__Original Research Article__

Orthogonal polynomials and Fourier series for functions of vector variable: Multidimensional-matrix approach

ABSTRACT

In the article, the theory of the Fourier series on the orthogonal multidimensional-matrix polynomials is developed. The known results from the theory of the orthogonal polynomials of the vector variable and the Fourier series are given and the new results are presented. In particular, the known results of the Fourier series are extended to the case of the multidimensional-matrix functions, what allows us to solve more general approximation problems. The general case of the approximation of the multidimensional-matrix function of the vector argument by the Fourier series on the orthogonal multidimensional-matrix polynomials is realized programmatically as the program function and its efficiency is confirmed. The analytical expressions for the coefficients of the second degree orthogonal polynomials and Fourier series for the potential studies are obtained.

Keywords: Fourier series; multidimensional-matrix orthogonal polynomials; multivariate polynomial regression.

1. INTRODUCTION

The most important tool for research of the real systems and processes is approximation. The mathematical models of the real systems and processes are their approximate mathematical images. The various methods of approximation there exist, one of which is approximation by Fourier series by the orthogonal polynomials.

The history of the orthogonal polynomials of both one and several variables dates back to Hermite [1]. Hermite in [1] and then Appel P. and Kampe de Feriet in [2] studied in details the properties of the so-called Hermite polynomials of one and two variables. The general theory of the orthogonal polynomials of many variables is developed in paper [3]. As per the works [1], [2], [3], this theory is constructed as the theory of two bi-orthonormal sequences of the polynomials: basic and conjugate. In work [4], it is proposed to choose a polynomial with the unit coefficient at the highest degree as the basic polynomial. This theory uses the classical (scalar) mathematical approach and is therefore the classical theory. The classical theory can be found also in [5], [6].

The foundations of the theory of the multidimensional matrices were laid in work [7] and developed in work [8]. The results of works [3], [4] were combined in works [9], [8] on the basis of the multidimensional-matrix mathematical approach. It is haw the multidimensional-matrix theory of the orthogonal polynomials of the vector variable arose.

In this article, the theory of the orthogonal multidimensional-matrix polynomials is developed in the direction of its practical use. Since the theory is created on the basis of the multidimensional-matrix mathematical approach, the multidimensional-matrix notation is used in this article. The basic definitions of the theory of the multidimensional matrices in English can be found in the appendix to the article [10].

2. ORTHOGONAL POLYNOMIALS OF THE VECTOR VARIABLE

Let Ω be some closed region of the space R^n, $\rho(x)$, $x \in \Omega$, be nonnegative function (weight function) such that the integrals (the moments of the weight function $\rho(x)$)

$$\nu_{x^i} = \int_{\Omega} x^i \rho(x) dx < \infty, \;\; i = 0,1,2,... \;, \qquad (1)$$

exist, and $L_2(\rho, \Omega)$ be the space of the functions with integrable square in Ω with the weight $\rho(x)$. Here x^i is the $(0,0)$-rolled degree of the one-dimensional matrix x: $x^i = {}^{0,0}(x^i) = {}^{0,0}(x \cdot x \cdots x)$ [8, 10].

The theory of the orthogonal polynomials of the vector variable is created as the theory of two bi-orthonormal sequences of the polynomials.

A multidimensional-matrix r degree polynomial $Q_r(x)$ of the vector (one-dimensional) variable $x \in \Omega$ is defined as follows [7, 8]:

$$Q_r(x) = \sum_{k=0}^{r} {}^{0,k}(C^*_{(r,k)} x^k) = \sum_{k=0}^{r} {}^{0,k}(x^k C^*_{(k,r)}), \quad r = 0,1,2,..., \qquad (2)$$

where $C^*_{(r,k)}$ are the $(r+k)$-dimensional matrices of the coefficients,

$$C^*_{(r,k)} = (c^*_{i_1,...,i_r,j_1,...,j_k}), \; r = 0,1,2,..., \;\; k = 0,1,2,...,r,$$

symmetrical with respect to the indices of their two multiindices $(i_1,...,i_r)$, $(j_1,...,j_k)$ and satisfying the conditions

$$C^*_{(r,k)} = (C^*_{(k,r)})^{H_{r+k,k}}, \;\; C^*_{(k,r)} = (C^*_{(r,k)})^{B_{k+r,k}} .$$

The notations $H_{r+k,k}$ and $B_{r+k,k}$ mean the transpose substitutions of the types “back” and “forward” respectively [8, 10]. Each of the indices of the multiindices $(i_1,...,i_r)$, $(j_1,...,j_k)$ takes the values $1,2,...,n$.

Definition. The sequence of the multidimensional-matrix polynomials $Q_r(x)$ (2) is called orthogonal in $L_2(\rho, \Omega)$ if the following conditions are satisfied:

$$\int_{\Omega} Q_r(x) Q_k(x) \rho(x) dx \; \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r. \end{cases} \qquad (3)$$

The two sequences of the orthogonal polynomials of many variables are considered: the basic sequence $P_r(x)$ and the sequence $Q_r(x)$ conjugate of $P_r(x)$, $r = 0,1,2,...$.

Definition. The multidimensional-matrix r degree polynomial in $L_2(\rho, \Omega)$ of the following form

$$P_r(x) = \sum_{k=0}^{r-1} {}^{0,k}(C_{(r,k)}x^k) + x^r = \sum_{k=0}^{r-1} {}^{0,k}(x^k C_{(k,r)}) + x^r, \ r = 0,1,2,..., \tag{4}$$

is called the basic polynomial, where $C_{(r,k)}$ are $(r+k)$-dimensional matrices of the coefficients,

$$C_{(r,k)} = (c_{i_1,...i_r,j_1,...,j_k}), \quad r = 0,1,2,..., \quad k = 0,1,2,...,r-1,$$

symmetrical with respect to the indices of their two multiindices $(i_1,...,i_r)$, $(j_1,..., j_k)$ and satisfying the conditions

$$C_{(r,k)} = C_{(k,r)}^{H_{r+k,k}}, \ C_{(k,r)} = C_{(r,k)}^{B_{k+r,k}}.$$

Definition. The multidimensional-matrix polynomial $P_r(x)$ (4) is called the basic orthogonal r degree polynomial in $L_2(\rho,\Omega)$ if it is orthogonal to the homogeneous polynomials $1, x, x^2,..., x^{r-1}$, $x^k = {}^{0,0}(x^k)$:

$$\int_\Omega P_r(x) x^k \rho(x) dx \ \begin{cases} = 0, & k = 0,1,...,r-1, \\ \neq 0, & k = r. \end{cases} \tag{5}$$

Definition. The sequences of the multidimensional-matrix polynomials $P_r(x)$ (4) and $Q_r(x)$ (2) are called the completely orthonormal in $L_2(\rho,\Omega)$ if the conditions (3), (5) are satisfied and the following condition

$$\int_\Omega Q_r(x) P_k(x) \rho(x) dx = \begin{cases} 0, & k = 0,1,...,r-1, \\ D_{(r,r)}, & k = r. \end{cases} \tag{6}$$

is satisfied too. There $D_{(r,r)}$ is the $2r$-dimensional order n matrix with the following structure:

$$D_{(r,r)} = (d_{i_1,i_2,...,i_r,j_1,j_2,...,j_r}), \ i_1, i_2,..., i_r, j_1, j_2,..., j_r = 1,2,...,n. \tag{7}$$

The elements of this matrix are defined by the expression

$$d_{i_1,i_2,...i_r,j_1,j_2,...,j_r} = \begin{cases} r_1!r_2!...r_n!, & perm(i_1,i_2,...,i_r) = (j_1, j_2,..., j_r), \\ 0, & perm(i_1,i_2,...,i_r) \neq (j_1, j_2,..., j_r), \end{cases} \tag{8}$$

in which $perm(i_1,i_2,...,i_r)$ means any permute of the values of the indices $i_1,i_2,...,i_r$, $r_1 + r_2 + ... + r_n = r$, and r_k is the number of repetitions of the k-value, $k = 1,2,...,n$.

The matrix $D_{(r,r)}$ (7), (8) has such a useful property that for any q-dimensional matrix $C = (c_{i_1,i_2,...i_{q-r},j_1,j_2,...,j_r})$ with $q \geq r$ symmetrical with respect to the indices $j_1, j_2,..., j_r$ the following equality is fulfilled [8]:

$${}^{0,r}(CD_{(r,r)}) = r!C.$$

Let us introduce the initial i-th order moments ν_{x^i} and the initial-central and central-initial $(i+j)$-th order moments $\nu_{x^i x_c^j}$, $\nu_{x_c^i x^j}$ of the weight function $\rho(x)$:

$$\nu_{x^0} = \int_\Omega \rho(x)dx \,,$$

$$\nu_{x^i x^j} = \nu_{x^{i+j}} = \int_\Omega x^{i+j}\rho(x)dx \,, \; i+j = 1,2,... \,, \tag{9}$$

$$\nu_{x^i x^j_c} = \int_\Omega x^i (x^j - \nu_{x^j})\rho(x)dx \,, \; i+j = 1,2,... \,, \tag{10}$$

$$\nu_{x^i_c x^j} = \int_\Omega (x^i - \nu_{x^i}) x^j \rho(x)dx \,, \; i+j = 1,2,... \,, \tag{11}$$

where x^i, $x^i(x^j - \nu_{x^j})$, $(x^i - \nu_{x^i})x^j$ are the $(0,0)$-rolled degrees and $(0,0)$-rolled products of the matrices [8]. We will often avoid the notation $(0,0)$-rolled product and will write yx instead of ${}^{0,0}(yx)$.

The moments (10), (11) have the following properties:

$$\nu_{x^i x^j_c} = \nu_{x^i_c x^j} = \nu_{x^i_c x^j_c} = \nu_{x^{i+j}} - \nu_{x^i}\nu_{x^j} \,, \tag{12}$$

$$\nu_{x^i_c x^j_c} = (\nu_{x^j_c x^i_c})^{B_{iq+jq,iq}} = \nu_{x^{j+i}} - \nu_{x^j}\nu_{x^i} \,,$$

where $B_{iq+jq,iq}$ is the transpose substitution of the type “forward” [8]. These properties are proved by calculation of the formulae (10), (11). The properties (12) allow us to use the following notations:

$$\mu_{x^i x^j} = \nu_{x^{i+j}} - \nu_{x^i}\nu_{x^j} \,, \; \mu_{x^j x^i} = \nu_{x^{j+i}} - \nu_{x^j}\nu_{x^i} \,,$$

$$\mu_{x^i x^j} = (\mu_{x^j x^i})^{B_{iq+jq,iq}} \,.$$

Let us introduce also the mutual moments

$$\nu_{y^i x^j} = \int_\Omega y^i(x) x^j \rho(x)dx \tag{13}$$

with the properties

$$\nu_{y^i x^j_c} = \nu_{y^i_c x^j} = \nu_{y^i_c x^j_c} = \int_\Omega y^i(x)(x^j - \nu_{x^j})\rho(x)dx = \nu_{y^i x^j} - \nu_{y^i}\nu_{x^j} = \mu_{y^i x^j} \,.$$

The weight function $\rho(x)$ in the case $\nu_{x^0} = 1$ represents the probability density function of the some random vector ξ.

Theorem [8]. If the sequences of the multidimensional-matrix polynomials $P_r(x)$ (4) and $Q_r(x)$ (2) are completely orthonormal in $L_2(\rho,\Omega)$, i.e. they satisfied the conditions (3), (5), (6), then the coefficients $C_{(r,k)}$ of the basic sequence $P_r(x)$ (4) are defined by the following multidimensional-matrix system of the linear algebraic equations

$$\nu_{x^r x^p} + \sum_{k=0}^{r-1} {}^{0,k}(C_{(r,k)}\nu_{x^k x^p}) = 0, \;\; r = 0,1,..., \;\; p = 0,1,...,r-1 \,, \tag{14}$$

and the coefficients $C^*_{(r,k)}$ of the conjugate sequence $Q_r(x)$ (2) are defined by the expression

$$C^*_{(r,k)} = r!\, {}^{0,r}({}^{0,r}B^{-1}_{(r,r)} C_{(r,k)}) \,,$$

where ${}^{0,r}B^{-1}_{(r,r)}$ *is the matrix* $(0,r)$ *-inverse to the following matrix* $B_{(r,r)}$ *:*

$$B_{(r,r)} = \nu_{x^r x^r} + \sum_{k=0}^{r-1} {}^{0,k}(C_{(r,k)} \nu_{x^k x^r}) + \sum_{k=0}^{r-1} {}^{0,k}(\nu_{x^r x^k} C_{(k,r)}) + \sum_{k=0}^{r-1}\sum_{q=0}^{r-1} {}^{0,k}(C_{(r,k)} {}^{0,q}(\nu_{x^k x^q} C_{(q,r)})) \quad (15)$$

The coefficients $C_{(k,r)}$ *of the basic sequence* $P_r(x)$ *(4) are defined by the following multidimensional-matrix system of the linear algebraic equations*

$$\nu_{x^p x^r} + \sum_{k=0}^{r-1} {}^{0,k}(\nu_{x^p x^k} C_{(k,r)}) = 0, \; r = 0,1,..., \; p = 0,1,..., r-1, \quad (16)$$

and the coefficients $C^*_{(k,r)}$ *of the conjugate sequence* $Q_r(x)$ *(2) are defined by the expression*

$$C^*_{(k,r)} = r!\, {}^{0,r}(C_{(k,r)} {}^{0,r}B^{-1}_{(r,r)}).$$

3. THE FOURIER SERIES ON THE ORTHOGONAL POLYNOMIALS

The Fourier series for the p-dimensional-matrix function $y(x)$ of the vector (one-dimensional-matrix) variable $x \in \Omega \subseteq R^n$ on the conjugate orthogonal polynomials $Q_r(x)$ (2) has the following form:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!}\, {}^{0,r}(B_r Q_r(x)), \quad (17)$$

where $B_r = (b_{j_1,j_2,...,j_p,i_1,i_2,...i_r})$ are $(p+r)$-dimensional symmetrical when $r \geq 2$ with respect to the indices $i_1, i_2, ..., i_r$ matrices of the n degree of the coefficients. They are defined by the expressions [8]

$$B_r = \int_\Omega {}^{0,0}\big(y(x)P_r(x)\big)\rho(x)dx. \quad (18)$$

Substitution the polynomial $P_r(x)$ (4) into (18) give the following expression for the coefficients B_r:

$$B_r = \int_\Omega {}^{0,0}\big(y(x)P_r(x)\big)\rho(x)dx = \int_\Omega {}^{0,0}\left(y(x)(x^r + \sum_{k=0}^{r-1} {}^{0,k}(x^k C_{(k,r)}))\right)\rho(x)dx =$$
$$= E\left({}^{0,0}(yx^r) + \sum_{k=0}^{r-1} {}^{0,k}({}^{0,0}(yx^k)C_{(k,r)}) \right) = \nu_{yx^r} + \sum_{k=0}^{r-1} {}^{0,k}(\nu_{yx^k} C_{(k,r)}), \; r = 0,1,2,... \quad (19)$$

The Fourier series on the basic orthogonal polynomials $P_r(x)$ (4) is obtained analogously:

$$y(x) \sim \sum_{r=0}^{\infty} \frac{1}{r!}\, {}^{0,r}(C_r P_r(x)), \quad (20)$$

where

$$C_r = \int_\Omega {}^{0,0}\big(y(x)Q_r(x)\big)\rho(x)dx.$$

Since

$$Q_r(x) = r!\,{}^{0,r}(P_r(x)^{0,r}B^{-1}_{(r,r)}) \ ,$$

then

$$C_r = \int_\Omega {}^{0,0}\big(y(x)Q_r(x)\big)\rho(x)dx = r!\int_\Omega {}^{0,0}\Big(y(x)^{0,r}(P_r(x)^{0,r}B^{-1}_{(r,r)})\Big)\rho(x)dx =$$

$$= r!\int_\Omega {}^{0,r}\Big({}^{0,0}\big(y(x)P_r(x)\big)^{0,r}B^{-1}_{(r,r)}\Big)\rho(x)dx = r!^{0,r}(B_r\,{}^{0,r}B^{-1}_{(r,r)}) \,. \qquad (21)$$

The approximation of the scalar (zero-dimensional-matrix) function $y(\xi)$ of the random vector ξ with the probability density function $\rho(x)$ by the finite sum of the Fourier series

$$s_m(\xi) = \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}(B_r Q_r(\xi)) = \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}(C_r P_r(\xi))$$

provides the minimum of the mean square error (m.s.e.) of the approximation

$$r_m^2 = E\Big({}^{0,0}\big(y(\xi) - s_m(\xi)\big)^2\Big) = \int_\Omega {}^{0,0}\big(y(x) - s_m(x)\big)^2 \rho(x)dx \,.$$

The minimal value $r^2_{m\min}$ of the m.s.e. is defined by the expression [8]

$$r^2_{m\min} = E(y^2(\xi)) - \sum_{r=0}^{m} \frac{1}{r!}{}^{0,r}(B_r C_r) \,,$$

where $E(\cdot)$ means the mathematical expectation.

4. THE POLYNOMIALS ORTHOGONAL WITH THE DISCRETE WEIGHT FUNCTION

The theory of the polynomials orthogonal with the continuous weight function $\rho(x)$ outlined above coincides with the theory of the polynomials orthogonal with the discrete weight function (p_k, x_k), when the l distinct points are given in the region $\Omega \subseteq R^n$ with positive weights $p_1, p_2, ..., p_l$ and the measure μ of the region Ω is define by the formula $\mu(\Omega) = \sum_{x_k \in \Omega} p_k$ [11]. One talks in this case about the polynomials orthogonal on the system of the points. The moments (9) is defined in this case by the expression

$$\nu_{x^i x^j} = \nu_{x^{i+j}} = \int_\Omega x^{i+j} d\mu = \sum_{k=1}^{l} x_k^{i+j} p_k \ , \ i + j = 1,2,... \ ,$$

and the mutual moments (13) is defined by the expression

$$\nu_{y^i x^j} = \int_\Omega y^i(x) x^j d\mu = \sum_{k=1}^{l} y_k^i x_k^j p_k \ , \ i + j = 1,2,... \ ,$$

where $y_k = y(x_k)$, $k = 1,2,...,l$.

We will call the discrete weight function with $\nu_{x^0} = 1$ as the discrete distribution of some random variable ξ. The important discrete distribution is so called empirical, or sample distribution, when x_i are the sample values of the random variable ξ and $p_i = 1/l$, where

l is the length of the sample. If the empirical distribution is used then the approximation is called empirical.

5. THE MULTIDIMENSIONAL-MATRIX APPROXIMATION BY THE FOURIER APPROXIMATION

It is of interest to obtain the coefficients $c_{(m,k)}$ of the approximation of the function $y(x)$ by the multidimensional-matrix m degree polynomial

$$y(x) \sim \sum_{k=0}^{m} {}^{0,k}(c_{(p,k)}x^k) \tag{22}$$

in the case when the Fourier approximation (20) the same degree of this function is obtained

$$y(x) \sim \sum_{k=0}^{m} \frac{1}{k!}\,{}^{0,k}(C_k P_k(x)). \tag{23}$$

The polynomial $P_k(x)$ of the fixed degree k provides in the expression (23) the following summand:

$$\frac{1}{k!}\,{}^{0,k}(C_k P_k(x)) = \frac{1}{k!}\,{}^{0,k}\left(C_k\left(\sum_{i=0}^{k} {}^{0,i}(C_{(k,i)}x^i)\right)\right) = \sum_{i=0}^{k}\frac{1}{k!}\,{}^{0,i}({}^{0,k}(C_k C_{(k,i)})x^i). \tag{24}$$

The variable x of the degree l, $l \le k$, appears in the expression (24) in the summand ${}^{0,l}({}^{0,k}(C_k C_{(k,l)})x^l)/k!$. Summation of the coefficients at x^l by k from l to m gives the following formula for the desired coefficients:

$$c_{(p,l)} = \sum_{k=l}^{m} \frac{1}{k!}\,{}^{0,k}(C_k C_{(k,l)}),\ l = 0,1,2,..,m. \tag{25}$$

If one takes in account that $C_{(i,i)} = E_s(0,i)$ is the symmetrical identity matrix which ensures the equality ${}^{0,i}(C_i C_{(i,i)}) = C_i$ then instead (25) we will have the expression

$$c_{(p,l)} = \frac{1}{l!}C_l + \sum_{k=l+1}^{m} \frac{1}{k!}\,{}^{0,k}(C_k C_{(k,l)}),\ l = 0,1,2,..,m. \tag{26}$$

6. COMPUTER SIMULATION

The algorithm of the approximation of the functions by the Fourier series was realized programmatically in the form of the standard Matlab function for general case and was checked on many functions.

We show the empirical approximation (according the p. 3) of the scalar regression function y of the two arguments x_1, x_2 as the polynomial (22) of the 7 degree ($p = 0$, $q = 1$, $m = 7$). The scalar values of the coefficients $c_{(m,k)}$ of the polynomial are random integer from -5 to 5. The measurement errors are independent normal with zero mean and variation 0.2. The approximating polynomial has the degree 7 too.

We will call the approximation by the algorithm developed in this article as the Fourier multidimensional-matrix approximation (Fmdm-approximation) in opposite to the multidimensional-matrix approximation (mdm-approximation) of the work [12].

Figure 1 shows three surfaces: real function, mdm-approximation and Fmdm-approximation. The design of the experiment is random, the values of the variables x_1, x_2 are choose from the uniform distribution $U(-1,1)$. The number of runs is 255. Both of the methods have the high accuracy of the approximation. However, the program of the Fmdm-approximation turned out to be faster-acting compared to the program of the mdm-approximation. Other benefits are to be found out.

It should be noted that the classical approximation for the considered case is impossible because it is very cumbersome and not developed.

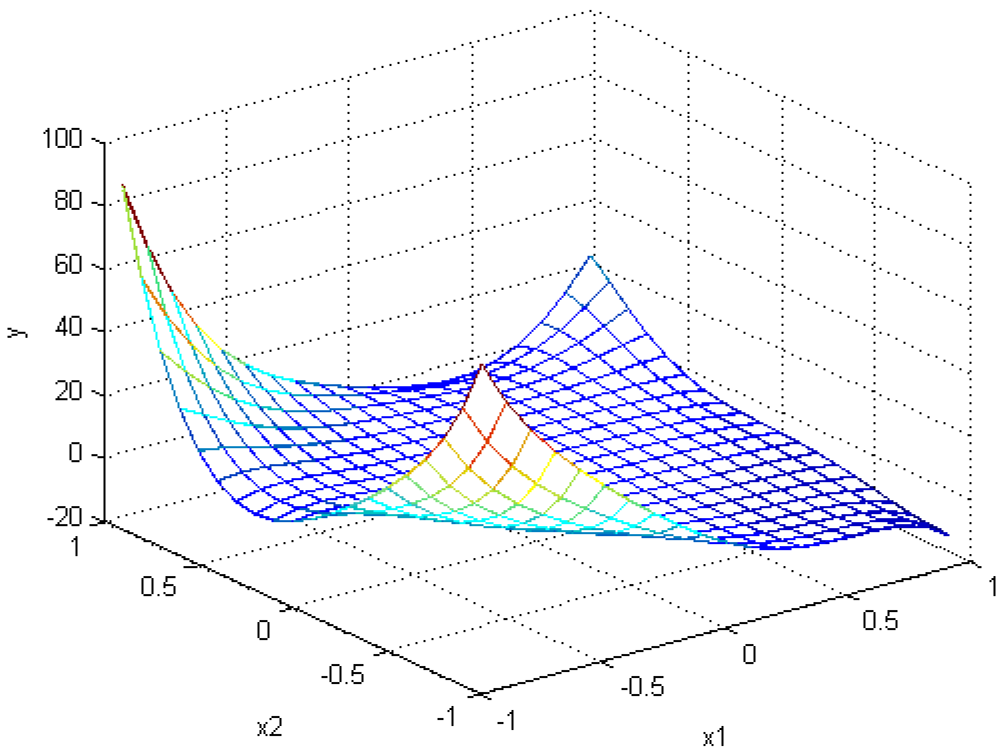

Fig. 1. Real function and its two empirical approximations

The considered approximations have the undoubted advantages compared to the classical approach: algorithmical generality and extensive possibilities. However, they have the certain hardware limitations: out of memory and unacceptably long calculation time for the personal computer in the case of big data.

7. THE ORTHOGONAL 0–2 DEGREES POLYNOMIALS

In the work [8], the expressions of zero and first degree orthonormal polynomials and the particular cases of the second degree polynomials are obtained. These results are completed in this article by the general expressions of the second degree polynomials and Fourier series. The complete expressions are presented in the table 1 for the case $\nu_{x^0} = 1$. The necessary proofs are given in the appendix.

Table 1. Orthogonal polynomials and Fourier series up to second degree inclusive

Polynomials $P(x)$	Polynomials $Q(x)$
$P_0(x) = 1$	$Q_0(x) = 1$
$P_1(x) = C_{(1,0)} + x$, $C_{(1,0)} = -\nu_x$.	$Q_1(x) = {}^{0,1}({}^{0,1}B^{-1}_{(1,1)} P_1(x))$, $B_{(1,1)} = \mu_{xx} = \nu_{xx} - \nu_x \nu_x$.
$P_2(x) = C_{(2,0)} + {}^{0,1}(C_{(2,1)} x) + x^2$, $C_{(2,1)} = -{}^{0,1}(\mu_{x^2 x} \, {}^{0,1}\mu^{-1}_{xx})$, $C_{(2,0)} = -{}^{0,1}(C_{(2,1)} \nu_x) - \nu_{x^2}$, $\mu_{x^2 x} = \nu_{x^3} - \nu_{x^2} \nu_x$.	$Q_2(x) = 2!^{0,2}({}^{0,2}B^{-1}_{(2,2)} P_2(x))$, $B_{(2,2)} = \mu_{x^2 x^2} - {}^{0,1}({}^{0,1}(\mu_{x^2 x} \, {}^{0,1}\mu^{-1}_{xx}) \mu_{xx^2})$, $\mu_{x^2 x^2} = \nu_{x^4} - \nu_{x^2} \nu_{x^2}$, $\mu_{x^2 x} = \nu_{x^3} - \nu_{x^2} \nu_x$.
Fourier series on the polynomials $P(x)$	Fourier series on the polynomials $Q(x)$
$y(x) \sim {}^{0,0}(C_0 P_0(x)) + {}^{0,1}(C_1 P_1(x)) +$ $+ \frac{1}{2} \, {}^{0,2}(C_2 P_2(x))$, $C_0 = {}^{0,0}(B_0 \, {}^{0,0}B^{-1}_{(0,0)}) = \nu_y$, $C_1 = {}^{0,1}(B_1 \, {}^{0,1}B^{-1}_{(1,1)}) = {}^{0,1}(\mu_{yx} \, {}^{0,1}\mu^{-1}_{xx})$, $C_2 = 2^{0,2}(B_2 \, {}^{0,2}B^{-1}_{(2,2)})$.	$y(x) \sim {}^{0,0}(B_0 Q_0(x)) + {}^{0,1}(B_1 Q_1(x)) +$ $+ \frac{1}{2} \, {}^{0,2}(B_2 Q_2(x))$, $B_0 = \nu_y$, $B_1 = \mu_{yx}$, $B_2 = \mu_{yx^2} - {}^{0,1}(\mu_{yx} \, {}^{0,1}({}^{0,1}\mu^{-1}_{xx} \mu_{xx^2}))$.

8. CONCLUSION

The known results of the Fourier series on the orthogonal polynomials are extended to the case of the multidimensional-matrix functions, what allows us to solve new problems such as approximation of parametric curves and surfaces. The analytical expressions for the orthogonal polynomials and Fourier series of the second degree useful for the potential analytical studies are obtained. The theoretical results are realized as the single function of the programming language with many possibilities which we call the algorithmic generality. The efficiency of the program function is confirmed on the instance, performing of which is impossible by the classical approach.

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APPENDIX

A1. Calculation of the polynomials of the small degrees. Let us obtain the orthonormal polynomials of the small degrees by solving the system of the equations (14).

We get the zero degree polynomials by definition:

$$P_0(x) = 1,$$

$$Q_0(x) = 1.$$

The one degree polynomials are obtained when $m = 1$ in the expressions (14). The system of the equations (14) consists of one equation:

$${}^{0,0}(C_{(1,0)}\nu_{x^0}) = -\nu_x .$$

If $\nu_{x^0} = 1$, then

$$C_{(1,0)} = C_{(0,1)} = -\nu_x ,$$

$$P_1(x) = x + {}^{0,0}(C_{(1,0)}x^0) = x - \nu_x .$$

The expression for the matrix $B_{(1,1)}$ from the expression (15) will look like this:

$$B_{(1,1)} = \nu_{xx} + {}^{0,0}(C_{(1,0)}\nu_x) + {}^{0,0}(\nu_x C_{(0,1)}) + {}^{0,0}(C_{(1,0)}(\nu_{x^0} C_{(0,1)})) .$$

Taking into account the expressions for $C_{(0,1)}$ and $C_{(1,0)}$ when $\nu_{x^0} = 1$ we get

$$B_{(1,1)} = \nu_{xx} - \nu_x \nu_x = \mu_{xx} .$$

Then

$$Q_1(x) = {}^{0,1}({}^{0,1}B^{-1}_{(1,1)} P_1(x)) = {}^{0,1}({}^{0,1}\mu^{-1}_{xx}(x - \nu_x)) .$$

The calculation of the first degree polynomials is completed.

The second degree polynomials are obtained when $m = 2$ in the expressions (14). We have now the following system consists of two equations (when $\nu_{x^0} = 1$):

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx} ,$$

$${}^{0,0}(C_{(2,0)}\nu_x) + {}^{0,1}(C_{(2,1)}\nu_{xx}) = -\nu_{x^2 x} .$$

We will solve this system by the Gauss elimination method. For this purpose, we subtract the first equation multiplied on the right by ν_x in the sense of the $(0,0)$ -rolled product from the second equation. We will get the following system:

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx} ,$$

$${}^{0,1}(C_{(2,1)}(\nu_{xx} - \nu_x \nu_x)) = -(\nu_{x^2 x} - \nu_{xx}\nu_x) ,$$

or in other notation

$$C_{(2,0)} + {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx} ,$$

$${}^{0,1}(C_{(2,1)}\mu_{xx}) = -\mu_{x^2 x} .$$

We get the expression for the coefficient $C_{(2,1)}$ from the second equation:

$$C_{(2,1)} = -{}^{0,1}(\mu_{x^2 x} {}^{0,1}\mu^{-1}_{xx}) .$$

Substituting this expression into first equation we get the expression for the coefficient $C_{(2,0)}$:

$$C_{(2,0)} = -\nu_{xx} - {}^{0,1}(C_{(2,1)}\nu_x) = -\nu_{xx} + {}^{0,1}({}^{0,1}(\mu_{x^2 x} {}^{0,1}\mu^{-1}_{xx})\nu_x) .$$

The coefficients $C_{(1,2)}$ and $C_{(0,2)}$ are obtained from the following system of the equations

$$C_{(0,2)} + {}^{0,1}(\nu_x C_{(1,2)}) = -\nu_{xx} ,$$

$${}^{0,0}(\nu_x C_{(0,2)}) + {}^{0,1}(\nu_{xx} C_{(1,2)}) = -\nu_{xx^2} .$$

which follows from (16) when $m = 2$. Solving this system by Gauss elimination method we get

$$C_{(1,2)} = -{}^{0,1}({}^{0,1}\mu^{-1}_{xx}\mu_{xx^2}) ,$$

$$C_{(0,2)} = -\nu_{xx} - {}^{0,1}(\nu_x C_{(1,2)}) = -\nu_{xx} + {}^{0,1}(\nu_x {}^{0,1}({}^{0,1}\mu^{-1}_{xx}\mu_{xx^2})) .$$

The second degree polynomial $P_2(x)$ of the basic sequence of the orthogonal polynomials has the form

$$P_2(x) = x^2 +^{0,1}(C_{(2,1)}x) + C_{(2,0)} = x^2 +^{0,1}(xC_{(1,2)}) + C_{(0,2)}.$$

The second degree polynomial $Q_2(x)$ of the conjugate sequence of the orthogonal polynomials is defined by the formula

$$Q_2(x) = 2!^{0,2}(\,{}^{0,2}B^{-1}_{(2,2)}P_2(x)),$$

where, from (15),

$$B_{(2,2)} = \nu_{x^2x^2} + \sum_{k=0}^{1}{}^{0,k}(C_{(2,k)}\nu_{x^{k+2}}) + \sum_{k=0}^{1}{}^{0,k}(\nu_{x^{2+k}}C_{(k,2)}) + \sum_{k=0}^{1}\sum_{q=0}^{1}{}^{0,k}(C_{(2,k)}\,{}^{0,q}(\nu_{x^{k+q}}C_{(q,2)})).$$

Let us find the matrix $B_{(2,2)}$ for the case $\nu_{x^0} = 1$.

$$B_{(2,2)} = \nu_{x^2x^2} + \underline{\underline{{}^{0,0}(C_{(2,0)}\nu_{xx})}} +^{0,1}(C_{(2,1)}\nu_{xx^2}) + \underline{{}^{0,0}(\nu_{xx}C_{(0,2)})} +^{0,1}(\nu_{x^2x}C_{(1,2)}) +$$
$$+^{0,0}(C_{(2,0)}C_{(0,2)}) + \underline{\underline{{}^{0,0}(C_{(2,0)}\,{}^{0,1}(\nu_xC_{(1,2)}))}} +$$
$$+\underline{{}^{0,0}({}^{0,1}(C_{(2,1)}\nu_x)C_{(0,2)})} +^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_{xx}C_{(1,2)})).$$

Combining the similar terms highlighted in the previous expression we get

$$B_{(2,2)} = \nu_{x^4} +^{0,0}(C_{(2,0)}(\nu_{xx} +^{0,1}(\nu_xC_{(1,2)}))) +^{0,0}((\nu_{xx} +^{0,1}(C_{(2,1)}\nu_x))C_{(0,2)}) +$$
$$+^{0,0}(C_{(2,0)}C_{(0,2)}) +^{0,1}(C_{(2,1)}\nu_{xx^2}) +^{0,1}(\nu_{x^2x}C_{(1,2)}) +^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_{xx}C_{(1,2)})) =$$
$$= e1 + (e2 + e3) + e4 + (e5 + e6) + e7.$$

Taking into account the expressions $C_{(2,0)} = -^{0,1}(C_{(2,1)}\nu_x) - \nu_{xx}$, $C_{(0,2)} = -\nu_{xx} -^{0,1}(\nu_xC_{(1,2)})$ gives

$$e2 = -\underline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\nu_{xx}\right)} - \overline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\,{}^{0,1}(\nu_xC_{(1,2)})\right)} -^{0,0}(\nu_{xx}\nu_{xx}) - \underline{\underline{{}^{0,0}\left(\nu_{xx}\,{}^{0,1}(\nu_xC_{(1,2)})\right)}},$$

$$e3 = -\underline{\underline{{}^{0,0}\left(\nu_{xx}\,{}^{0,1}(\nu_xC_{(1,2)})\right)}} - \overline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\,{}^{0,1}(\nu_xC_{(1,2)})\right)} -^{0,0}(\nu_{xx}\nu_{xx}) - \underline{{}^{0,0}\left({}^{0,1}(C_{(2,1)}\nu_x)\nu_{xx}\right)},$$

$$e4 = \overline{{}^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_x\nu_xC_{(1,2)}))} + \underline{{}^{0,1}(C_{(2,1)}(\nu_x\nu_{xx}))} + \underline{\underline{{}^{0,1}((\nu_{xx}\nu_x)C_{(1,2)})}} +^{0,0}(\nu_{xx}\nu_{xx}),$$

$$e5 = \underline{{}^{0,1}(C_{(2,1)}\nu_{xx^2})},$$

$$e6 = \underline{\underline{{}^{0,1}(\nu_{x^2x}C_{(1,2)})}},$$

$$e7 = \overline{{}^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_{xx}C_{(1,2)}))}.$$

Summation of the terms $e1 - e7$ and combining the highlighted similar terms leads to the expression

$$B_{(2,2)} = \nu_{x^2x^2} + \underline{{}^{0,1}(C_{(2,1)}\nu_{xx^2})} - \underline{{}^{0,1}(C_{(2,1)}(\nu_x\nu_{xx}))} + \overline{{}^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_{xx}C_{(1,2)}))} -$$
$$-\overline{{}^{0,1}(C_{(2,1)}\,{}^{0,1}(\nu_x\nu_xC_{(1,2)}))} + \underline{\underline{{}^{0,1}(\nu_{x^2x}C_{(1,2)})}} - \underline{\underline{{}^{0,1}((\nu_{xx}\nu_x)C_{(1,2)})}} -^{0,0}(\nu_{xx}\nu_{xx}),$$

or

$$B_{(2,2)} = \nu_{x^2x^2} -^{0,0}(\nu_{xx}\nu_{xx}) +^{0,1}(C_{(2,1)}(\nu_{xx^2} - \nu_x\nu_{xx})) +^{0,1}((\nu_{x^2x} - \nu_{xx}\nu_x)C_{(1,2)}) +$$
$$+^{0,1}(C_{(2,1)}\,{}^{0,1}((\nu_{xx} - \nu_x\nu_x)C_{(1,2)})),$$

or

$$B_{(2,2)} = \mu_{x^2x^2} +^{0,1}(C_{(2,1)}\mu_{xx^2}) +^{0,1}(\mu_{x^2x}C_{(1,2)}) +^{0,1}(C_{(2,1)}\,{}^{0,1}(\mu_{xx}C_{(1,2)})).$$

Taking into account the expressions $C_{(2,1)} = -^{0,1}(\mu_{x^2x}\,{}^{0,1}\mu^{-1}_{xx})$, $C_{(1,2)} = -^{0,1}(\,{}^{0,1}\mu^{-1}_{xx}\,\mu_{xx^2})$ we get

$$B_{(2,2)} = \mu_{x^2x^2} -^{0,1}({}^{0,1}(\mu_{x^2x}\,{}^{0,1}\mu^{-1}_{xx})\mu_{xx^2}) -^{0,1}(\mu_{x^2x}\,{}^{0,1}(\,{}^{0,1}\mu^{-1}_{xx}\,\mu_{xx^2})) +$$
$$+^{0,1}(\mu_{x^2x}\,{}^{0,1}({}^{0,1}\mu^{-1}_{xx}\,\mu_{xx^2})),$$

or finally

$$B_{(2,2)} = \mu_{x^2x^2} -^{0,1}({}^{0,1}(\mu_{x^2x}\,{}^{0,1}\mu^{-1}_{xx})\mu_{xx^2}).$$

A2. Calculation of the Fourier series of the small degrees. The Fourier series (17) with three terms on the conjugate polynomials $Q_r(x)$ for the scalar function of the vector variable $y(x)$ has the form

$$y(x) \sim {}^{0,0}(B_0 Q_0(x)) + {}^{0,1}(B_1 Q_1(x)) + \frac{1}{2}\, {}^{0,2}(B_2 Q_2(x)) .$$

Let us find the coefficients B_i of this series by the formula (19).

$$B_0 = \int_\Omega {}^{0,0}(y(x) P_0(x)) \rho(x) dx = \int_\Omega y(x) \rho(x) dx = \nu_y ,$$

$$B_1 = \int_\Omega {}^{0,0}(y(x) P_1(x)) \rho(x) dx = \int_\Omega {}^{0,0}(y(x)(x - \nu_x)) \rho(x) dx = \mu_{yx} ,$$

$$B_2 = \int_\Omega {}^{0,0}(y(x) P_2(x)) \rho(x) dx = \int_\Omega {}^{0,0}(y(x)(x^2 + {}^{0,1}(x C_{(1,2)}) + C_{(0,2)})) \rho(x) dx =$$

$$= \nu_{yx^2} + {}^{0,1}(\nu_{yx} C_{(1,2)}) + \nu_y C_{(0,2)} =$$

$$= \nu_{yx^2} - {}^{0,1}(\nu_{yx}\, {}^{0,1}({}^{0,1}\mu_{xx}^{-1} \mu_{xx^2})) - {}^{0,1}(\nu_y \nu_x C_{(1,2)}) - \nu_y \nu_{xx} =$$

$$= \nu_{yx^2} - {}^{0,1}(\nu_{yx}\, {}^{0,1}({}^{0,1}\mu_{xx}^{-1} \mu_{xx^2})) + {}^{0,1}(\nu_y \nu_x (\, {}^{0,1}({}^{0,1}\mu_{xx}^{-1} \mu_{xx^2}))) - \nu_y \nu_{xx} =$$

$$= \mu_{yx^2} - {}^{0,1}(\mu_{yx}\, {}^{0,1}({}^{0,1}\mu_{xx}^{-1} \mu_{xx^2})) .$$

The Fourier series (17) with three terms on the basic polynomials $P_r(x)$ for the scalar function of the vector variable $y(x)$ has the form

$$y(x) \sim {}^{0,0}(C_0 P_0(x)) + {}^{0,1}(C_1 P_1(x)) + \frac{1}{2}\, {}^{0,2}(C_2 P_2(x)) .$$

We get in accordance with the formula $C_r = r!^{0,r}(B_r\, {}^{0,r} B_{(r,r)}^{-1})$ (21):

$$C_0 = {}^{0,0}(B_0\, {}^{0,0} B_{(0,0)}^{-1}) = \nu_y ,$$

$$C_1 = {}^{0,1}(B_1\, {}^{0,1} B_{(1,1)}^{-1}) = {}^{0,1}(\mu_{yx}\, {}^{0,1} \mu_{xx}^{-1}) ,$$

$$C_2 = 2^{0,2}(B_2\, {}^{0,2} B_{(2,2)}^{-1}) .$$

The coefficients of the approximation of the function $y(x)$ by the series (22) on the degrees of the variable x up to second degree inclusive ($m = 2$) are defined by the following expressions defined by the formula (26):

$$c_{(2,0)} = \frac{1}{0!} C_0 + \sum_{k=1}^{2} \frac{1}{k!}\, {}^{0,k}(C_k C_{(k,0)}) = C_0 + {}^{0,1}(C_1 C_{(1,0)}) + \frac{1}{2!}\, {}^{0,2}(C_2 C_{(2,0)}) ,$$

$$c_{(2,1)} = \frac{1}{1!} C_1 + \sum_{k=2}^{2} \frac{1}{k!}\, {}^{0,k}(C_k C_{(k,1)}) = C_1 + \frac{1}{2!}\, {}^{0,2}(C_2 C_{(2,1)}) ,$$

$$c_{(2,2)} = \frac{1}{2!} C_2 .$$