On Suborbits and Graphs Associated with Action of Alternating Groups on Cartesian Product of Two Sets

Abstract

In this paper, the suborbits and graphs associated with the action of direct product of two Alternating groups on the Cartesian product of two sets are studied. It is shown that the suborbits are self-paired and the associated graphs are undirected and regular with girth 3.

Keywords: Alternating Group, Suborbit, Suborbital graph, Self-paired, Connected component, Girth, Undirected, Direct Product, Cartesian Product.

1 Introduction

The idea of suborbital graphs corresponding to non-trivial suborbits of a group acting on a set X was first investigated by Sims in 1967 on graphs and finite permutation groups, see [18]. He defined a suborbital graph $\mathcal{G}_i$ corresponding to the suborbital $O_i \subseteq X \times X$ as a graph whose vertex set is X and edge set E consists of directed edges from x to y where $(x, y) \in O_i$. Since then, there has been an intensive study on these graphs by several researchers including; [9], [10], [8], [11],[12], [17],[1], [2],[16],[13],[15],[4],[5],[19] , among others.

Suborbital graphs and their properties for ordered triples in $A_n,\ (n = 5, 6, 7)$ through rank and subdegree determination were investigated by [14]. It was shown that if A_n $(n \geq 5)$ acts on the ordered pairs, the suborbital graphs corresponding to the non-trivial suborbits are connected. Further, it was proved that the suborbital graphs $\mathcal{G}_i$ corresponding to the suborbits $\triangle_i,\ i = 1, 2, 5, 6, 7, 14, 15, 22, 23, 24, 28, 32, 33, 34$ are un directed (since the suborbits are self-paired) and the graphs $\mathcal{G}_{jk}$ where $j = 3, 8, 9, 10, 11, 16, 17, 25, 26, 29$ and $k = 4, 12, 13, 19, 18, 20, 21, 27, 30, 31$ are directed for each j and k (since the suborbits $\triangle_j$ and $\triangle_k$ are respectively paired).

Gikunju [6] constructed and investigated the suborbital graphs corresponding to the action of direct products of symmetric groups S_n on a Cartesian product of three sets. It was showed that the suborbital graphs $\mathcal{G}_i,\ i = 1, 2, \cdots, n$ corresponding to the non-trivial suborbits $O_i,\ i = 1, 2, \cdots, 6$ are disconnected but $\mathcal{G}_7$ is connected each with girth 3 for all $n > 2$. It was further showed that the suborbital graphs are undirected and the graphs $\mathcal{G}_i,\ i = 1, 2, \cdots, 7$ are regular with the respective degrees $(n-1), (n-1), (n-1), (n-1)^2, (n-1)^2, (n-1)^3,$ and $(n-1)^3$ for all $n > 2$.

This paper investigates the properties of suborbital graphs associated with the action of direct products of Alternating groups A_n on Cartesian products of two sets.

1.1 Definitions and Preliminary results

Definition 1.1. [Product Action][3, p.3] Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the the Cartesian product $X_1 \times X_2$ by the rule

$$(g_1, g_2)(x_1, x_2) = (g_1x_1, g_2x_2) \;\forall\; g_1 \in G_1, g_2 \in G_2 \text{ and } x_1 \in X_1,\; x_2 \in X_2.$$

Remark 1.1. Through out this paper, the group action defined is in a similar way as in Definition 1.1 as

$$(g_1, g_2,)(x_1, x_2) = (g_1x_1, g_2x_2) \;\forall\; g_1, g_2 \in G \text{ and } x_i \in X_i$$

where $G = A_n \times A_n$ and, $X_1 = \{1, 2, \cdots, n\},\; X_2 = \{n + 1, n + 2, \cdots, 2n\}$.

Definition 1.2. Let $\triangle$ be an orbit of G_x on X. Define $\triangle^* = \{gx : \; g \in G,\; x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x and is called the G_x-orbit paired with $\triangle$. Wielandt [20] proved that if $\triangle^* = \triangle$, then $\triangle$ is called a self-paired orbit of G_x.

Definition 1.3. Suppose G is a group acting transitively on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \cdots \triangle_{k-1}$ of G_x on X are known as suborbits of G. The rank of G in this case is k. The sizes $n_i = |\triangle_i|$ $(i = 0, 1, 2, \cdots, k - 1)$ are known as the subdegrees of G. It was proved by [7] that the rank and subdegrees of the suborbits $\triangle_i$ $(i = 0, 1, 2, \cdots, k - 1)$ are independent of the choices of $x \in X$.

Theorem 1.1. [18] Let G be transitive on X and let suborbit $\triangle_i$ $(i = 1, 2, \cdots, k - 1)$ correspond to suborbital O_i. Then the corresponding suborbital graph $\mathcal{G}_i$ is

1. directed if $\triangle_i$ is self-paired and undirected if $\triangle_i$ is not self-paired
2. connected if and only if G is primitive.

2 Main Results

We recall that for $n \geq 3$, the action of $A_n \times A_n$ on $X_1 \times X_2$ is transitive and imprimitive with 3^2 suborbitals for $n = 3$ and 2^2 suborbitals for $n \geq 4$.

2.1 Suborbital graphs of $G = A_3 \times A_3$ acting on the $X_1 \times X_2$

We notice that the suborbit $\triangle_0$ has only one element (coordinate), and therefore clearly self-paired and the graph corresponding to this suborbit is a null graph with no interesting properties to study.

Lemma 2.1. The suborbits $\triangle_1, \triangle_2, \cdots,$ and $\triangle_8$ of G are self-paired.

Proof. By Definition 1.2, consider for example $\triangle_2 = \{(1,6)\}$ and let $g_1, g_2 \in G$. Then $(g_1, g_2)(1,6) = (1,4)$ implies $g_1(1) = 1$ and $g_2(6) = 4$. Thus $(g_1, g_2) = ((1), (6\ 4)$. So, $(g_1, g_2)(1,4) = (1,6) \in \triangle_3$. Hence $\triangle_3^* = \triangle_3$ i.e. self-paired. Similarly, $\triangle_i : \ i = 1, 3, 4, \cdots, 8$ are self-paired □

Corollary 2.1. The suborbital graphs $\mathcal{G}_i, i = 0, 1, 2, \cdots, 8$ corresponding to suborbits $\triangle_i$ are undirected.

Proof. Since the suborbits, $\triangle_i$ are self-paired, then by Theorem 1.1, we are done. □

The suborbital graphs $\mathcal{G}_i : \ i = 1, 2, \cdots, 8$ corresponding to the non-trivial suborbits $\triangle_i$ in the following way.

Let A and B be distinct points in $X_1 \times X_2$. Then the suborbital O_1 corresponding to the suborbit $\triangle_1$ is given as;

$$O_1 = \{(g_1, g_2)(1, 4), (g_1, g_2)(1, 5) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_1$ from A to B if the first coordinates in A and B are identical but the second coordinates are different.

Remark 1. The graph $\mathcal{G}_2$ corresponding to $\triangle_2$ is same as $\mathcal{G}_1$ as they both have have edges between points with similar conditions.

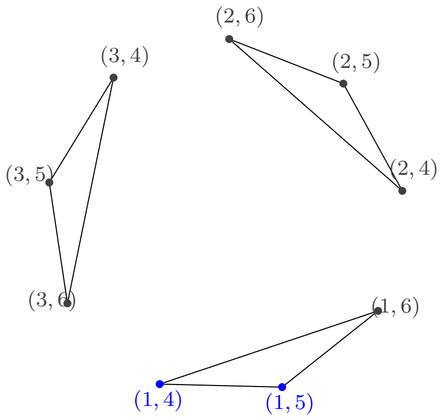

Figure 1: Suborbital graph $\mathcal{G}_1$ corresponding to suborbit $\triangle_1$ of $A_3 \times A_3$ acting on $X_1 \times X_2$

The suborbital graphs corresponding to suborbits $\triangle_3$, and, $\triangle_4$ are same as they both have edges between points with similar conditions. For example, suborbital O_4

corresponding to the suborbit $\triangle_4$ is given as;

$$O_4 = \{(g_1, g_2)(1, 4), (g_1, g_2)(3, 4) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_4$ from A to B if the first coordinates in A and B are different but the second coordinates are identical.

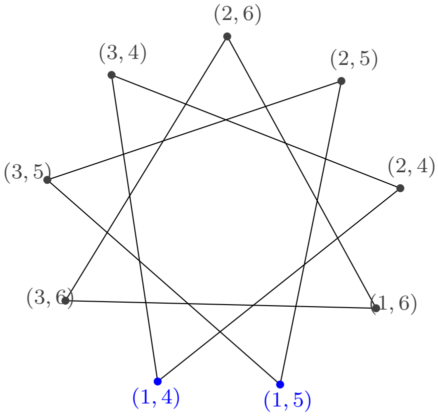

Figure 2: Suborbital graph $\mathcal{G}_4$ corresponding to suborbit $\triangle_4$ of $A_3 \times A_3$ acting on $X_1 \times X_2$

The suborbital graphs corresponding to suborbits $\triangle_i$: $i = 5, 6, 7, 8$ are same as they all have edges between points with similar conditions. For example, suborbital O_8 corresponding to the suborbit $\triangle_8$ is given as;

$$O_7 = \{(g_1, g_2)(1, 4), (g_1, g_2)(3, 6) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_4$ from A to B if the first and second coordinates in A are different from the first and second coordinates in B.

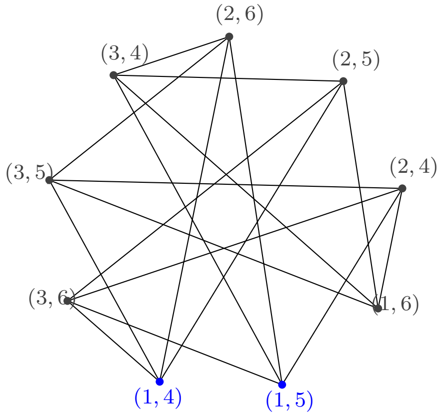

Figure 3: Suborbital graph $\mathcal{G}_8$ corresponding to suborbit $\triangle_8$ of $A_3 \times A_3$ acting on $X_1 \times X_2$

2.2 Suborbital graphs of $G = A_4 \times A_4$ acting on the $X_1 \times X_2$

For the four suborbits of this action, self-pairedness is checked for each of the suborbit and the corresponding suborbital graphs $\mathcal{G}_i, \ i = 0, 1, 2, 3$ constructed.

Since the suborbit $\triangle_0$ has only one element (coordinate), then it is clearly self-paired and the graph corresponding to this suborbit is a null graph with no interesting properties to study.

Lemma 2.2. The remaining suborbits $\triangle_1, \triangle_2,$ and $\triangle_3$ of G are self-paired.

Proof. By Definition 1.2, consider for example $\triangle_1 = \{(1,6),(1,7),(1,8)\}$ and let $g_1, g_2 \in G$. Then $(g_1, g_2)(1,6) = (1,5)$ implies $g_1(1) = 1$ and $g_2(6) = 5$. Thus $(g_1, g_2) = ((1),(6\ 5)$. So, $(g_1, g_2)(1,5) = (1,6) \in \triangle_1$. Hence $\triangle_1^* = \triangle_1$ i.e. self-paired. Similarly, $\triangle_i : \ i = 2, 3$

are self-paired ☐

Corollary 2.2. The suborbital graphs $\mathcal{G}_i, i = 0, 1, 2, 3$ of corresponding to suborbits $\triangle_i$ are undirected.

Proof. By Lemma 2.2, suborbits $\triangle_i$ are self-paired. Now by Theorem 1.1, we are done. ☐

The suborbital graphs $\mathcal{G}_i : \; i = 1, 2, 3$ corresponding to the non-trivial suborbits $\triangle_i$ in the following way.

Let A and B be distinct points in $X_1 \times X_2$. Then the suborbital O_1 corresponding to the suborbit $\triangle_1$ is given as;

$$O_1 = \{(g_1, g_2)(1, 5), (g_1, g_2)(1, 6) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_1$ from A to B if the first coordinates in A and B are identical but the second coordinates are different.

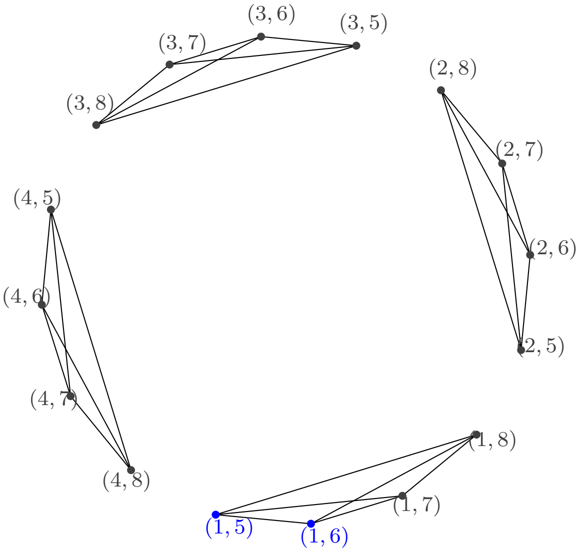

Figure 4: Suborbital graph $\mathcal{G}_1$ corresponding to suborbit $\triangle_1$ of $A_4 \times A_4$ acting on $X_1 \times X_2$

The suborbital O_2 corresponding to the suborbit $\triangle_2$ is given as;

$$O_2 = \{(g_1, g_2)(1, 5), (g_1, g_2)(2, 5) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_2$ from A to B if the first coordinates in A and B are different but the second coordinates are the same.

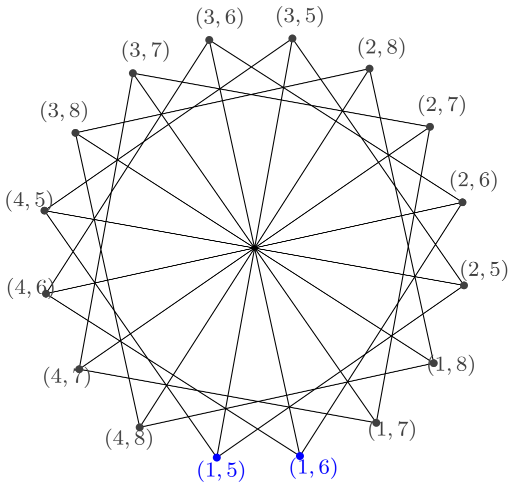

Figure 5: Suborbital graph $\mathcal{G}_2$ corresponding to suborbit $\triangle_2$ of $A_4 \times A_4$ acting on $X_1 \times X_2$

Suborbital O_3 corresponding to the suborbit $\triangle_2$ is given as;

$$O_3 = \{(g_1, g_2)(1, 5), (g_1, g_2)(2, 6) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_3$ from A to B if the first and second coordinates in A are different from the first and second coordinates in B.

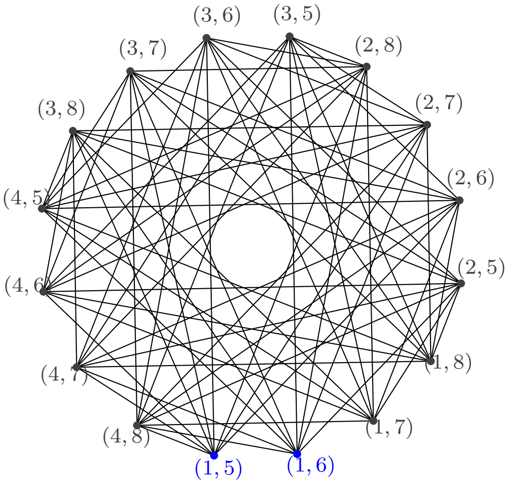

Figure 6: Suborbital graph $\mathcal{G}_3$ corresponding to suborbit $\triangle_3$ of $A_4 \times A_4$ acting on $X_1 \times X_2$

2.3 Suborbital graphs of $G = A_5 \times A_5$ acting on the $X_1 \times X_2$

Since there are four suborbits for this action, we again check self-pairedness for each of the suborbit and also construct the corresponding suborbital graphs $\mathcal{G}_i,\ i = 0, 1, 2, 3$.

The suborbit $\triangle_0$ has only one element (coordinate), and therefore clearly self-paired and the graph corresponding to this suborbit is a null graph with no interesting properties to study.

Lemma 2.3. The suborbits $\triangle_1, \triangle_2,$ and $\triangle_3$ of G are self-paired.

Proof. By Definition 1.2, consider for example $\triangle_2 = \{(2, 6), (3, 6), (4, 6), (5, 6)\}$ and let $g_1, g_2 \in G$. Then $(g_1, g_2)(2, 6) = (1, 6)$ implies $g_1(2) = 1$ and $g_2(6) = 6$. Thus $(g_1, g_2) =$

$((2\ 1),(6)$. So, $(g_1,g_2)(1,6)=(2,6)\in\triangle_2$. Hence $\triangle_2^*=\triangle_1$ i.e. self-paired. Similarly, $\triangle_1$ and $\triangle_3$ are self-paired. □

Corollary 2.3. The suborbital graphs $\mathcal{G}_i, i=1,2,3$ of corresponding to suborbits $\triangle_i$ are undirected.

Proof. By Lemma 2.3, $\triangle_i$ are self-paired. So by Theorem 1.1, the proof is complete. □

The suborbital graphs $\mathcal{G}_i:\ i=1,2,3$ corresponding to the non-trivial suborbits $\triangle_i$ in the following way.

Let A and B be distinct points in $X_1\times X_2$. Then the suborbital O_1 corresponding to the suborbit $\triangle_1$ is given as;

$$O_1=\{(g_1,g_2)(1,6),(g_1,g_2)(1,7)|(g_1,g_2)\in G\}.$$

Thus there exists an edge in $\mathcal{G}_1$ from A to B if the first coordinates in A and B are identical but the second coordinates are different.

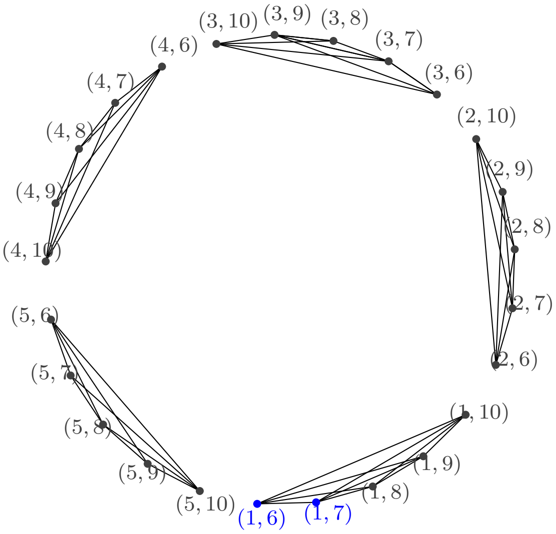

Figure 7: Suborbital graph $\mathcal{G}_1$ corresponding to suborbit $\triangle_1$ of $A_5 \times A_5$ acting on $X_1 \times X_2$

The suborbital O_2 corresponding to the suborbit $\triangle_2$ is given as;

$$O_2 = \{(g_1, g_2)(1, 6), (g_1, g_2)(2, 6) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_2$ from A to B if the first coordinates in A and B are different but the second coordinates are the same.

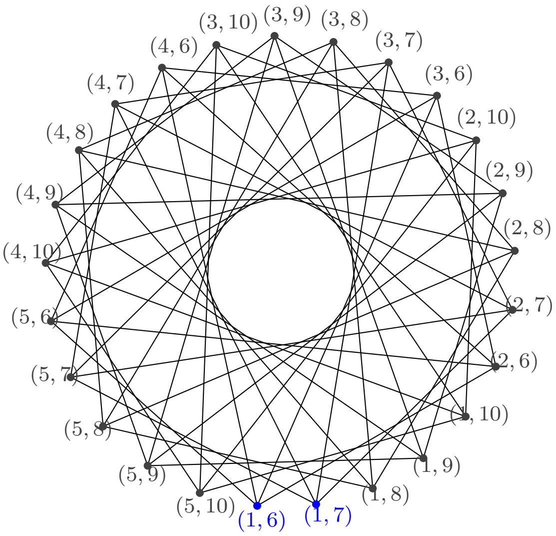

Figure 8: Suborbital graph $\mathcal{G}_2$ corresponding to suborbit $\triangle_2$ of $A_5 \times A_5$ acting on $X_1 \times X_2$

Suborbital O_3 corresponding to the suborbit $\triangle_2$ is given as;

$$O_3 = \{(g_1, g_2)(1, 6), (g_1, g_2)(2, 7) | (g_1, g_2) \in G\}.$$

Thus there exists an edge in $\mathcal{G}_3$ from A to B if the first and second coordinates in A are different from the first and second coordinates in B.

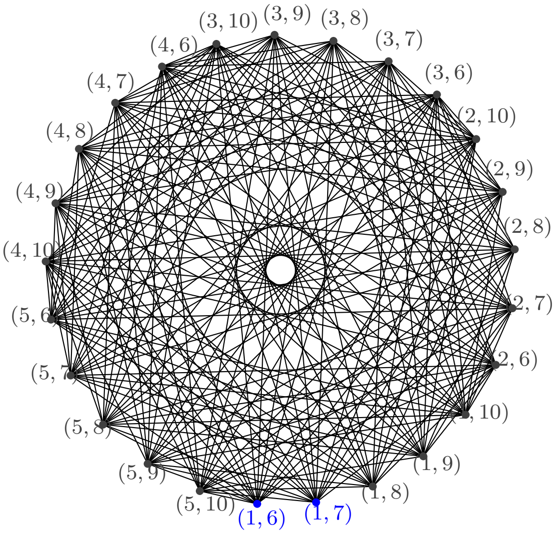

Figure 9: Suborbital graph $\mathcal{G}_3$ corresponding to suborbit $\triangle_3$ of $A_5 \times A_5$ acting on $X_1 \times X_2$

2.4 Suborbital graphs of $G = A_n \times A_n$ acting on the $X_1 \times X_2$

In this Subsection, we generalize results from Subsections 2.1,2.2, and 2.3 as follows;

Lemma 2.4. For $n > 3$, the suborbits $\triangle_0, \triangle_1, \triangle_2, \triangle_3$ of G are self-paired.

Proof. By Definition 1.2, the trivial suborbit $\triangle_0$ is clearly self-paired since it has only one element. Now consider $\triangle_1 = \{(1, n+2), (1, n+3), \cdots, (1, 2n)\}$ and let $g_1, g_2 \in G$. Then we have $(g_1, g_2)\{(1, n+2), (1, n+3), \cdots, (1, 2n)\}$. Since the first coordinates are constant, we consider; $g_2\{n+2, n+3, \cdots, 2n\} = \{1, 2, \cdots, n-1\}$ implying $g_2 = \{(n+2\ 1), (n+3\ 2), \cdots, (2n, n-1)\}$. Thus $g_2\{1, 2, \cdots, n-1\} = \{n+2, n+3, \cdots, 2n\} \in \triangle_1$. Hence $\triangle_1^* = \triangle_1$ i.e. self-paired.

For $\triangle_2 = \{(2, n{+}1), (3, n{+}1), \cdots, (n, n{+}1)\}$ if $(g_1, g_2)\{(2, n{+}1), (3, n{+}1), \cdots, (n, n{+}1)\}$, then since the second coordinate is constant, we have $g_1\{2, 3, \cdots, n\} = \{1, 2, \cdots, n-1\}$ considered. This implies that $g_1 = \{(2\ 1)(3\ 2), \cdots, (n\ n-1)\}$ and consequently $g_1\{1, 2, \cdots, n-1\} = \{2, 3, \cdots, n\} \in \triangle_2$ i.e., $\triangle_2$ is also self-paired.

Lastly, for $\triangle_3 = \{(2, n{+}2), (3, n{+}2), \cdots, (n, n{+}2), (2, n{+}3), (2, n{+}4), \cdots, (2, 2n), \cdots, (n, 2n)\}$. If $(g_1, g_2)\{(2, n+2), (3, n+2), \cdots, (n, 2n)\}$, we consider $g_1\{2, 3, \cdots, n\}$ $g_2\{n+2, n+3, \cdots, 2n\} = \{1, 2, \cdots, n-1\}$. But g_1 and g_2 are already obtained and moreover, $(g_1, g_2)\{1, 2, \cdots, n-1\} = \{(2, n+2), (3, n+2), \cdots, (n, n+2), (2, n+3), (2, n+4), \cdots, (2, 2n), \cdots, (n, 2n)\} \in \triangle_3$. Hence $\triangle_3$ is self-paired. □

Corollary 2.4. For $n > 3$, the suborbital graphs $\mathcal{G}_i, i = 0, 1, 2, 3$ corresponding to suborbits $\triangle_i$ for the action $A_n \times A_n$ on $X_1 \times X_2$ are undirected.

Proof. Since by Lemma 2.4 the suborbits are self-paired, then by Theorem 1.1, the proof is complete. □

3 Conclusion

The graph $\mathcal{G}_0$ corresponding to trivial suborbit, $\triangle_0$ is a null graph for all $n \geq 3$ with no properties to explore.

For $n = 3$, the graphs $\mathcal{G}_1,$ and $\mathcal{G}_2$ are same and $\mathcal{G}_3,$ and $\mathcal{G}_4$ are the same too with properties; regular of degree $n-1$, girth 3 and disconnected with $n-1$ connected components. Also, graphs $\mathcal{G}_k:\ k = 5, 6, \cdots, 8$ are also same and are regular of degree $n+1$ with girth 3.

For $n > 3$, there is agreement of the subdegrees of the action of $A_n \times A_n$ on $X_1 \times X_2$ with regularity of the graphs: $\mathcal{G}_1,$ and $\mathcal{G}_2$ are regular with degree $n-1$, and $\mathcal{G}_3$ is regular of degree $(n-1)^2$ with all graphs having a girth of 3.

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