)$$

Bayes estimators under gamma prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF} = \frac{n+b}{a-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \quad (3.17)$$

$$\alpha_{QLF} = \frac{n+b-2}{a-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \quad (3.18)$$

and

$$\alpha_{PLF} = \frac{\left[\left(n+b+1\right)\left(n+b\right)\right]^{0.5}}{a-\beta^{-1}\sum_{i=1}^{n}\left\{1-\left[\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]^{-\beta}\right\}} \quad (3.19)$$

4. Results and Discussions

Here, we conducted a Monte Carlo simulation with R software under 10,000 replications using inverse transformation method of simulation to generate random samples of sizes n = (25, 45, 85, 125, 175, 225) from the GomLinD under varying parameter values. The results of this simulation study was presented in the following tables by listing the true parameter values and the average estimates of the shape parameter with their respective Mean Square Errors (MSEs) under the appropriate estimation methods which include the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under Uniform Jeffrey and gamma priors respectively. The measure used for checking the efficiency of the estimators is the Mean Square Error (MSE): $MSE = \frac{1}{n}E\left(\hat{\alpha}-\alpha\right)^2$.

Table 1: Estimates and Mean Squared Errors (MSEs) for $\alpha = 1.8, \theta = 1.5, \beta = 1.2, a = 0.5$ and $b = 2.0$ under different priors, loss functions and sample sizes

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.8735	1.9485	1.7986	1.9856	1.8735	1.7237	1.9106	1.9474	1.8031	1.9831
	MSE	0.1593	0.1885	0.1418	0.2073	0.1593	0.1361	0.1723	0.1747	0.1312	0.1922
45	Estimate	1.8375	1.8784	1.7967	1.8987	1.8375	1.7559	1.8578	1.8799	1.7999	1.8998
	MSE	0.0787	0.0869	0.0739	0.0923	0.0787	0.0725	0.0824	0.0839	0.0710	0.0891
85	Estimate	1.8226	1.8440	1.8011	1.8547	1.8226	1.7797	1.8333	1.8455	1.8030	1.856
	MSE	0.0408	0.0432	0.0394	0.0448	0.0408	0.0389	0.0419	0.0425	0.0386	0.044
125	Estimate	1.8159	1.8304	1.8013	1.8376	1.8159	1.7868	1.8231	1.8315	1.8027	1.8387
	MSE	0.0273	0.0284	0.0266	0.0291	0.0273	0.0263	0.0278	0.0281	0.0262	0.0288
175	Estimate	1.8120	1.8223	1.8016	1.8275	1.8120	1.7913	1.8171	1.8232	1.8026	1.8283
	MSE	0.0189	0.0195	0.0186	0.0199	0.0189	0.0185	0.0192	0.0194	0.0184	0.0197
225	Estimate	1.8086	1.8167	1.8006	1.8207	1.8086	1.7925	1.8126	1.8174	1.8013	1.8214
	MSE	0.0150	0.0153	0.0148	0.0156	0.0150	0.0147	0.0152	0.0153	0.0147	0.0155

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 2: Estimates and Mean Squared Errors (MSEs) for $\alpha = 0.5, \theta = 1.5, \beta = 1.2, a = 0.5$ and $b = 2.0$ under different priors, loss functions and sample sizes

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	0.5204	0.5412	0.4996	0.5516	0.5204	0.4788	0.5307	0.5560	0.5148	0.5662
	MSE	0.0123	0.0145	0.0109	0.0160	0.0123	0.0105	0.0133	0.0164	0.0116	0.0181
45	Estimate	0.5104	0.5218	0.4991	0.5274	0.5104	0.4877	0.5161	0.5300	0.5075	0.5356
	MSE	0.0061	0.0067	0.0057	0.0071	0.0061	0.0056	0.0064	0.0073	0.0059	0.0078

85	Estimate	0.5063	0.5122	0.5003	0.5152	0.5063	0.4944	0.5092	0.5166	0.5048	0.5196
	MSE	0.0032	0.0033	0.0030	0.0035	0.0032	0.0030	0.0032	0.0035	0.0031	0.0036
125	Estimate	0.5044	0.5084	0.5004	0.5105	0.5044	0.4963	0.5064	0.5114	0.5034	0.5134
	MSE	0.0021	0.0022	0.0021	0.0022	0.0021	0.0020	0.0021	0.0023	0.0021	0.0023
175	Estimate	0.5033	0.5062	0.5005	0.5076	0.5033	0.4976	0.5048	0.5083	0.5026	0.5098
	MSE	0.0015	0.0015	0.0014	0.0015	0.0015	0.0014	0.0015	0.0015	0.0014	0.0016
225	Estimate	0.5024	0.5046	0.5002	0.5057	0.5024	0.4979	0.5035	0.5063	0.5018	0.5074
	MSE	0.0012	0.0012	0.0011	0.0012	0.0012	0.0011	0.0012	0.0012	0.0012	0.0012

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 3: Estimates and Mean Squared Errors (MSEs) for $\alpha = 1.8, \theta = 0.2, \beta = 1.2, a = 0.5$ and $b = 2.0$ under different priors, loss functions and sample sizes

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.8735	1.9485	1.7986	1.9856	1.8735	1.7237	1.9106	1.9474	1.8031	1.9831
	MSE	0.1593	0.1885	0.1418	0.2073	0.1593	0.1361	0.1723	0.1747	0.1312	0.1922
45	Estimate	1.8375	1.8784	1.7967	1.8987	1.8375	1.7559	1.8578	1.8799	1.7999	1.8998
	MSE	0.0787	0.0869	0.0739	0.0923	0.0787	0.0725	0.0824	0.0839	0.0710	0.0891
85	Estimate	1.8226	1.8440	1.8011	1.8547	1.8226	1.7797	1.8333	1.8455	1.8030	1.856
	MSE	0.0408	0.0432	0.0394	0.0448	0.0408	0.0389	0.0419	0.0425	0.0386	0.044
125	Estimate	1.8159	1.8304	1.8013	1.8376	1.8159	1.7868	1.8231	1.8315	1.8027	1.8387
	MSE	0.0273	0.0284	0.0266	0.0291	0.0273	0.0263	0.0278	0.0281	0.0262	0.0288
175	Estimate	1.8120	1.8223	1.8016	1.8275	1.8120	1.7913	1.8171	1.8232	1.8026	1.8283
	MSE	0.0189	0.0195	0.0186	0.0199	0.0189	0.0185	0.0192	0.0194	0.0184	0.0197
225	Estimate	1.8086	1.8167	1.8006	1.8207	1.8086	1.7925	1.8126	1.8174	1.8013	1.8214
	MSE	0.0150	0.0153	0.0148	0.0156	0.0150	0.0147	0.0152	0.0153	0.0147	0.0155

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 4: Estimates and Mean Squared Errors (MSEs) for $\alpha = 1.8, \theta = 1.5, \beta = 0.3, a = 0.5$ and $b = 2.0$ under different priors, loss functions and sample sizes

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.8735	1.9485	1.7986	1.9856	1.8735	1.7237	1.9106	1.9474	1.8031	1.9831
	MSE	0.1593	0.1885	0.1418	0.2073	0.1593	0.1361	0.1723	0.1747	0.1312	0.1922
45	Estimate	1.8375	1.8784	1.7967	1.8987	1.8375	1.7559	1.8578	1.8799	1.7999	1.8998
	MSE	0.0787	0.0869	0.0739	0.0923	0.0787	0.0725	0.0824	0.0839	0.0710	0.0891
85	Estimate	1.8226	1.8440	1.8011	1.8547	1.8226	1.7797	1.8333	1.8455	1.8030	1.856
	MSE	0.0408	0.0432	0.0394	0.0448	0.0408	0.0389	0.0419	0.0425	0.0386	0.044
125	Estimate	1.8159	1.8304	1.8013	1.8376	1.8159	1.7868	1.8231	1.8315	1.8027	1.8387
	MSE	0.0273	0.0284	0.0266	0.0291	0.0273	0.0263	0.0278	0.0281	0.0262	0.0288
175	Estimate	1.8120	1.8223	1.8016	1.8275	1.8120	1.7913	1.8171	1.8232	1.8026	1.8283
	MSE	0.0189	0.0195	0.0186	0.0199	0.0189	0.0185	0.0192	0.0194	0.0184	0.0197
225	Estimate	1.8086	1.8167	1.8006	1.8207	1.8086	1.7925	1.8126	1.8174	1.8013	1.8214
	MSE	0.0150	0.0153	0.0148	0.0156	0.0150	0.0147	0.0152	0.0153	0.0147	0.0155

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 5: Estimates and Mean Squared Errors (MSEs) for $\alpha = 1.8, \theta = 1.5, \beta = 1.2, a = 2.5$ and $b = 2.0$ under different priors, loss functions and sample sizes

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.8735	1.9485	1.7986	1.9856	1.8735	1.7237	1.9106	1.6945	1.5689	1.7256
	MSE	0.1593	0.1885	0.1418	0.2073	0.1593	0.1361	0.1723	0.0972	0.1272	0.0948

45	Estimate	1.8375	1.8784	1.7967	1.8987	1.8375	1.7559	1.8578	1.7381	1.6641	1.7565
	MSE	0.0787	0.0869	0.0739	0.0923	0.0787	0.0725	0.0824	0.0601	0.0701	0.0594
85	Estimate	1.8226	1.8440	1.8011	1.8547	1.8226	1.7797	1.8333	1.7695	1.7288	1.7797
	MSE	0.0408	0.0432	0.0394	0.0448	0.0408	0.0389	0.0419	0.0350	0.0376	0.0349
125	Estimate	1.8159	1.8304	1.8013	1.8376	1.8159	1.7868	1.8231	1.7798	1.7518	1.7868
	MSE	0.0273	0.0284	0.0266	0.0291	0.0273	0.0263	0.0278	0.0245	0.0257	0.0245
175	Estimate	1.8120	1.8223	1.8016	1.8275	1.8120	1.7913	1.8171	1.7862	1.7660	1.7912
	MSE	0.0189	0.0195	0.0186	0.0199	0.0189	0.0185	0.0192	0.0175	0.0181	0.0175
225	Estimate	1.8086	1.8167	1.8006	1.8207	1.8086	1.7925	1.8126	1.7886	1.7728	1.7925
	MSE	0.0150	0.0153	0.0148	0.0156	0.0150	0.0147	0.0152	0.0142	0.0145	0.0141

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 6: Estimates and Mean Squared Errors (MSEs) for $\alpha = 1.8, \theta = 1.5, \beta = 1.2, a = 0.5$ and $b = 0.1$ under different priors, loss functions and sample sizes

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.8735	1.9485	1.7986	1.9856	1.8735	1.7237	1.9106	1.8103	1.6661	1.8461
	MSE	0.1593	0.1885	0.1418	0.2073	0.1593	0.1361	0.1723	0.1323	0.1299	0.1396
45	Estimate	1.8375	1.8784	1.7967	1.8987	1.8375	1.7559	1.8578	1.8039	1.7239	1.8238
	MSE	0.0787	0.0869	0.0739	0.0923	0.0787	0.0725	0.0824	0.0714	0.0710	0.0735
85	Estimate	1.8226	1.8440	1.8011	1.8547	1.8226	1.7797	1.8333	1.8051	1.7627	1.8157
	MSE	0.0408	0.0432	0.0394	0.0448	0.0408	0.0389	0.0419	0.0387	0.0383	0.0394
125	Estimate	1.8159	1.8304	1.8013	1.8376	1.8159	1.7868	1.8231	1.8041	1.7753	1.8113
	MSE	0.0273	0.0284	0.0266	0.0291	0.0273	0.0263	0.0278	0.0263	0.0261	0.0266
175	Estimate	1.8120	1.8223	1.8016	1.8275	1.8120	1.7913	1.8171	1.8036	1.7830	1.8088
	MSE	0.0189	0.0195	0.0186	0.0199	0.0189	0.0185	0.0192	0.0185	0.0183	0.0186

225	Estimate	1.8086	1.8167	1.8006	1.8207	1.8086	1.7925	1.8126	1.8021	1.7861	1.8061
	MSE	0.0150	0.0153	0.0148	0.0156	0.0150	0.0147	0.0152	0.0147	0.0146	0.0148

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Judging the results from table 1 to table 6, it can be explained that the estimators of the shape parameter using QLF under Gamma, uniform and Jeffrey priors are better than the other estimators the reason is that they have the smallest MSEs irrespective of the differences in the samples sizes and the allotted values of the parameter. We also discovered that there is consistency in the efficiency of the QLF under gamma prior and this consistency in the result for Bayesian estimators (using QLF under Uniform, Jeffrey and gamma priors) is an indication that the method is the most suitable for estimating the shape parameter compared to MLE and Bayesian method with the other two loss functions considered in this study. Also, judging all the prior distributions, we can clearly state that the QLF under the gamma prior has the smallest MSEs as compared to uniform and Jeffrey priors irrespective of the parameter values and the sample sizes and this level of performance of the QLF is found to be consistent despite all differences.

Finally, the results in the tables above has shown that the average estimates of the shape parameter get closer to its true value when sample size increases and the mean square errors (MSEs) all decrease as sample size increases which satisfies the first-order asymptotic theory. Also, Bayesian estimators and maximum likelihood estimators (MLEs) all become better when the sample size increases. In fact, for very large sample sizes the performances of these estimators are observed to be closely similar for all the methods of estimation.

5. Summary and Conclusion

In this article, we have derived Bayesian estimators for the shape parameter of the Gompertz-Lindley distribution with the assumption of the Uniform, Jeffrey and gamma prior distributions using three loss functions which are squared error loss function, quadratic loss function and precautionary loss function. The posterior distributions and Bayes estimators of the shape parameter of the Gompertz-Lindley were derived using the aforementioned priors and loss functions respectively. We checked efficiency of the proposed estimators using their mean square errors by means of Monte Carlo Simulations with different parameter values and sample sizes. The results revealed that using quadratic loss function gives estimators with the lowest MSEs under all the prior distributions (gamma, Jeffreys and uniform). Specifically speaking, it was discovered that Bayesian Method using Quadratic Loss Function under gamma prior gives the most efficient estimators of the shape parameter compared to estimators of Maximum Likelihood method, Squared Error Loss Function and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the differently chosen parameters values and the sample sizes. It was also clear that changing values of the scale parameter of the distribution does not affect or change the efficiency of the estimators of the estimated shape parameter.

Recommendation: We recommend that since this study considered only one shape parameter of the GomLinD, future studies should consider the scale parameter of the Gompertz-Lindley distribution because in statistical applications of the model it will be very important to identify and understand the best method for estimating both the scale and shape parameters of the model.

REFERENCES

[1] Lindley, D.V. (1958). Fiducial distributions and Bayes Theorem. J. Roy. Stat. Soc. Series B (Methodological), 102–107.

[2] Ghitany, M.E., Atieh, B., Nadarajah, S., 2008. Lindley distribution and its application. Math. Comput. Simul. 78 (4), 493–506.

[3] Mazucheli, J., Achcar, J.A., 2011. The Lindley distribution applied to competing risks lifetime data. Comput. Methods Programs Biomed. 104, 188–192.

[4] Krishna, H., Kumar, K., 2011. Reliability estimation in Lindley distribution with progressively type-ii right censored sample. Math. Comput. Simul 82, 281–294.

[5] Singh, B., Gupta, P.K., 2012. Load-sharing system model and its application to the real data set. Math. Comput. Simul 82, 1615–1629.

[6] Al-Mutairi, D.K., Ghitany, M.E., Kundu, D., 2013. Inferences on stress-strength reliability from Lindley distributions. Commun. Stat. - Theory and Methods 42, 1443–1463.

[7] Sharma, V.K., Singh, S.K., Singh, U., 2014. A new upside-down bathtub shaped hazard rate model for survival data analysis. Appl. Math. Comput. 239, 242–253.

[8] Merovci, F. (2013). Transmuted Lindley distribution. Int. J. Open Problems Compt. Math., Vol. 6, No. 2, 63-74.

[9] Ashour SK. & Eltehiwy MA. Exponentiated power Lindley distribution. J. Adv. Res. 2015; 6, 895–905.

[10] Nadarajah, S., Bakouch, H. S. and Tahmasbi, R. (2011). A generalized Lindley distribution. Sankhya B , 73:331–359.

[11] Elgarhy, M., M.Rashed, M. and Shawki, A.W. (2016). Transmuted Generalized Lindley Distribution. *International Journal of Mathematics Trends and Technology, 29(2): 145-154*

[12] Alkarni, S. H. (2015). Extended Power Lindley Distribution: A New Statistical Model For Non-Monotone Survival Data. European Journal of Statistics and Probability, 3(3), pp.19-34.

[13] Al-khazaleh, M., Al-Omari, A. I. and Al-khazaleh, A.M.H. (2016). Transmuted Two-Parameter Lindley Distribution. Journal of Statistics Applications & Probability, 5(3): 421-432.

[14] Shanker, R., Shukla, K. K., Shanker, R. and Leonida, T. A. (2017). A Three-Parameter Lindley Distribution. American Journal of Mathematics and Statistics, 7(1): 15-26.

[15] Ghitany M, Al Mutairi D, Balakrishnan N, and Al Enezi I (2013) Power Lindley distribution and associated inference. Computational Statistics and Data Analysis 64: 20-33.

[16] Merovci, F., Elbatal, I. Transmuted Lindley-geometric distribution and its applications. J. Stat. Appl. Probability Lett., 2014; 3, 77–91.

[17] Merovci, F., Sharma, V.K., 2014. The beta Lindley distribution: properties and applications. J. Appl. Math., 1–10.

[18] Akmakyapan S, Kadlar GZ. A new customer lifetime duration distribution: the Kumaraswamy Lindley distribution. Int. J. Trade, Economics Finance, 2014; 5, 441–444.

[19] Koleoso, P. O., Chukwu, A. U. & Bamiduro, T. A. A three-parameter Gompertz-Lindley distribution: Its properties and applications. *J. of Math. Theo. & Mod.*, 2019; 9(4): 29-42.

[20] Cordeiro GM, Afify AZ, Ortega EMM, Suzuki AK & Mead ME. The odd Lomax generator of distributions: Properties, estimation and applications. Journal of Computational and Applied Mathematics, 2019; 347: 222–237.

[21] Cordeiro GM, Ortega EMM, Popovic BV & Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. *Applied Mathematics and Computation*, 2014; 247: 465-486.

[22] Afify MZ, Yousof HM, Cordeiro GM, Ortega EMM. & Nofal ZM. The Weibull Frechet Distribution and Its Applications. Journal of Applied Statistics, 2016; 1-22.

[23] Tahir MH, Zubair M, Mansoor M, Cordeiro GM & Alizadeh M. A New Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, 2016; 45(2), 629-647.

[24] Ieren TG & Yahaya A. The Weimal Distribution: its properties and applications. *Journal of the Nigeria Association of Mathematical Physics,* 2017; 39: 135-148.

[25] Bourguignon M, Silva RB & Cordeiro G. M. The weibull-G family of probability distributions. Journal of Data Science, 2014; 12: 53-68.

[26] Gomes-Silva F, Percontini A, De Brito E, Ramos MW, Venancio R & Cordeiro GM. The Odd Lindley-G Family of Distributions. Austrian Journal of Statistics, 2017; 46, 65-87.

[27] Ieren TG, Koleoso PO, Chama AF, Eraikhuemen IB & Yakubu N. A Lomax-inverse Lindley Distribution: Model, Properties and Applications to Lifetime Data. *Journal of Advances in Mathematics and Computer Science,* 2019; 34(3-4): 1-28.

[28] Alzaatreh A, Famoye F & Lee C. A new method for generating families of continuous distributions. *Metron*, 2013; 71: 63–79.

[29] Ieren TG, Kromtit FM, Agbor BU, Eraikhuemen IB & Koleoso PO. A Power Gompertz Distribution: Model, Properties and Application to Bladder Cancer Data. *Asian Research Journal of Mathematics*, 2019; 15(2): 1-14.

[30] Ieren TG, Oyamakin SO & Chukwu AU. Modeling Lifetime Data With Weibull-Lindley Distribution. *Biometrics and Biostatistics International Journal,* 2018; 7(6): 532–544.

[31] Martz HF & Waller RA. *Bayesian reliability analysis*. New York, NY: John Wiley, 1982.

[32] Ibrahim JG, Chen MH & Sinha D. *Bayesian survival analy.*. New York, NY: Springer-Verlag, 2001.

[33] Singpurwalla ND. *Reliability and risk: A Bayesian perspective*. Chichester: John Wiley, 2006.

[34] Dey S. Bayesian estimation of the shape parameter of the generalized exponential distribution under different loss functions. *Pakistan Journal of Statistics and Operation Research*, 2010; *6*(2): 163-174.

[35] Ahmed AOM, Ibrahim NA, Arasan J & Adam MB. Extension of Jeffreys' prior estimate for weibull censored data using Lindley's approximation. *Austr. J. of B. and Appl. Sci.*, 2011; 5(12): 884–889.

[36] Pandey BN, Dwividi N, Pulastya B. Comparison between Bayesian and maximum likelihood estimation of the scale parameter in Weibull distribution with known shape under linex loss function, *J. of Sci. Resr.*, 2011; 55: 163–172.

[37] Mabur TM, Omale A, Lawal A, Dewu MM & Mohammed S. Bayesian Estimation of the Scale Parameter of the Weimal Distribution. Asian Journal of Probability and Statistics, 2018; 2(4): 1-9.

[38] Ahmad K, Ahmad SP & Ahmed A. On parameter estimation of erlang distribution using bayesian method under different loss functions, *Proc. of Int. Conf. on Adv. in Computers, Com., and Electronic Eng.*, 2015; 200–206.

[39] Ieren TG, Chama AF, Bamigbala OA, Joel J, Kromtit FM & Eraikhuemen IB. On A Shape Parameter of Gompertz Inverse Exponential Distribution Using Classical and Non Classical Methods of Estimation. Journal of Scientific Research & Reports, 2020; 25(6): 1-10.

[40] Eraikhuemen IB, Bamigbala OA, Magaji UA, Yakura BS & Manju KA. Bayesian Analysis of Weibull-Lindley Distribution Using Different Loss Functions. Asian Journal of Advanced Research and Reports, 2020; 8(4): 28-41.

[41] Ieren TG & Oguntunde PE. A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution. *Asian J. of Prob. and Stat.*, 2018; 1(1): 1-12.

[42] Ieren TG & Chukwu AU. Bayesian Estimation of a Shape Parameter of the Weibull-Frechet Distribution. Asian Journal of Probability and Statistics, 2018; 2(1):1-19, 2018.

[43] Krishna H & Goel N. Maximum Likelihood and Bayes Estimation in Randomly Censored Geometric Distribution. *Journal of Probability and Statistics*, 2017. 12 pages.

[44] Gupta PK & Singh AK. Classical and Bayesian estimation of Weibull distribution in presence of outliers. *Cogent Math.*, 2017; 4: 1300975.

[45] Gupta I. Estimation of Parameter And Reliability Function of Exponentiated Inverted Weibull Distribution using Classical and Bayesian Approach. *Int. J. of Recent Sci. Res.*, 2017; 8(7): 18117-18819.

[46] Ahmad K, Ahmad SP & Ahmed A. Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *J. of Prob. and Stat.*, Article ID 7581918, 8, 2016. pages http://dx.doi.org/10.1155/2016/7581918.

[47] Preda V, Eugenia P & Alina C. Bayes Estimators of Modified-Weibull Distribution parameters using Lindley's approximation. *WSEAS TRANSACTIONS on MATHEMATICS*, 2010; 9 (7), 539-549.

[48] Abbas K, Abbasi NY, Ali A, Khan SA, Manzoor S, Khalil A, Khalil U, Khan DM, Hussain Z & Altaf M. Bayesian Analysis of Three-Parameter Frechet Distribution with Medical Applications. Computational and Mathematical Methods in Medicine, Volume 2019, Article ID 9089856, 8 pages, https://doi.org/10.1155/2019/9089856.

[49] Ramos PL, Louzada F & Ramos E. Posterior Properties of the Nakagami-m Distribution Using Noninformative Priors and Applications in Reliability. IEEE Transactions on reliability, 67(1): 105-117, 2017.

[50] Ramos PL & Louzada F. Bayesian reference analysis for the Generalized Gamma distribution. IEEE Communications Letters, 2018; 22(9), 1950-1953.

[51] Tomazella VLD, de Jesus SR, Louzada F, Nadarajah S & Ramos PL. Reference Bayesian analysis for the generalized lognormal distribution with application to survival data. Statistics and Its Interface, 2020; 13(1): 139-149.

[52] Azam Z, & Ahmad SA. Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Pakistan J. of Stat. and Operat. Res.*, 2014; 10(2), 217-228.

[53] Norstrom JG. The use of precautionary loss functions in risk analysis. IEEE Trans. Reliab., 1996; 45(3): 400–403, 1996. http://dx.doi.org/10.1109/24.536992.