An Investigation into the Order of Difference of Product of Consecutive Integral Perfect Powers.

Abstract
The article established an analysis on the difference of the product of order of integral perfect powers. The proof on the analysis were achieved by the use of property of difference operator, combinatorial technique, mathematical induction principle. The results proved conclusively that if any number of consecutive integers are raised to a positive power k, then the $(2k)^{th}$ difference of the product of the kth power of two consecutive integers is equal to (2k)!
Keywords: *mathematical induction, finite difference, positive powers, integral order, factorial differential operator.*

1. Introduction

An integral perfect power is the irrational root that is an integer. A perfect power is a number which has rational root Chase [1]. A finite difference is a mathematical expression of the form $f\,(x + b) - f\,(x + a)$ and a forward difference is of the form.
$\Delta h\,[f](x) = f\,(x + h) - f(x)$[2]

In this article, an examination on the set of positive integers {0,1,2,3,4,5,6,7,8,9} reveals that the product of consecutive elements of the set of the integers are {0,2,6,12,20,30,42,56,72}. The first difference of the set of product of this integers is the set of even numbers {2,4,6,8,10,12,14,16}. Furthermore, the set of the differences of the precedingdifference set is {2,2,2,2,2,2,2} = {2}, a single ton set. Clearly, the set of differences of the above second difference set is {0}. The emerging pattern motivates the investigation of whether or not this pattern will persist for all perfect squares and higher power greater than two resulting integers.

In their contributions in Exponential Diophantine equations, Shorey and Tijdeman[3] investigated perfect powers at integral values of a polynomial, and obtain in particular the following result; let $f\,(x)$be a polynomial with rational integer coefficients and with a least two simple rational zeros suppose $b \neq 0$, $m \geq 0$, x and y with $|Y| > 1$ are rational integers.Then the equation $f\,(x) = by^m$implies that m is bounded by computable number depending only on b and f. There was also discussion on orders of difference of perfect powers by Ladan, Ukwu and Apine [4]. They established a relation between the difference operator and factorial on the set of integers. Their investigation proved that, for any positive integer k, the k^{th} order difference of the k^{th}power of any integer is equal to k!

That is;

$$\Delta^k\left(j^k\right) = k! \text{ for any positive integer j and for any positive integer k.}$$

More generally, in the sequel, the question at the heart of the matter is the following; what is the computation disposition of the orders of difference of the product of consecutive integral perfect powers. Review of literature shows that no such investigation has been undertaken. Thus, this article adds to the existing body of knowledge, by providing answers to the above questions.

2 Methods

2.1 PreliminaryDefinitions

In what follows, differences of finite orders will be defined.

2.1.1 Differences of Order One (1)

Given a sequence$\{f_j\}_1^\infty$, defined the difference of order one at j with respect to the sequence by:

$\Delta\ (f_j) = f_{j+1}$-f_j, for every integral j.

2.1.2 Higher Order Differences

Higher Order differences can be defined recursively by:

$$\Delta^k(f_j) = \Delta(\Delta^{k-1}(f_j))$$
$$= \Delta^{k-1}(\Delta(f_j))$$
$$=\Delta^{k-1}[f_{j+1} - f_j] \text{ for } k \geq 2$$

3 Results and Discussion

3.1 Preliminary Theorem

Suppose that $f_j = j$ for integral j, such that j ≥0, we have that;

i. $\Delta\left[\left(f_j\right)\left(f_{j+1}\right)\right]$is even

ii. $\Delta^2\left[\left(f_j\right)\left(f_{j+1}\right)\right]$=2

iii. $\Delta^k\left[\left(f_j\right)\left(f_{j+1}\right)\right]$=0, k $\epsilon\{3,4\ldots.\}$.

Proof:

Let j = {0,1,2,3,4,5,6,7,8,9,}

Then the table below yields values of $\Delta^k\left(f_j^p\right)$for selected values of k and P in the set {1}and {0,1,2,3} respectively, when $\Delta^0\left(f_j^p\right)= f_j^p$.

Table 1. Difference of Product Order Table for selected Consecutive Positive Integers.

f_j	$(f_j)(f_{j+1}) = i$	Δ i	Δ^2i	Δ^3i
0	0	2	2	0
1	2	4	2	0
2	6	6	2	0
3	12	8	2	0
4	20	10	2	0
5	30	12	2	0
6	42	14	2	
7	56	16		
8	72			
9				

It is clear that $\Delta^k\left[(f_j)(f_{j+1})\right] = 0$

For all k ≥ 4 and Jϵ{0,1, ……..,8,9}.

Obviously, the theorem is valid Jϵ{0,1,2,3,4,5,6,7,8,9}as observed from columns 4 and 5. The proofs of (i), (ii) and (iii)are direct.

i. $\Delta\left[(f_j)(f_{j+1})\right]$ $= \Delta\ (j)(j+1) = \Delta(j^2 + j)$

$= \Delta j^2 + \Delta j$

$= [(j+1)^2 - (j)^2] + [(j+1) - (j)]$

$= j^2 + 2_j + 1 - j^2 + j + 1 - j$

$= 2_j + 2 = 2(j+1) = \text{even.}$

It shows that is even for positive integers j. proving (i)

ii. $\Delta^2\left[(f_j)(f_{j+1})\right] = \Delta\ [2\ (j+1)] = 2\ [\Delta\ (j+1)]$

$= 2\left(\Delta_j + \Delta_j\right)$

$= 2\ \Delta_j = 2\ (j+1-j)$

$= 2\ (1) = 2$

Proving (ii)

iii. The principle of mathematical induction is needed in the proof of (iii)

For k=3,

$\Delta^3\left[(f_j)(f_{j+1})\right] = \Delta\left(\Delta^2(j^2+j)\right) = \Delta\ (2) = 2 - 2 = 0$

Consequently $\Delta^k(j^2+j) = \Delta^{k-3}\left(\Delta^3(j^2+j)\right)$

$= \Delta^k\ (0) = 0$

For all k≥ 4, proving

$\Delta^k(j^2+j) = 0,$ for all positive integerForall k ≥ 3. This established the proof of (iii).

In the sequel, we examine the computational disposition of $\Delta^{k}\left[\left(f_{j}^{p}\right)\left(f_{j+1}^{p}\right)\right]$ for every positive integral k and p.

The results are summarized as in the following theorem.

3.2 Main Theorem.

Let $f_j = j$ where j is any positive integer. Then for arbitrary positive integer k and p,

$\Delta^{k}\left[\left(f_{j}^{p}\right)\left(f_{j+1}\right)^{p}\right]$ is given by $\Delta^{k}\left[\left(f_{j}^{p}\right)\left(f_{j+1}^{p}\right)\right] = \begin{cases} 0, \text{ if } k > 2p \ldots\ldots\ldots\ldots (a) \\ (2\,k)!, \text{if } k = 2p \ldots\ldots. (b). \end{cases}$

3.2.1 Proof of (a)

$$\Delta^{3}\left[\left(f_{j}\right)\left(f_{j+1}\right)\right] = \Delta^{3}[(j)(j+1)] = \Delta^{3}(j^{2}+j) =$$
$$\Delta^{2}(\Delta j^{2} + \Delta j) = \Delta^{2}\left[[(j+1)^{2} - j^{2}] + [(j+1) - j]\right] =$$
$$\Delta^{2}\left[\left(j^{2} + 2_{j} + 1 - j^{2}\right) + (j + 1 - j)\right] = \Delta^{2}\left(2_{j} + 1 + 1\right) =$$
$$\Delta^{2}(2j+2) = \Delta\,[\Delta(2j+2)] = \Delta(2\Delta j + \Delta 2) =$$
$$\Delta\left[2\left((j+1) - j\right)\right] = \Delta[2(j+1-j)] = \Delta[2(1)] = \Delta 2 = 0.$$

So (a) is valid for k=3 and P=1, which are the minimal values for the respective exponents. Assume that (a) is valid for all pairs of integer $\tilde{k}, \hat{p}$ for which $\tilde{k}\hat{p} \leq k + p$ for some positive integers $\tilde{k}$ and $\hat{p}$ such that $p \geq 1, k \geq 3, p < k$. Then,

$$\Delta^{k+1}\left[\left(f_{j}\right)\left(f_{j+1}\right)\right] = \Delta\left[\left[\Delta^{k}\left(f_{j}\right)\left(f_{j+1}\right)\right]\right] = \Delta(0) = 0$$

(by induction hypothesis). Therefore, the validity of (a) is established.

3.2.2 Proof of (b)

We examine $\Delta^{2k}\left[\left(f_{j}^{k}\right)\left(f_{j+1}^{k}\right)\right]$: $k = 1$

$\Rightarrow \Delta^{2}\left[\left(f_{j}\right)\left(f_{j+1}\right)\right] = \Delta^{2}[(j)(j+1)] = \Delta^{2}(j^{2}+j) =$

$\Delta[\Delta j^{2} + \Delta j] = \Delta[(j+1)^{2} - j^{2} + (j+1) - j]$

$= \Delta(j^{2} + 2j + 1 - j^{2} + j + 1 - j)$

$= \Delta(2j + j) = 2\Delta j + \Delta 2$

$= 2[(j+1) - j + 2 - 2] = 2(1+0)$

$= 2(1) = [2(1)]! = 2!$

$k = 2 \Rightarrow \Delta^{4}\left[\left(f_{j}^{2}\right)\left(f_{j+1}^{2}\right)\right] = [2(2)]! = 4!$

(by induction hypothesis)

$\Rightarrow$The theorem is valid for $k \epsilon \{1,2\}$.

Assume the validity of the theorem for $k \epsilon \{3,\ldots q\}$ for some integer $q \geq 4$.

Then $\Delta^{2q}\left[\left(f_{j}^{q}\right)\left(f_{j+1}^{q}\right)\right] = [2\,(q)]!$

(by induction hypothesis).Finally,we need to prove that:

$\Delta^{2(q+1)}\left[\left(f_{j}^{q+1)}\right)\left(f_{j+1}^{q+1}\right)\right] = [2\,(q+1)]!$

Furthermore, there is a strong relationship between the difference operator Δ and the differential operator D.

$$D^k(j^p) = 0 \text{ if } k > P$$

$$D^k(j^p) = k! \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (I)$$

$$\Delta^k(j^p) = k! \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (II)$$

By comparism of (I) and (II)

$$\Rightarrow D^k\left(J^k\right) = \Delta^k\left(J^k\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (III)$$

We shall apply the differential operator to show the validity of proof (b) for p = {1, 2, 3}.

We examine the sequence,

$$\left(f_j^p\right)\left(f_{j+1}^p\right) = (J^P)(j+1)^P$$

$$= J^P\left[\sum_{i=0}^{P}\binom{P}{i} j^{P-i}\right]$$

$$= \left[\sum_{i=0}^{P}\binom{P}{i} j^{2P-i}\right]$$

Case 1, P = 1

$$\left(f_j\right)\left(f_{j+1}\right) = \left[\sum_{i=0}^{1}\binom{1}{i} j^{2-i} = j^{2-j}\right]$$

$$\Delta\left[\left(f_j\right)\left(f_{j+1}\right)\right] = \frac{d}{dj}\left(j^{2-j}\right) = 2j^{-1}$$

$$\Delta^2\left[\left(f_j\right)\left(f_{j+1}\right)\right] = \frac{d}{dj}\left(2j^{-1}\right) = 2\,[2(1)]! = 2!$$

$$\Delta\left[\sum_{i=0}^{1}\binom{1}{i} j^{2-i}\right] = [2(1)]!$$

Case 2, P = 2

$$\Rightarrow \sum_{i=0}^{2}\binom{2}{i} j^{4-i} = j^4 + 2j^3 + j^2$$

$$\Delta\sum_{i=0}^{2}\binom{2}{i} j^{4-i} = \frac{d}{dj}\left(j^4 + 2j^3 + j^2\right) = 4j^3 + 6j^2 + 2j$$

$$\Delta^2 \sum_{i=0}^{2} \binom{2}{i} j^{4-i} = \frac{d}{dj}(4j^3 + 6j^2 + 2j) = 12j^2 + 12j + 2$$

$$\Delta^3 \sum_{i=0}^{2} \binom{2}{i} j^{4-i} = \frac{d}{dj}\left(12j^2 + 12j + 2\right) = 24j + 12$$

$$\Delta^4 \sum_{i=0}^{2} \binom{2}{i} j^{4-i} = \frac{d}{dj}\left(24j + 12\right) = 24 = 4!$$

$\Rightarrow 4! = [2(2)]! = 24$

which established the validity of the proof of (b).

Case 3, P = 3

$$\Rightarrow \sum_{i=0}^{3} \binom{3}{i} j^{6-i} = j^6 + 3j^5 + 3j^4 + j^3$$

$$\Delta\left[\sum_{i=0}^{3} \binom{3}{i} j^{6-i}\right] = 6j^5 + 15j^4 + 12j^3 + 3j^2$$

$$\Delta^{2}\left[\sum_{i=0}^{3} \binom{3}{i} j^{6-i}\right] = 30j^4 + 60j^3 + 36j^2 + 6j$$

$$\Delta^{3}\left[\sum_{i=0}^{3} \binom{3}{i} j^{6-i}\right] = 120j^3 + 180j^2 + 72j + 6$$

$$\Delta^{4}\left[\sum_{i=0}^{3} \binom{3}{i} j^{6-i}\right] = 360j^2 + 360j + 72$$

$$\Delta^{5}\left[\sum_{i=0}^{3} \binom{3}{i} j^{6-i}\right] = 720j + 360$$

$$\Delta^{\mathbf{6}}\left[\sum_{i=0}^{3}\binom{3}{i}j^{6-i}\right] = 720 = 6! = [2(3)]!$$

$$\Rightarrow \Delta^{\mathbf{2P}}\left[\sum_{i=0}^{P}\binom{p}{i}j^{2p-i}\right] = [2(p)]!$$

$$= (2P)!$$

$$\Rightarrow \Delta^{k}\left[\left(f_j^{P)}\right)\left(f_{j+1}^{P}\right)\right] = (2k)!, \text{if } k = 2p.$$

Thus the validity of (b) is established by the application of the difference operator Δ with respect to the differential operator D.

3.2.3 Pattern Recognition

The factorial of even numbers can be generated through a polynomial function with respect to the coefficients of Pascal's triangle in a triangular array.

The powers of the variable of the polynomial functions are distributed with respect to the Even and Odd positive integers.

→**Pattern** with respect to Pascal's coefficient for some P ≥ 1.

$$j^2 + j$$

$$j^4 + 2j^3 + j^2$$

$$j^6 + 3j^5 + 3j^4 + j^3$$

$$j^8 + 4j^7 + 6j^6 + 4j^5 + j^4$$

$$j^{10} + 5j^9 + 10j^8 + 10j^7 + 5j^6 + j^5$$

$$j^{12} + 6j^{11} + 15j^{10} + 20j^9 + 15j^8 + 6j^7 + j^6$$

$$j^{14} + 7j^{13} + 21j^{12} + 35j^{11} + 35j^{10} + 21j^9 + 7j^8 + j^7$$

$$j^{16} + 8j^{15} + 24j^{14} + 56j^{13} + 70j^{12} + 56j^{11} + 28j^{10} + 8j^9 + j^8$$

→**Pattern** of powers of variable

2 1
4 3 2
6 5 4 3
8 7 6 5 4
10 9 8 7 6 5
12 11 10 9 8 7 6
14 13 12 11 10 9 8 7
16 15 14 13 12 11 10 9 8

POWERS OF POLYNOMIAL with respect to PASCAL'S TRIANGLE

Observe that the power of the first term, that is the highest power of each polynomial is equal to K = 2p and the least power of the polynomial is equal to P.

$$\Rightarrow \Delta^{k}\left[\left(f_j^{P)}\right)\left(f_{j+1}^{P}\right)\right] = k!, = [2(P)]!$$

such that k = 2p, p ≥ 1

Universality of Mathematics

Pascal's triangle, credited to French mathematician Blaise Pascal (1623-1662), appeared in a Chinese document printed in 1303. The Binomial Theorem was known in Eastern Cultures prior to its discovery in Europe. The same mathematics is often discovered or invented by independent researchers separated by time, place and culture. Robort [10]

Finally, let = p + 1 = q + 1

$$\Rightarrow \Delta^{2(q+1)}\left[\left(f_j^{q+1)}\right)\left(f_{j+1}^{q+1}\right)\right]$$

$$= \Delta^{2(q+1)}\left[\sum_{i=0}^{q+1}\binom{q+1}{i} j^{2(q+1)-i}\right]$$

$$= \Delta^{2q+2)}\left[\sum_{i=0}^{q+1}\binom{q+1}{i} j^{2q+2-i}\right]$$

$$\Rightarrow \Delta^{2q}\left[\Delta^{2}\sum_{i=0}^{q+1}\binom{q+1}{i} j^{2q+2-i}\right]$$

Let q = 1

$$\Rightarrow \Delta^{2q}\left[\Delta^2 \sum_{i=0}^{2} \binom{2}{i} j^{4-i}\right] = \Delta^{\mathbf{2}}\left(12j^2 + 12j + 2\right)$$

$$= \Delta[\Delta(12j^2 + 12j + 2)] = \Delta(24j + 12)$$

$= \Delta(24j + 12) = 24 = 4! = 2(2!)$.

3.3 Corollary

$$\Delta^{2k}\left[\left(j^k\right)(j+1)^k\right] = \Delta^{\mathbf{2k}}\left[\sum_{i=0}^{k} \binom{k}{i} j^{2k-i}\right]$$

$$= [2(k)]!$$

For every positive integer J and k = 0 such that i ≥ 0.

In words, for any positive integer k, the $2k^{th}$ order difference of the product of the k^{th} power of any positive integers is equal to $(2k^{th})$!

The implication of (b) in theorem 3.2 is the following: if any number of set of consecutive integers are raised to a positive integral power K, then the $2k^{th}$ difference of the product of any two consecutive elements of the set is equal to (2k)!

In their contribution, Ladan, Ukwuand Apineinvestigated difference of integral perfect powers and obtain in particular.

$\Delta^k\left(j^k\right) = k!$ for any integer j and for any positive integer k. Theoretically, for any positive integerk, the k^{th} order difference of the k^{th} power of any integer is equal to k!

4. CONCLUSION

This article established the structures of finite difference orders with respect product of powers of consecutive integers. In particular, the result portrays a startling similarity between the difference orders and the D (differential) operator powers of polynomials with positive integral powers. The limitation of work of Ladan, Ukwu and Apine was restricted at the coefficient of k! in the k^{th} difference operator to be equal to one (1). This article investigated in deft, the existence of a constant even coefficient (2)! for all k belonging to positive integers. The computation of the difference of product of powers of consecutive integers was programmed using Java Programming language. The result of the application of the algorithm in a computer system was able to capture the case of k = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The result was not able to capture the case k ≥ 11, which is the constraints restriction due to memory capacity of Java Programming language, on the result of the computation of $\Delta^{2k}\left[\left(j^k\right)(j+1)^k\right] = [2(k)]!$

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