***Original Research Article***

# Boole-Shannon Method for Pseudo-BooleanFunctions


***Abstract:***

This papershowsthe significant simplicitiesthatcan be realized for network analysis when the edge capacity is big enough to carry a largenumber of flows.We used a network decomposition method, which substantially simplifies network analysis.The pseudo-Boolean functions are repeatedly subjected to Boole-Shannon expansion about appropriate keystone variables until subexpressions with statistically independent terms are found.This method focuses on conditioning a complex network based on the possible states of a keystone element or the possible combinations of states of many keystone elements. Two demonstration examples with ample details are provided to demonstrate the proposed method's applicability.



***Keywords:*** Pseudo-Boolean function; Probability-ready expression; Capacitated networks; System reliability.



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## 1. INTRODUCTION

Many novel approaches for studying a capacitated network have been developed over the last decade. [4-9]. A broad class of these approaches is based on the repeated application of Bayesian decomposition to the network graph [1, 10], or equivalently, of Boole-Shannon expansion to the switching (Boolean) function of network success or network failure[11,12].The Boole-Shannon expansion methods yield probability-ready expressions (PREs) with statistically independent ANDed subexpressions and disjoint ORed subexpressions, whereas the Bayesian decomposition methods reduce the complicated graph of the system to simple series-parallel subgraphs [2].This paper demonstrates a method from the aforementioned category of methods. This method, like the ones in [11, 12], applies the Boole-Shannon expansion to the (switching) Boolean expression and looks for sub-expressions with statistically independent terms.

This method, however, varies from the others in that it works with pseudo-Boolean functions rather than switching or Boolean functions.An efficient strategy for transforming a Boolean formula into a PRE-form is to (recurrently) use the Boole-Shannon Expansion [2], which has the following form when implemented with respect to a single variable $X_k$

$$f(\boldsymbol{X}) = (\bar{X}_k \wedge f(\boldsymbol{X}|0_k)) \vee (X_k \wedge f(\mathbf{X}|1_k)). \quad (1)$$

The Boolean function $f(\boldsymbol{X})$ is expressed in terms of its two subfunctions $f(\boldsymbol{X}|0_k)$ and $f(\boldsymbol{X}|1_k)$ in this Boole-Shannon Expansion.These subfunctions are equal to the Boolean quotients $f(\boldsymbol{X})/\bar{X}_k$ and $f(\boldsymbol{X})/X_k$, respectively, and can be produced by setting $X_k$ in the expression $f(\boldsymbol{X})$ to 0 and 1.The two subfunctions $f(\boldsymbol{X}|0_k)$ and $f(\boldsymbol{X}|1_k)$ are functions of at most $(n-1)$ variables If $f(\boldsymbol{X})$ is a sum-of-products (sop) expression of $n$ variables. A formal proof of (1) is obtained by considering only two exhaustive cases: (a) the case $\{X_k = 0\}$, for which the L.H.S. = the R.H.S = $f(X|O_k)$ ; and (b) the case $\{X_k = 1\}$, for which the L.H.S. = the R.H.S. = $f(X|1_k)$.The Boole-Shannon Expansion (1) accomplishes the objectives of having a PRE-formvery well. PRE-expressions will be used to express the subfunctions in (1), and $f(X)$ will be in PRE-form as well, Because (a) The R.H.S. of (1) has two disjoint parts, the first of which contains the complemented literal $\overline{X}_k$ and the second of which contains the un-complemented literal $X_k$; and (b) Each of these two parts is theresult of two statistically independent entities.

The Total Probability Theorem [3] in the probability domain and the Factoring Theorem in the "Graph Domain" are both equivalent to the Boole–Shannon Expansion in the Boolean domain.The Variable-Entered Karnaugh Map (VEKM) serves as a powerful visual assist or tool for implementing this expansion [5-9].The expansion tree is a complete binary tree (commonly called a Binary Decision Diagram) with $2^n$ leaves when the Boole-Shannon expansion is applied in sequence to the $n$ variables of $f(\boldsymbol{X})$. These leaves are functions with no variables or constants, and they correspond to the entries of a typical K-Map of $f(\boldsymbol{X})$. The entries of a variable-entered (or map-entered) K-map of $f(\boldsymbol{X})$ are Sibling nodes (nodes at the same level) of this expansion tree [4].

The remainder of this paper is organized in the following manner.The fundamental assumptions for our model, as well as the notation and some relevant nomenclature, are presented in Section 2.The nomenclature sectiondiscusses the arithmetic and Boolean representations used to describe a pseudo-Boolean function, also known as a pseudo-switching function. This pseudo-switching function represents the network's general capacity function as well as the algebraic decomposition formula, which is the most fundamental theorem for a pseudo-switching function that is used to get the network's general capacity function and its mean.To clearlyillustratethe recommended method, Section 3 of the paperpresentstwo didactic examples. The conclusion of the paper is summarized in Section 4.

## 2. ASSUMPTIONS, NOTATION AND NOMENCLATURE:

### 2.1. Assumptions

(1) The linear graph of our network consists of nodes with unrestricted capacity and perfect reliability, as well as transmission branches that are limited in capacity and not completely reliable

(2) Each branch in the network can be in one of two states: successful state or failed state. The success of branch is statistically independent.

(3) Each branch $(i\ ,j)$is given a certain value for its reliability $p_{ij}$ and capacity$c_{ij}$, where $0 \leq p_{ij} \leq 1$ , $c_{ij} \geq 0$.The maximum amount of flow that can pass through a branch in either direction is determined by the branch capacity.

(4) Each branch in the network has a specific direction. If a branch is bidirectional, it is split into two directed branches that are opposite to each other, and their failures are completely dependent on each other. These two branches have the same level of reliability, but their capacities may differ.

### 2.2. Notation

$n$The total number of links in the network.

$X_i, \overline{X}_i$ These variables indicate whether or not link $i$ has operated successfully or failed. They are binary random variables that can only take the values of 0 or 1, representing failure or success, respectively.$X_i = 1$ and $\overline{X}_i = 0$ if $i$ is successful, and $X_i = 0$ and $\overline{X}_i = 1$ if $i$ is unsuccessful. The anti-parallel successes for a bidirectional branch $ij$ are the same $X_{ij} = X_{ji}$.

$S\ ,\overline{S}$These Variables are used as indicators for whether the network has succeeded or failed. Success can mean either connectivity or meeting a flow requirement.

$p_i, q_i$ Reliability and unreliability of link $i$: $p_i \equiv \Pr\{X_i = 1\}, q_i \equiv \Pr\{\overline{X}_i = 1\} = 1 - p_i$ .

$R, U$ Network reliability and unreliability; $R = \Pr\{S = 1\} = E\{S\}$ , $U = \Pr\{\overline{S} = 1\} = 1.0 - R$, $0.0 \leq R$ , $U \leq 1.0$ .

$c_i$ Flow capacity of link $i$; $c_i \geq 0$ .

$\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{c}$ n-dimensional vectors of branch successes, reliabilities and capacities:

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$\boldsymbol{X} = (X_1 X_2 \dots X_n)^T; \boldsymbol{p} = (p_1 p_2 \dots p_n)^T ; \boldsymbol{c} \equiv (c_1 c_2 \dots c_n)^T$.

$C_{ij}(\boldsymbol{X})$Capacity function that determines the highest flow that can be transmitted between nodes $i$ and $j$ in state $\boldsymbol{X}$, without breaching the link capacities, $C_{ij}(\boldsymbol{X}) \geq 0$. For a link $(i, j)$ : $C_{ij} = c_{ij} X_{ij}$.

$s, t$refer to the source node and the terminal node, respectively.

$C_{ij}(\boldsymbol{X}|1_l), C_{ij}(\boldsymbol{X}|0_l)$ The function $C_{ij}(\boldsymbol{X})$when $X_l$is set to 1 or 0. Meanings of $C_{ij}(\boldsymbol{X}|1_l, 1_m), etc.$ follow similarity.

### 2.3. Nomenclature

#### 2.3.1. Pseudo-Boolean (pseudo-switching) function $\boldsymbol{C}(\boldsymbol{X})$:

A Boolean function, also known as a Switching function, is a type of mapping that takes as input a combination of $n$ binary digits (either 0 or 1) and produces a single output digit that is also binary (either 0 or 1), $\{0, 1\}^n \to \{0, 1\}$. In other words, $S(\boldsymbol{X})$ represents a unique combination of 0's and 1's for every possible combination of $n$ binary digit[4, 5, 7-9]. However, a pseudo-switching (pseudo-Boolean) function $C(\boldsymbol{X})$ is a mapping $\{0, 1\}^n \to R$ where $R$ is the field of real numbers, i.e., $C(\boldsymbol{X})$ assigns a real number to each of the $2^n$ possible $\boldsymbol{X}$ values. For binary capacitated networks and other applications, pseudo-Boolean functions play an essential role [4-9].This section discusses pseudo-switching functions briefly and provides some ideas on their utility in the analysis of binary capacitated networks [4].The capacity function of a binary flow network, which represents the source-to-terminal flow, can be expressed by the function $C_{ij}(\boldsymbol{X})$. This function can be algebraically decomposed with respect to one of its input variables, $X_l$, where $l$ takes on values of $1, 2, \dots, n$.

$$C_{ij}(\boldsymbol{X}) = \overline{X}_l C_{ij}(\boldsymbol{X}|O_l) + X_l C_{ij}(\boldsymbol{X}|1_l)$$
$$= (1 - X_l) C_{ij}(\boldsymbol{X}|0_l) + X_l C_{ij}(\boldsymbol{X}|1_l) = C_{ij}(\boldsymbol{X}|O_l) + \left[C_{ij}(\boldsymbol{X}|1_l) - C_{ij}(\boldsymbol{X}|O_l)\right] X_l \,. \quad (2)$$

Equation (2) can be proved by perfect induction of all possible situations or values of $\boldsymbol{X}$,namely, $\{\boldsymbol{X}|0_l\}$ and $\{\boldsymbol{X}|1_l\}$.By utilizing this decomposition relationship of $C_{ij}(\boldsymbol{X})$, it is possible to represent various characteristics of the function as a pseudo-Boolean function. This includes, notably,

(a) Proof exists indicating that $C_{ij}(\boldsymbol{X})$ is a multi-affine function [4, 5], i.e., It is an algebraic function with each variable being a first-degree polynomial. This indicates that if any $(n - 1)$ variables are given fixed values, the function in the remaining variables reduces to a first-degree polynomial.

(b) The existence of a multitude of representations for $C_{ij}(\boldsymbol{X})$ can be demonstrated by repeatedly applying (2) to different input variables, resulting in an expansion tree for $C_{ij}(\boldsymbol{X})$.

The capacity function $C_{ij}(\boldsymbol{X})$ is represented at each level of the expansion tree by a variable-entered Karnaugh map (VEKM).

(c) The binary flow network's capacity function $C_{ij}(\boldsymbol{X})$demonstrates coherence within the system through the characteristics of causality, monotonicity, and relevancy. Causality is expressed as [4]

$$C_{ij}(\boldsymbol{0}) = 0, \quad (3a)$$

$$C_{ij}(\boldsymbol{1}) = \sum_{l=1}^{n} c_l. \quad (3b)$$

Monotonicity refers to the fact that the function $C_{ij}(\boldsymbol{X})$is monotone and non-decreasing in $\boldsymbol{X}$, and therefore, the coefficient of $X_l$ in (2) is non-negative, i.e.,

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$C_{ij}\,(\boldsymbol{X}|1_l) \;\geq\; C_{ij}\,(\boldsymbol{X}|O_l), l = 1,2, \ldots, n. for\ all\ \boldsymbol{X}.$ (4)

Relevancy denotes that the coefficient of $X_l$ in (2) is non-negative for a given value(s) of $\boldsymbol{X}$, i.e.,

$C_{ij}\,(\boldsymbol{X}|1_l) > C_{ij}\,(\boldsymbol{X}|O_l), l = 1,2, \ldots, n. for\ at\ least\ one\ particular\ \boldsymbol{X}\ .$ (5)

(d) It has been determined that $C_{ij}\,(\boldsymbol{X})$ can be expressed as a polynomial representation, which means it can be represented as a sum of products. The terms 'sum' and 'product' refer to their conventional meanings of real addition and multiplication, respectively, rather than logical addition (OR) and logical products (AND) of literals.Furthermore, according to (4), the sum-of-products expression of $C_{ij}\,(\boldsymbol{X})$includes only uncomplemented literals $X_l$. Since the expected value of a sum of real numbers is equal to the sum of their expected values, we can calculate the mean value (or expected value) of the random function $C_{ij}\,(\boldsymbol{X})$ in its sumofproducts form using the following expression:

$E\{C_{ij}\,(\boldsymbol{X})\} = E\{C_{ij}\}(\boldsymbol{p})\ ,$ (6)

and may be obtained directly (on a one-to-one basis) from $C_{ij}\,(\boldsymbol{X})$(s-o-p) by substituting the component reliabilities $p_l = \mathrm{E}\{X_l\}$ and $q_l = E\{\overline{X}_l\}$, for the corresponding Boolean inputs $X_l$, and $\overline{X}_l$.The resultant polynomial representation has traditionally been utilized to handle pseudo-Boolean functions [4-9].Pseudo-Boolean functions are fundamentally equal to set functions, which are mappings of finite set subsets into the real field. The term pseudo-Boolean function was established in [13] to represent the similarity of these functions to Boolean functions.

(e) A modified Karnaugh map which is a very effective manual tool that provides pictorial insight into the many functional features and techniques, can be used to specify the pseudo-switching function $C_{ij}\,(\boldsymbol{X})$.The variables in the map correspond to the individual elements of $\boldsymbol{X}$, while the entries in the map represent real values denoted as $C_{ij}\,(X_k)$that reflect the s-t capacity for states $X_k$ but are not necessarily 1's and 0's. Any of the approaches in [5, 8] can be used to acquire these numbers separately or collectively.In order to represent $C_{ij}\,(\boldsymbol{X})$in a near-minimal sum-of-products (s-o-p) form, it is crucial to envelop the non-zero elements in the map through the least number of map loops feasible. Each loop should be as expansive as possible, encompassing $2^i$adjacent cells (where $i$ ranges from 0 to $n$) while ensuring the inclusion of at least one previously uncovered value. The contribution of a loop to the s-o-p representation of $C_{ij}\,(\boldsymbol{X})$is determined by the product of the loop's covered value and the standard loop term.To facilitate the selection of larger loops, it may be necessary to divide a cell entry into multiple values, which can then be covered by several loops. This division is generally achievable for integer-valued entries in maps that depict compact networks. Once a portion of an entry has been covered, it is substituted with its remaining uncovered portion; if an entry is entirely covered, it is replaced with a zero. The process concludes when all entries in the map have been reduced to zeros.
The map procedure yields capacity expressions that are more straightforward than those obtained via the direct state-enumeration method.This method proves particularly beneficial when the map entries are part of a small set of integer values, which is typical when branch capacities are integer-valued. Although the map procedure is restricted to handling smaller networks (with six branches or fewer), it can be expanded to accommodate moderate-sized networks through the implementation of variable-entered Karnaugh maps (VEKMs) [6].

**2.3.2. Thedecompositiontechnique:**

The decomposition method entails configuring a complex network (complicated system) to adopt the possible states of a keystone element $K_1$ or the possible combinations of states of multiple keystone elements [9].viz, this technique requires applying the law of total probability once or several times.In

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its most basic form, it entails picking a keystone element and then determining network's reliability twice: once as if the keystone element failed (R=0) and once as if the keystone element succeeded (R=1).The technique then combines these two probabilities to determine the system's reliability, because the key component will either fail or be operational at any given time.The event $S$ is depicted in the Venn graph in Fig. 1, indicating that the system is operational.This event might be categorized as a union of two mutually exclusive dependent events, viz.(i) $K_1 \cap S$,which shows that the keystone link is operational and thus the system is working and (ii)$\overline{K}_1 \cap S$,which shows that the keystone link is in a failed state while the rest of the system is working properly.

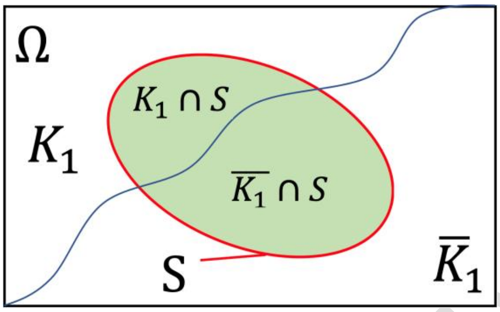


Fig. 1. Event S, which indicates a fully functional system, partitioned as the union of a pair of mutually exclusive events: $\boldsymbol{K_1} \cap \boldsymbol{S}$ and $\overline{\boldsymbol{K}}_{\boldsymbol{1}} \cap \boldsymbol{S}$

The primary purpose of this technique is to breakdown the system graph into two subgraphs $K_1 \cap S$ and $\overline{K}_1 \cap S$, each of which has a simpler structure than the original network, in order to achieve the desired result.According to the total probability theorem, the probability $Pr(S)$ of event $S$ occurring while the system is in the operating state can be calculated as the sum of the probabilities of the two mutually-exclusive events that are indicated above:

$$\Pr(\mathrm{S}) = \Pr(\mathrm{S} \cap \mathrm{K}_1) + \Pr(\mathrm{S} \cap \overline{\mathrm{K}}_1) \qquad (8)$$

In order to simplify Equation (8), each of the conjunctive probabilities in it can be expressed as a product of a conditional probability and a marginal probability, i.e.

$\Pr(\mathrm{S} \cap \mathrm{K}_1) = \Pr(\mathrm{S}|\mathrm{K}_1)\ \Pr(\mathrm{K}_1)$ and$\Pr(\mathrm{S} \cap \overline{\mathrm{K}}_1) = \Pr(\mathrm{S}|\overline{\mathrm{K}}_1)\ \Pr(\overline{\mathrm{K}}_1)$, to finally obtain

$$\Pr(\mathrm{S}) = \Pr(\mathrm{S}|\mathrm{K}_1) \Pr(\mathrm{K}_1) + \Pr(\mathrm{S}|\overline{\mathrm{K}}_1) \Pr(\overline{\mathrm{K}}_1) \quad (9)$$

In equation (9) $\Pr(S|K_1)$ denotes the conditional probability that the system is operational given that the keystone branch is operationaland $\Pr(S|\overline{K}_1)$ denotes the conditional probability that the system is operational given that the keystone element is in the unsuccessful state, whereas $\Pr(\mathrm{K}_1)$ and $\Pr(\overline{K}_1)$ denote the marginal probabilities of the keystone branch being in the operating state and failing state, respectively.Similarly, if two independent keystone elements $K_1$ and $K_2$ are chosen (instead of just one keystone element), the probability of network success $\Pr(S)$ (the system's reliability) is calculated as the sum of the probabilities of four mutually exclusive events.

$$\Pr(S) = \Pr(S|K_1K_2) \Pr(K_1)\Pr(K_2) + \Pr(S|K_1\overline{K}_2) \Pr(K_1)\Pr(\overline{K}_2) + \Pr(S|\overline{K}_1K_2) \Pr(\overline{K}_1) \Pr(K_2) + \Pr(S|\overline{K}_1\overline{K}_2) \Pr(\overline{K}_1) \Pr(\overline{K}_2) \qquad (10)$$

### 2.3.3. Reliability-Ready Expression (RRE):

A reliability-ready expression is a switching (Boolean) function in which entities that have been logically multiplied (ANDed) are statistically independent, whereas entities that have been logically

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added (ORed) are disjointed. This expression could be successfully converted to the algebraic or probability domain on a one-to-one basis by substituting Boolean (switching) indicators for corresponding statistical expectations and by substituting logical addition and multiplication for their arithmetic counterparts.The PRE concept originated for Boolean functions [2], but it is applicable for Boolean disjunctive normal forms of pseudo-Boolean functions as well.

## 3. EXAMPLES:

### 3.1. Example 1:

We examine an example discussed previously in [4].This particular example pertains to the capacity function $C_{st}(\boldsymbol{X})$ of a flow network represented in Figure 2. The links within this network are binary, and each link has a specific capacity assigned to it, with $C_1 = 10, C_2 = 3, C_3 = C_4 = 4$ and $C_5 = 5$ (in appropriate units for the flow commodity involved). The minimal disjunctive normal form (DNF) for the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ is given by [4]

$C_{st}(\boldsymbol{X}) = 7(X_1X_2X_3X_4 \vee X_1X_2X_4X_5) \vee 4(X_3X_4 \ \vee X_1X_4X_5) \vee 3(X_1X_2 \vee X_2X_3X_5)$.(11)

Equation (11) is formulated using the operations of join (disjunction, ORing), and meet (conjunction, ANDing), which are represented by $\boldsymbol{X} \vee \boldsymbol{Y}, and \boldsymbol{XY} = \boldsymbol{X} \wedge \boldsymbol{Y}$, respectively. These operations, which are used to combine or compare different values, also correspond to the maximum (max) and minimum (min) operators in R.The Boole-Shannon expansion in (1,2) is applicable to the function in (11) though it is pseudo-Boolean rather than Boolean. Expanding $C_{st}(\boldsymbol{X})$about $X_5$ , we obtain

$C_{st}(\boldsymbol{X}) = \bar{X}_5\left(\left(X_1X_2X_3X_4 \vee X_1X_2X_4(\boldsymbol{0})\right) \vee 4\left(X_3X_4 \ \vee X_1X_4(\boldsymbol{0})\right) \vee 3\left(X_1X_2 \vee X_2X_3(\boldsymbol{0})\right)\right) \vee$
$X_5\left(\left(X_1X_2X_3X_4 \vee X_1X_2X_4(\boldsymbol{1})\right) \vee 4\left(X_3X_4 \ \vee X_1X_4(\boldsymbol{1})\right) \vee 3\left(X_1X_2 \vee X_2X_3(\boldsymbol{1})\right)\right) = \bar{X}_5\,(7(X_1X_2X_3X_4) \vee$
$4(X_3X_4) \vee 3(X_1X_2)) \vee X_5\,(7(X_1X_2X_3X_4 \vee X_1X_2X_4) \vee 4(X_3X_4 \ \vee X_1X_4) \vee 3(X_1X_2 \vee X_2X_3))$. (12)

Here, the pseudo-Boolean literal $7X_1X_2X_3X_4$ subsumes (and is absorbed in) the pseudo-Boolean literal $7X_1X_2X_4$ in the $X_5$ subfunction. However, it does not subsume (and is not absorbed in) $3\,X_1X_2$ or $4\,X_3X_4$ in the $\bar{X}_5$ subfunction. Therefore, (12) reduces to

$C_{st}(\boldsymbol{X}) = \bar{X}_5\,(7(X_1X_2X_3X_4) \vee 4(X_3X_4) \vee 3(X_1X_2)) \vee X_5\,(7(X_1X_2X_4) \vee 4(X_3X_4 \ \vee X_1X_4) \vee$
$3(X_1X_2 \vee X_2X_3))$. (13)
Further expansion about $X_3$ together with due absorptions results in

$C_{st}(\boldsymbol{X}) = \bar{X}_3\bar{X}_5\,(3(X_1X_2)) \vee \bar{X}_3X_5\,(7(X_1X_2X_4) \vee 4(X_1X_4) \vee 3(X_1X_2))$
$\vee X_3\bar{X}_5\,(7(X_1X_2X_4) \vee 4(X_4) \vee 3(X_1X_2)) \vee X_3X_5\,(7(X_1X_2X_4) \vee 4(X_4) \vee 3(X_2))$. (14)

Further expansion about $X_1$ together with due absorptions results in

$C_{st}(\boldsymbol{X}) = \bar{X}_1\bar{X}_3\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_3X_5\ (0) \vee \bar{X}_1X_3\bar{X}_5\,(4(X_4)) \vee \bar{X}_1X_3X_5\,(4(X_4) \vee 3(X_2)) \vee$
$X_1\bar{X}_3\bar{X}_5\,(3(X_2)) \vee X_1\bar{X}_3X_5\,(7(X_2X_4) \vee 4(X_4) \vee 3(X_2))$
$\vee X_1X_3\bar{X}_5\,(7(X_2X_4) \vee 4(X_4) \vee 3(X_2)) \vee X_1X_3X_5\ (7(X_2X_4) \vee 4(X_4) \vee 3(X_2))$. (15)

Further expansion about $X_4$ together with due absorptions results in

$C_{st}(\boldsymbol{X}) = \bar{X}_1\bar{X}_3\bar{X}_4\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_3\,\bar{X}_4X_5\ (0) \vee \bar{X}_1X_3\bar{X}_4\bar{X}_5\,(0) \vee \bar{X}_1X_3\,\bar{X}_4X_5\,(3(X_2))$.
$X_1\bar{X}_3\bar{X}_4\bar{X}_5\,(3(X_2)) \vee X_1\bar{X}_3\bar{X}_4X_5\,(3(X_2)) \vee X_1X_3\bar{X}_4\bar{X}_5\,(3(X_2)) \vee X_1X_3\bar{X}_4X_5\ (3(X_2)) \vee$
$\bar{X}_1\bar{X}_3X_4\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_3X_4X_5\ (0) \vee \bar{X}_1X_3X_4\bar{X}_5\,(4) \vee \bar{X}_1X_3X_4X_5\,(4)$.
$X_1\bar{X}_3X_4\bar{X}_5\,(3(X_2)) \vee X_1\bar{X}_3X_4X_5\,(7(X_2) \vee 4) \vee X_1X_3X_4\bar{X}_5\,(7(X_2) \vee 4) \vee X_1X_3X_4X_5\ (7(X_2) \vee 4)$.
(16)

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Final expansion about $X_2$ together with due absorptions results in

$C_{st}(\boldsymbol{X}) = \bar{X}_1\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_2\bar{X}_3\,\bar{X}_4X_5\ (0) \vee \bar{X}_1\bar{X}_2X_3\bar{X}_4\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_2X_3\,\bar{X}_4X_5\ (0).$
$X_1\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5\ (0) \vee X_1\bar{X}_2\bar{X}_3\bar{X}_4X_5\ (0) \vee X_1\bar{X}_2X_3\bar{X}_4\bar{X}_5\ (0) \vee X_1\bar{X}_2X_3\bar{X}_4X_5\ (0) \vee$
$\bar{X}_1\bar{X}_3X_4\bar{X}_5\ (0) \vee \bar{X}_1\bar{X}_2\bar{X}_3X_4X_5\ (0) \vee \bar{X}_1\bar{X}_2X_3X_4\bar{X}_5\ (4) \vee \bar{X}_1\bar{X}_2X_3X_4X_5\ (4).$
$X_1\bar{X}_2\bar{X}_3X_4\bar{X}_5\ (0) \vee X_1\bar{X}_2\bar{X}_3X_4X_5\ (4) \vee X_1\bar{X}_2X_3X_4\bar{X}_5\ (4) \vee X_1\bar{X}_2X_3X_4X_5\ (4)$
$\vee\ \bar{X}_1X_2\bar{X}_3\bar{X}_4\bar{X}_5\ (0) \vee \bar{X}_1X_2\bar{X}_3\,\bar{X}_4X_5\ (0) \vee \bar{X}_1X_2X_3\bar{X}_4\bar{X}_5\ (0) \vee \bar{X}_1X_2X_3\,\bar{X}_4X_5\ (3).$
$X_1X_2\bar{X}_3\bar{X}_4\bar{X}_5\ (3) \vee X_1X_2\bar{X}_3\bar{X}_4X_5\ (3) \vee X_1X_2X_3\bar{X}_4\bar{X}_5\ (3) \vee X_1X_2X_3\bar{X}_4X_5\ (3) \vee$
$\bar{X}_1X_2\bar{X}_3X_4\bar{X}_5\ (0) \vee \bar{X}_1X_2\bar{X}_3X_4X_5\ (0) \vee \bar{X}_1X_2X_3X_4\bar{X}_5\ (4) \vee \bar{X}_1X_2X_3X_4X_5\ (4).$
$X_1X_2\bar{X}_3X_4\bar{X}_5\ (3) \vee X_1X_2\bar{X}_3X_4X_5\ (7) \vee X_1X_2X_3X_4\bar{X}_5\ (7) \vee X_1X_2X_3X_4X_5\ (7).$
(17)

Equation (17) is a minterm expansion of the given pseudo-Boolean capacity function, which might be immediately be represented by the multi-valued (value-entered) Karnaugh map in Fig. 3. In retrospect, it is possible to verify that each of the equations (11) to (16) is exactly equivalent to that figure [4].

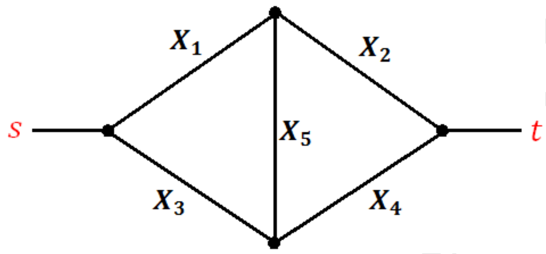


Fig. 2. A capacitated 5-branch $st$ network with a capacity vector $c = [\,10\ \ 3\ \ 4\ \ 4\ \ 5]^T$.

$X_5$
$X_1$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 3 | 0 | 0 | 3 | 0 | 0 |
| 0 | 0 | 3 | 0 | 4 | 7 | 0 | 0 |
| 4 | 4 | 7 | 4 | 4 | 7 | 4 | 4 |
| 0 | 0 | 3 | 0 | 0 | 3 | 3 | 0 |

$X_4$
$X_3$
$X_2$ $X_2$

$C_{st}(\boldsymbol{X})$

Fig. 3. The representation of multi valued Karnaugh map for the network $C_{st}(X)$ in Fig. 2

### 3.2. Example 2:

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Another example will be revisited herein that has been considered earlier in [4]. The capacity function $C_{st}(\boldsymbol{X})$for the flow network shown in Figure 4 is discussed in this example. The network has links with binary capacities, namely $C_1 = 6, C_2 = 7, C_3 = 4, C_4 = 10,\ C_5 = 5, C_6 = 3\ and\ C_7 = 4$ (in appropriate units for the flow commodity involved). The minimal disjunctive normal form (DNF) for the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ is given by [4].

$C_{st}(\boldsymbol{X}) = 7(X_2X_5X_6X_7 \vee X_1X_2X_4X_6X_7 \vee X_2X_3X_4X_6X_7) \vee 6(X_1X_4X_5X_6X_7 \vee X_1X_3X_4X_6X_7) \vee$
$4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)$.(18)

The equation (18) is formulated using the join operation (also known as disjunction or ORing) represented by $\boldsymbol{X} \vee \boldsymbol{Y}$, and the meet operation (also known as conjunction or ANDing) represented by $\boldsymbol{XY} = \boldsymbol{X} \wedge \boldsymbol{Y}$.
The Boole-Shannon expansion in (1, 2) is applicable to the function in (18) though it is pseudo-Boolean rather than Boolean. Decomposing the s-t capacity function $C_{st}(\boldsymbol{X})$ with respect to the indicator variables $X_3 and X_5$, which represent the bridging branches in the flow network of Figure 4, yields the following expression, which is a special case of the decomposition formula (1,2).

$C_{st}(\boldsymbol{X}) = \overline{X}_3\overline{X}_5\ C_{st}(X|0_3\ , 0_5) + \overline{X}_3X_5C_{st}(X|0_3\ , 1_5) + X_3\overline{X}_5\ C_{st}(X|1_3\ , 0_5) + X_3X_5C_{st}(X|1_3\ , 1_5)$(19)

The sub-functions in (19) are obtained via (18) as:

$C_{st}(\boldsymbol{X}|0_3, 0_5) = 7X_1X_2X_4X_6X_7 \vee 4\ X_2X_7 \vee 3X_1X_4X_6$(20)
Further expansion about $X_1$ together with due absorptions results in

$C_{st}(\boldsymbol{X}|0_3, 0_5) = \overline{X}_1(4X_2X_7) \vee X_1(7X_2X_4X_6X_7 \vee 4X_2X_7 \vee 3X_4X_6)$ (21)

Further expansion about $X_2$ together with due absorptions results in

$C_{st}(\boldsymbol{X}|0_3, 0_5) = \overline{X}_2\big(X_1(3X_4X_6)\big) \vee X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7 \vee 3X_4X_6)\right)$(22)

Further expansion about $X_4$ together with due absorptions results in

$C_{st}(\boldsymbol{X}|0_3, 0_5) = \overline{X}_4\left(X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7)\right)\right) \vee X_4\left(\overline{X}_2(3X_6X_1) \vee X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7 \vee\right.\right.$
$\left.\left.3X_6)\right)\right)$(23)

Further expansion about $X_6$ together with due absorptions results in

$C_{st}(\boldsymbol{X}|0_3, 0_5) = \overline{X}_6\left(\overline{X}_4\left(X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7)\right)\right) \vee X_4\left(X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7)\right)\right)\right) \vee$
$X_6\left(\overline{X}_4\left(X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7)\right)\right) \vee X_4\left(\overline{X}_2(3X_1) \vee X_2\left(\overline{X}_1(4X_7) \vee X_1(4X_7 \vee 3)\right)\right)\right)$(24)

Final expansion about $X_7$ together with due absorptions results in

$$
\begin{aligned}
C_{st}(\boldsymbol{X}|0_3, 0_5) = {} & \overline{X}_7\left(X_6\left(X_4\left(\overline{X}_2(3X_1) \vee X_2(X_1(3))\right)\right)\right) \\
& \vee X_7\left(\overline{X}_6\left(\overline{X}_4\left(X_2\left(\overline{X}_1(4) \vee X_1(4)\right)\right) \vee X_4\left(X_2\left(\overline{X}_1(4) \vee X_1(4)\right)\right)\right)\right. \\
& \left.\vee X_6\left(\overline{X}_4\left(X_2\left(\overline{X}_1(4) \vee X_1(4)\right)\right) \vee X_4\left(\overline{X}_2(3X_1) \vee X_2\left(\overline{X}_1(4) \vee X_1(7)\right)\right)\right)\right)
\end{aligned}
$$

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$C_{st}(\boldsymbol{X}|0_3,0_5) = 3X_1\overline{X}_2X_4X_6\overline{X}_7 \vee 3X_1X_2X_4X_6\overline{X}_7 \vee 4\overline{X}_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2X_4\overline{X}_6X_7 \vee 4X_1X_2X_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2\overline{X}_4X_6X_7 \vee 4X_1X_2\overline{X}_4X_6X_7 \vee 3X_1\overline{X}_2X_4X_6X_7 \vee 4\overline{X}_1X_2X_4X_6X_7 \vee 7X_1X_2X_4X_6X_7$(25)

$C_{st}(\boldsymbol{X}|1_3,1_5) = 7(X_2X_6X_7 \vee X_1X_2X_4X_6X_7 \vee X_2X_4X_6X_7) \vee 6(X_1X_4X_6X_7 \vee X_1X_4X_6X_7) \vee 4(X_2X_7 \vee X_1X_7 \vee X_1X_4X_7) \vee 3(X_1X_4X_6 \vee X_2X_6 \vee X_2X_4X_6 \vee X_1X_6)$(26)

Expanding $C_{st}(\boldsymbol{X}|1_3,1_5)$ about $X_1$ :

$C_{st}(\boldsymbol{X}|1_3,1_5) = \overline{X}_1(7(X_2X_6X_7) \vee 4(X_2X_7) \vee 3(X_2X_6)) \vee X_1 (7(X_2X_6X_7) \vee 6(X_4X_6X_7) \vee 4(X_7) \vee 3(X_6)))$ (27)

Expanding $C_{st}(\boldsymbol{X}|1_3,1_5)$ about $X_2$ :

$C_{st}(\boldsymbol{X}|1_3,1_5) = \overline{X}_2 (X_1(6(X_4X_6X_7) \vee 4(X_7) \vee 3(X_6))) \vee X_2 (\overline{X}_1(7(X_6X_7) \vee 4(X_7) \vee 3(X_6)) \vee X_1(7(X_6X_7) \vee 6(X_4X_6X_7) \vee 4(X_7) \vee 3(X_6)))$ (28)

Expanding $C_{st}(\boldsymbol{X}|1_3,1_5)$ about $X_4$ :

$C_{st}(\boldsymbol{X}|1_3,1_5) = \overline{X}_4 (\overline{X}_2 (X_1(4(X_7) \vee 3(X_6))) \vee X_2 (\overline{X}_1(7(X_6X_7) \vee 4(X_4) \vee 3(X_6)) \vee X_1(7(X_6X_7) \vee 4(X_7) \vee 3(X_6))))) \vee X_4 (\overline{X}_2 (X_1(6(X_6X_7) \vee 4(X_7) \vee 3(X_6))) \vee X_2(\overline{X}_1(7(X_6X_7) \vee 4(X_7) \vee 3(X_6)) \vee X_1( 7(X_6X_7) \vee 6(X_6X_7) \vee 3(X_6))))$ (29)

Expanding $C_{st}(\boldsymbol{X}|1_3,1_5)$ about $X_6$ :

$C_{st}(\boldsymbol{X}|1_3,1_5) = \overline{X}_6 (\overline{X}_4 (\overline{X}_2(X_1(4(X_7)) \vee X_2(\overline{X}_1(4(X_7)) \vee X_1(4(X_7)))) \vee X_4 (\overline{X}_2 (X_1(4(X_7))) \vee X_2 (\overline{X}_1(4(X_7)) \vee X_1(4(X_7)))))) \vee X_6 (\overline{X}_4 (\overline{X}_2(X_1(4(X_7) \vee 3)) \vee X_2 (\overline{X}_1(7(X_7) \vee 3) \vee X_1(7(X_7) \vee 3))) \vee X_4 (\overline{X}_2(X_1(6(X_7) \vee 3)) \vee X_2 (\overline{X}_1(7(X_7) \vee 3) \vee X_1(7(X_7) \vee 3))))$ (30)

Final expansion $C_{st}(\boldsymbol{X}|1_3,1_5)$ about $X_7$ :

$$
\begin{aligned}
C_{st}(\boldsymbol{X}|1_3,1_5) = \overline{X}_7 \Bigg( & X_6 \Big(\overline{X}_4 \big(\overline{X}_2(X_1(3)) \vee X_2 (\overline{X}_1(3) \vee X_1(3))\big) \\
& \vee X_4 \big(\overline{X}_2(X_1(3)) \vee X_2 (\overline{X}_1(3) \vee X_1(3))\big)\Big)\Bigg) \\
& \vee X_7 \Bigg(\overline{X}_6 \Big(\overline{X}_4 \big(\overline{X}_2(X_1(4)) \vee X_2 (\overline{X}_1(4) \vee X_1(4))\big) \\
& \vee X_4 \big(\overline{X}_2(X_1(4)) \vee \overline{X}_1X_2(4) \vee X_1(4)\big)\Big) \\
& \vee X_6 \Big(\overline{X}_4 \big(\overline{X}_2(X_1(4)) \vee X_2 (\overline{X}_1(7) \vee X_1(7))\big) \\
& \vee X_4 \big(\overline{X}_2(X_1(6)) \vee X_2 (\overline{X}_1(7) \vee X_1(7))\big)\Big)
\end{aligned}
$$

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$C_{st}(\boldsymbol{X}|1_3, 1_5) = 3X_1\overline{X}_2\overline{X}_4X_6\overline{X}_7 \vee 3\overline{X}_1X_2\overline{X}_4X_6\overline{X}_7 \vee 3X_1X_2\overline{X}_4X_6\overline{X}_7 \vee 3X_1\overline{X}_2X_4X_6\overline{X}_7 \vee 3\overline{X}_1X_2X_4X_6\overline{X}_7 \vee 3X_1X_2X_4X_6\overline{X}_7 \vee 4X_1\overline{X}_2\overline{X}_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1\overline{X}_2X_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2X_4\overline{X}_6X_7 \vee 4X_1X_2X_4\overline{X}_6X_7 \vee 4X_1\overline{X}_2\overline{X}_4X_6X_7 \vee 7\overline{X}_1X_2\overline{X}_4X_6X_7 \vee 7X_1X_2\overline{X}_4X_6X_7 \vee 6X_1\overline{X}_2X_4X_6X_7 \vee 7\overline{X}_1X_2X_4X_6X_7 \vee 7X_1X_2X_4X_6X_7$ (31)

$C_{st}(\boldsymbol{X}|1_3, 0_5) = 7(X_1X_2X_4X_6X_7 \vee X_2X_4X_6X_7) \vee 6(X_1X_4X_6X_7) \vee 4(X_2X_7 \vee X_1X_7) \vee 3(X_1X_4X_6 \vee X_2X_4X_6)$ (32)

$C_{st}(\boldsymbol{X}|1_3, 0_5) = 3X_1\overline{X}_2X_4X_6\overline{X}_7 \vee 3\overline{X}_1X_2X_4X_6\overline{X}_7 \vee 3X_1X_2X_4X_6\overline{X}_7 \vee 4X_1\overline{X}_2\overline{X}_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1\overline{X}_2X_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2X_4\overline{X}_6X_7 \vee 4X_1X_2X_4\overline{X}_6X_7 \vee 4X_1\overline{X}_2\overline{X}_4X_6X_7 \vee 4\overline{X}_1X_2\overline{X}_4X_6X_7 \vee 4X_1X_2\overline{X}_4X_6X_7 \vee 6X_1\overline{X}_2X_4X_6X_7 \vee 7\overline{X}_1X_2X_4X_6X_7 \vee 7X_1X_2X_4X_6X_7$(33)

$C_{st}(\boldsymbol{X}|0_3, 1_5) = 7(X_2X_6X_7 \vee X_1X_2X_4X_6X_7) \vee 6(X_1X_4X_6X_7) \vee 4(X_2X_7 \vee X_1X_4X_7) \vee 3(X_1X_4X_6 \vee X_2X_6)$ (34)

$C_{st}(\boldsymbol{X}|0_3, 1_5) = 3\overline{X}_1X_2\overline{X}_4X_6\overline{X}_7 \vee 3X_1X_2\overline{X}_4X_6\overline{X}_7 \vee 3X_1\overline{X}_2X_4X_6\overline{X}_7 \vee 3\overline{X}_1X_2X_4X_6\overline{X}_7 \vee 3X_1X_2X_4X_6\overline{X}_7 \vee 4\overline{X}_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1X_2\overline{X}_4\overline{X}_6X_7 \vee 4X_1\overline{X}_2X_4\overline{X}_6X_7 \vee 4\overline{X}_1X_2X_4\overline{X}_6X_7 \vee 4X_1X_2X_4\overline{X}_6X_7 \vee 7\overline{X}_1X_2\overline{X}_4X_6X_7 \vee 7X_1X_2\overline{X}_4X_6X_7 \vee 6X_1\overline{X}_2X_4X_6X_7 \vee 7\overline{X}_1X_2X_4X_6X_7 \vee 7X_1X_2X_4X_6X_7$(35)

Substituting these sub-functions into (19) yields the equivalent form (18). Additionally, they can be utilized to complete the map entries in Figure 5.

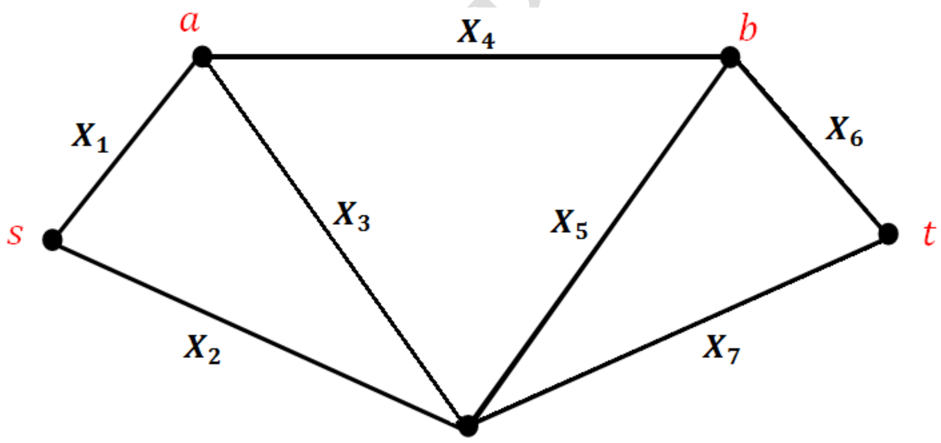


Fig. 4. A 7-branch bridge network of a capacity vector $c = [\,6\ \ 7\ \ 4\ \ 10\ \ 5\ \ 3\ \ 4\,]^T$

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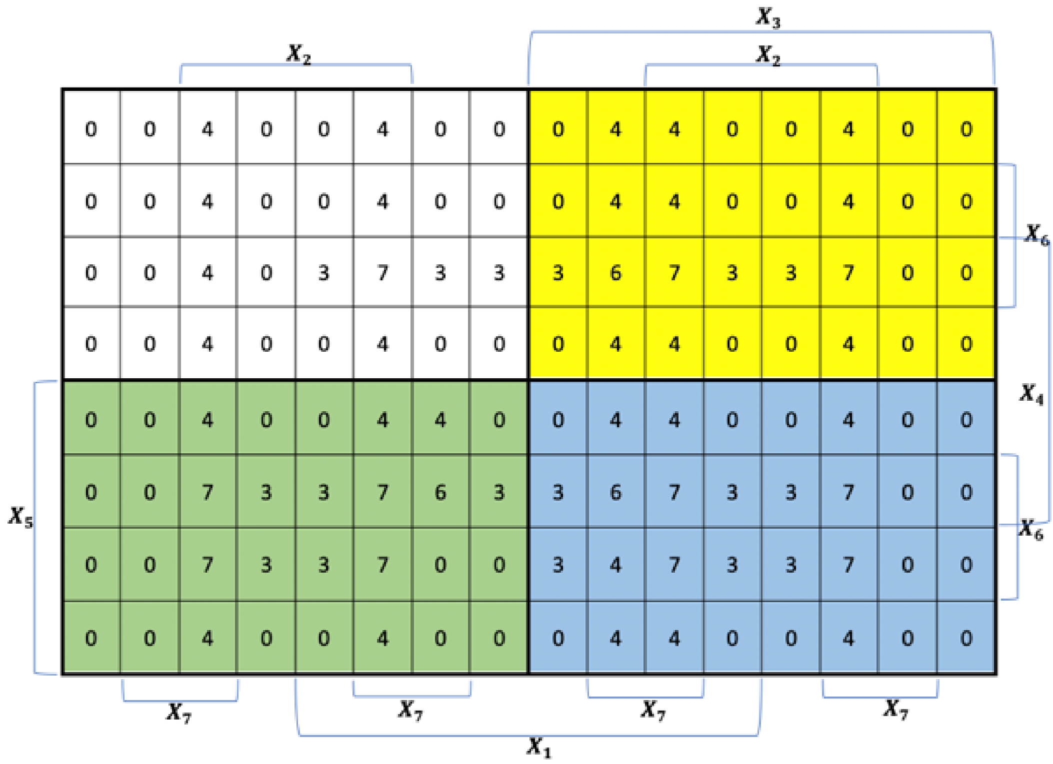


Fig. 5. The representation of multi valued Karnaugh map for the network $C_{st}(X)$ in Fig. 4

## 4. CONCLUSION:

The network decomposition approach has been presented in this paper, in which network analysis is made simple by applying this method.Therefore, our results help analyze or simplify a huge network by decomposing the original network into a simpler network. The method of network reliability by decomposition technique can be very efficient if proper choice of subdivision is made. This paper has provided two didactic examples that not only exemplify the aforementioned methodology but also underscore its computational benefits.

## DATA AVAILABILITY

Data sharing is not applicable to this article.

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