Original Research Article

Gaussian Generalized Woodall Numbers

Abstract

In this work, we define Gaussian generalized Woodall numbers and give properties of Gaussian modified Woodall, Gaussian modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen numbers as special cases. First, we present some background about Woodall numbers and Gaussian numbers before defining Gaussian generalized Woodall numbers.

Recently, there have been so many studies of the sequences of numbers in the literature which are defined recursively. Four of these type of sequences are the sequences of Gaussian modified Woodall, Gaussian modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen which are special case of generalized Woodall numbers.

Keywords. Gaussian Woodall numbers, Gaussian Cullen numbers, Gaussian generalized Woodall numbers, Gaussian modified Woodall numbers, Gaussian modified Cullen numbers, Woodall numbers.

1. Introduction

First, we recall some properties of generalized Woodall numbers. The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero.

A generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$. Therefore, recurrence (1.1) holds for all integer n. For more details, see [18].

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3], [4], [6], [7], [10], [12], [13], [14], [19], [23], [24].

Next, we give Binet formula of generalized Woodall numbers.

THEOREM 1.1. *[18, Theorem 1.1] Binet formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where A_1, A_2 and A_3 are defined by

$$\begin{aligned} A_1 &= -W_2 + 4W_1 - 3W_0, \\ A_2 &= \frac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 &= W_2 - 4W_1 + 4W_0, \end{aligned}$$

that is,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2 (x-1) = 0,$$

where

$$\begin{aligned} \alpha &= \beta = 2, \\ \gamma &= 1. \end{aligned}$$

Now, the first few generalized Woodall numbers with positive subscript and negative subscript are given in the following table.

Table 1. The first few generalized Woodall numbers with positive subscript and negative subscript.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$
6	$196W_0 - 324W_1 + 129W_2$	$\frac{1}{64}(247W_0 - 243W_1 + 60W_2)$

Now, we define four specific cases of the sequence $\{W_n\}$. Modified Woodall sequence $\{G_n\}_{n\geq 0}$, modified Cullen sequence $\{H_n\}_{n\geq 0}$, Woodall sequence $\{R_n\}$ and Cullen sequence $\{C_n\}$ are defined, respectively, by the third-order recurrence relations,

$$\begin{aligned} G_n &= 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 5, & (1.3) \\ H_n &= 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \qquad H_0 = 3, H_1 = 5, H_2 = 9, & (1.4) \\ R_n &= 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \qquad R_0 = -1, R_1 = 1, R_2 = 7, & (1.5) \\ C_n &= 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \qquad C_0 = 1, C_1 = 3, C_2 = 9. & (1.6) \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned} G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\ H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\ R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\ C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3)-(1.6) hold for all integer n.

For all integers n, modified Woodall, modified Cullen, Woodall and Cullen numbers (using initial conditions in (1.2)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= (n-1)\,2^n + 1, \\ H_n &= 2^{n+1} + 1, \\ R_n &= n \times 2^n - 1, \\ C_n &= n \times 2^n + 1, \end{aligned}$$

respectively.

Now we give some information about Gaussian sequence from literature.

- First we give Gaussian numbers with second order recurreance
  - Horadam [9] introduced Gussian Fibonacci numbers as

    $$GF_n = F_n + iF_{n-1}$$

    where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact,he defined these numbers as $GF_n = F_n + iF_{n-1}$ and he called these numbers as complex Fibonacci numbers.)
  - Pethe and Horadam [11] introduced generalized Gaussian Fibonacci numbers

    $$GF_n = F_n + iF_{n-1}$$

    where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

– Halıcı and Öz [8] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1}, \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

– Aşçı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + j_{n-1}, \end{aligned}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

– Taşcı [22] introduced and studied Gaussian Mersenne numbers and define by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

– Taşçı [20] introduced and studied Gaussian balancing and Lucas Balancing numbers and given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + C_{n-1}, \end{aligned}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6C_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

– Ertaş and Yılmaz [2] studied Gaussian Oresme numbers and given by

$$GT_n = T_n + iT_{n-1}$$

where $T_n = T_{n-1} - \frac{1}{4}T_{n-2}$, $T_0 = 0$, $T_1 = \frac{1}{2}$.

• Now, we present Gaussian numbers with third order reccurance relations.

– Soykan, Taşdemir, Okumuş and Göcen [15] presented Gaussian generalized Tribonacci numbers by given

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

– Taşcı [21] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GR_n &= R_n + iR_{n-1}, \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

– Cerda-Morales [5] defined Gaussian third-order Jacobsthal numbers by given

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

2. Gaussian Generalized Woodall Numbers

In this chapter, we define Gaussian generalized Woodall numbers and we give some properties.

Gaussian generalized Woodall numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 5GW_{n-1} - 8GW_{n-2} + 4GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i(\frac{1}{4}(8W_0 - 5W_1 + W_2)),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1,$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 2GW_{-(n-1)} - \frac{5}{4}GW_{-(n-2)} + \frac{1}{4}GW_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) hold for all integer n. Note that for $n \geq 0$, we get

$$GW_n = W_n + iW_{n-1} \tag{2.2}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.3}$$

The first few generalized Gaussian Woodall numbers with positive subscript and negative subscript are given in the following table.

Table 2. The first few generalized Gaussian Woodall numbers.

n	GW_n	GW_{-n}
0	$W_0 + i\frac{1}{4}(8W_0 - 5W_1 + W_2)$	$W_0 + i\frac{1}{4}(8W_0 - 5W_1 + W_2)$
1	$W_1 + iW_0$	$\frac{1}{4}(8W_0 - 5W_1 + W_2) + i\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
2	$W_2 + iW_1$	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2) + i\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
3	$4W_0 - 8W_1 + 5W_2 + iW_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2) + i\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$

We consider four special cases of GW_n :

$GW_n(0, 1, 5 + i) = GG_n$ is the sequence of Gaussian Modified Woodall numbers,

$GW_n(3 + 2i, 5 + 3i, 9 + 5i) = GH_n$ is the sequence of Gaussian Modified Cullen numbers,

$GW_n(-1 - \frac{3}{2}i, 1 - i, 7 + i) = GR_n$ is the sequence of Gaussian Woodall numbers and

$GW_n(1 + \frac{1}{2}i, 3 + i, 9 + 3i) = GC_n$ is the sequence of Gaussian Cullen numbers.

We formally define them as follows. Four special cases of GW_n with the initial conditions are defined by

$$\begin{aligned} GG_n &= 5GG_{n-1} - 8GG_{n-2} + 4GG_{n-3}, \quad GG_0 = 0, GG_1 = 1, GG_2 = 5+i, \\ GH_n &= 5GH_{n-1} - 8GH_{n-2} + 4GH_{n-3}, \quad GH_0 = 3+2i, GH_1 = 5+3i, GH_2 = 9+5i, \\ GR_n &= 5GR_{n-1} - 8GR_{n-2} + 4GR_{n-3}, \quad GR_0 = -1-\frac{3}{2}i, GR_1 = 1-i, GR_2 = 7+i, \\ GC_n &= 5GC_{n-1} - 8GC_{n-2} + 4GC_{n-3}, \quad GC_0 = 1+\frac{1}{2}i, GC_1 = 3+i, GC_2 = 9+3i. \end{aligned}$$

Note that for all integers n, we obtain

$$\begin{aligned} GG_n &= G_n + iG_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \\ GR_n &= R_n + iR_{n-1}, \\ GC_n &= C_n + iC_{n-1}. \end{aligned}$$

The first few values of Gaussian Modified Woodall numbers, Gaussian Modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen numbers with positive and negative subscript are given in the following table.

Table 3. The first few values of special cases of generalized Gaussian Woodall numbers.

n	0	1	2	3	4	5	6	7
GG_n	0	1	$5+i$	$17+5i$	$49+17i$	$129+49i$	$321+129i$	$769+321i$
GG_{-n}	0	$\frac{1}{4}i$	$\frac{1}{4}+\frac{1}{2}i$	$\frac{1}{2}+\frac{11}{16}i$	$\frac{11}{16}+\frac{13}{16}i$	$\frac{13}{16}+\frac{57}{64}i$	$\frac{57}{64}+\frac{15}{16}i$	$\frac{15}{16}+\frac{247}{256}i$
GH_n	$3+2i$	$5+3i$	$9+5i$	$17+9i$	$33+17i$	$65+33i$	$129+65i$	$257+129i$
GH_{-n}	$3+2i$	$2+\frac{3}{2}i$	$\frac{3}{2}+\frac{5}{4}i$	$\frac{5}{4}+\frac{9}{8}i$	$\frac{9}{8}+\frac{17}{16}i$	$\frac{17}{16}+\frac{33}{32}i$	$\frac{33}{32}+\frac{65}{64}i$	$\frac{65}{64}+\frac{129}{128}i$
GR_n	$-1-\frac{3}{2}i$	$1-i$	$7+i$	$23+7i$	$63+23i$	$159+63i$	$383+159i$	$895+383i$
GR_{-n}	$-1-\frac{3}{2}i$	$-\frac{3}{2}-\frac{3}{2}i$	$-\frac{3}{2}-\frac{11}{8}i$	$\frac{11}{8}-\frac{5}{4}i$	$-\frac{5}{4}-\frac{37}{32}i$	$-\frac{37}{32}-\frac{35}{32}i$	$-\frac{35}{32}-\frac{135}{128}i$	$\frac{135}{128}-\frac{33}{32}i$
GC_n	$1+\frac{1}{2}i$	$3+i$	$9+3i$	$25+9i$	$65+25i$	$161+65i$	$385+161i$	$897+385i$
GC_{-n}	$1+\frac{1}{2}i$	$\frac{1}{2}+\frac{1}{2}i$	$\frac{1}{2}+\frac{5}{8}i$	$\frac{5}{8}+\frac{3}{4}i$	$\frac{3}{4}+\frac{27}{32}i$	$\frac{27}{32}+\frac{29}{32}i$	$\frac{29}{32}+\frac{121}{128}i$	$\frac{121}{128}+\frac{31}{32}i$

We now present the Binet formula for the Gaussian generalized Woodall numbers.

THEOREM 2.1. *The Binet's formula for the Gaussian generalized Woodall numbers is*
$GW_n = (((-W_2+4W_1-3W_0)+\frac{W_2-3W_1+2W_0}{2}n)2^n + (W_2-4W_1+4W_0)) + i(((-W_2+4W_1-3W_0)+\frac{W_2-3W_1+2W_0}{2}(n-1))2^{n-1} + (W_2-4W_1+4W_0)).$

Proof. The proof follows from (1.2) and (2.2). □

The previous Theorem gives the following results, as special cases.

COROLLARY 2.2. *For all n we have the following Binet's Formulas*

(a): $GG_n = i2^{n-1}(n-2) + 2^n(n-1) + 1 + i.$

(b): $GH_n = 2i2^{n-1} + 2 \times 2^n + 1 + i$.

(c): $GR_n = i2^{n-1}(n-1) + 2^n n - 1 - i$.

(d): $GC_n = i2^{n-1}(n-1) + 2^n n + 1 + i$.

The following Theorem presents the generating function of Gaussian generalized Woodall numbers.

THEOREM 2.3. *The generating function of Gaussian generalized Woodall numbers is given as*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 5GW_0)x + (GW_2 - 5GW_1 + 8GW_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{2.4}$$

Proof. Let

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$$

be generating function of Gaussian generalized Woodall numbers. Then using the definition of Gaussian Woodall numbers, and substracting $xf(x)$, $x^2 f(x)$ and $x^3 f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1 - 5x + 8x^2 - 4x^3) f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 5x \sum_{n=0}^{\infty} GW_n x^n + 8x^2 \sum_{n=0}^{\infty} GW_n x^n - 4x^3 \sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 5 \sum_{n=0}^{\infty} GW_n x^{n+1} + 8 \sum_{n=0}^{\infty} GW_n x^{n+2} - 4 \sum_{n=0}^{\infty} GW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 5 \sum_{n=1}^{\infty} GW_{n-1} x^n + 8 \sum_{n=2}^{\infty} GW_{n-2} x^n - 4 \sum_{n=3}^{\infty} GW_{n-3} x^n, \\
&= (GW_0 + GW_1 x + GW_2 x^2) - 5(GW_0 x + GW_1 x^2) + 8GW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (GW_n - 5GW_{n-1} + 8GW_{n-2} - 4GW_{n-3}) x^n, \\
&= GW_0 + GW_1 x + GW_2 x^2 - 5GW_0 x - 5GW_1 x^2 + 8GW_0 x^2, \\
&= GW_0 + (GW_1 - 5GW_0)x + (GW_2 - 5GW_1 + 8GW_0)x^2.
\end{aligned}$$

Now, it follows that

$$f_{GW_n}(x) = \frac{GW_0 + (GW_1 - 5GW_0)x + (GW_2 - 5GW_1 + 8GW_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

This completes the proof. □

The previous Theorem gives the following results as particular examples:

$$f_{GG_n}(x) = \frac{x + ix^2}{1 - 5x + 8x^2 - 4x^3}, \tag{2.5}$$

$$f_{GH_n}(x) = \frac{(8 + 6i)x^2 - (10 + 7i)x + 3 + 2i}{1 - 5x + 8x^2 - 4x^3}, \tag{2.6}$$

$$f_{GR_n}(x) = \frac{-(6 + 6i)x^2 + (6 + \frac{13}{2}i)x - 1 - \frac{3}{2}i}{1 - 5x + 8x^2 - 4x^3}, \tag{2.7}$$

$$f_{GC_n}(x) = \frac{(2+2i)\,x^2 - (2+\frac{3}{2}i)x + 1 + \frac{1}{2}i}{1 - 5x + 8x^2 - 4x^3}. \tag{2.8}$$

3. Some Identities Connecting Gaussian modified Woodall, Gaussian modified Cullen, Gaussian Woodall and Gaussian Cullen Numbers

In this section, we obtain some identities on Gaussian modified Woodall, Gaussian modified Cullen, Gaussian Woodall and Gaussian Cullen numbers.

THEOREM 3.1. *The following equations hold for all integer n.*

$$GH_n = 2GG_{n+2} - 7GG_{n+1} + 6GG_n, \tag{3.1}$$

$$GH_n = 3GG_{n+1} - 10GG_n + 8GG_{n-1}, \tag{3.2}$$

$$GR_n = -2GC_{n+2} + 8GC_{n+1} - 7GC_n,$$

$$GG_n = -\frac{1}{2}GC_{n+2} + \frac{3}{2}GC_{n+1}, \tag{3.3}$$

$$GC_n = -\frac{7}{4}GR_{n+3} + \frac{27}{4}GR_{n+2} - 6GR_{n+1}, \tag{3.4}$$

$$GH_n = -\frac{1}{2}GR_{n+3} + \frac{5}{2}GR_{n+2} - 3GR_{n+1}, \tag{3.5}$$

$$GH_n = 5GG_n - 16GG_{n-1} + 12GG_{n-2}. \tag{3.6}$$

Proof. To proof identity (3.1), we can write

$$GH_n = aGG_{n+2} + bGG_{n+1} + cGG_n$$

and solving the system of equations

$$\begin{aligned} GH_0 &= aGG_2 + bGG_1 + cGG_0, \\ GH_1 &= aGG_3 + bGG_2 + cGG_1, \\ GH_2 &= aGG_4 + bGG_3 + cGG_2. \end{aligned}$$

We find that $a = 2$, $b = -7$, $c = 6$. Or using the relations $GH_n = H_n + iH_{n-1}$, $GG_n = G_n + iG_{n-1}$ and identity $H_n = 2G_{n+2} - 7G_{n+1} + 6G_n$, we obtain the identity (3.1). The others can be found similarly. □

LEMMA 3.2. *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$*. Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are given as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The next Theorem presents the generating functions of even and odd-indexed Gaussian generalized Woodall sequences.

THEOREM 3.3. *The generating functions of the sequences GW_{2n} and GW_{2n+1} are given by*

$$f_{GW_{2n}}(x) = \frac{GW_0 - (9GW_0 - GW_2)x + (44GW_0 - 36GW_1 + 8GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

and

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (4GW_0 - 17GW_1 + 5GW_2)x + (32GW_0 - 20GW_1 + 4GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

respectively.

Proof. Both statements are consequences of Lemma (3.2) applied to (2.4) and some lengthy algebraic calculations. □

The previous theorem gives the following corollaries as particular examples.

COROLLARY 3.4. *We have the followings:*

(a): $f_{GR_{2n}}(x) = -\frac{(24+22i)x^2-(16+\frac{29}{2}i)x+1+\frac{3}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GR_{2n+1}} = (x)\frac{-(24+24i)x^2+(14+16i)x+1-i}{1-9x+24x^2-16x^3}$.

(b): $f_{GC_{2n}}(x) = \frac{(8+10i)x^2-\frac{3}{2}ix+1+\frac{1}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GC_{2n+1}}(x) = \frac{(8+8i)x^2-2x+3+i}{1-9x+24x^2-16x^3}$.

(c): $f_{GG_{2n}}(x) = \frac{(4+8i)x^2+(5+i)x}{1-9x+24x^2-16x^3}$ *and* $f_{GG_{2n+1}}(x) = \frac{4ix^2+(8+5i)x+1}{1-9x+24x^2-16x^3}$.

(d): $f_{GH_{2n}}(x) = \frac{(24+20i)x^2-(18+13i)x+3+2i}{1-9x+24x^2-16x^3}$ *and* $f_{GH_{2n+1}}(x) = \frac{(32+24i)x^2-(28+18i)x+5+3i}{1-9x+24x^2-16x^3}$.

From Corollary (3.4) we can obtain the following corollary which presents the identities on Gaussian Woodall sequences.

COROLLARY 3.5. *We have the following identities:*

(a): $(4+8i)\,GH_{2n-4} + (5+i)\,GH_{2n-2} = (24+20i)GG_{2n-4} - (18+13i)\,GG_{2n-2} + (3+2i)GG_{2n}$.

(b): $(4+8i)\,GH_{2n-3} + (5+i)\,GH_{2n-1} = (32+24i)\,GG_{2n-4} - (28+18i)\,GG_{2n-2} + (5+3i)GG_{2n}$.

(c): $-(24+24i)\,GG_{2n-4} + (14+16i)\,GG_{2n-2} + (1-i)GG_{2n} = (4+8i)GR_{2n-3} + (5+i)\,GR_{2n-1}$.

(d): $-(24+24i)\,GG_{2n-3} + (14+16i)\,GG_{2n-1} + (1-i)GG_{2n+1} = 4iGR_{2n-3} + (8+5i)\,GR_{2n-1} + GR_{2n+1}$.

(e): $(8+10i)\,GG_{2n-4} - \frac{3}{2}iGG_{2n-2} + (1+\frac{1}{2}i)GG_{2n} = (4+8i)GC_{2n-4} + (5+i)\,GC_{2n-2}$.

(f): $(8+10i)\,GG_{2n-3} - \frac{3}{2}iGG_{2n-1} + (1+\frac{1}{2}i)GG_{2n+1} = 4iGC_{2n-4} + (8+5i)\,GC_{2n-2} + GC_{2n}$.

(g): $(8+8i)\,GG_{2n-4} - 2GG_{2n-2} + (3+i)GG_{2n} = (4+8i)GC_{2n-3} + (5+i)\,GC_{2n-1}$.

(h): $(8+8i)GG_{2n-3} - 2GG_{2n-1} + (3+i)GG_{2n+1} = 4iGC_{2n-3} + (8+5i)\,GC_{2n-1} + GC_{2n+1}$.

(i): $-(24+22i)GG_{2n-4} + (16+\frac{29}{2}i)GG_{2n-2} - (1+\frac{3}{2}i)GG_{2n} = (4+8iGR_{2n-4} + (5+i)\,GR_{2n-2}$.

(j): $-(24+22i)GG_{2n-3} + (16+\frac{29}{2}i)GG_{2n-1} - (1+\frac{3}{2}i)GG_{2n+1} = 4iGR_{2n-4} + (8+5i)\,GR_{2n-2} + GR_{2n}$.

(k): $4iGH_{2n-4} + (8+5i)\,GH_{2n-2} + GH_{2n} = (24+20i)\,GG_{2n-3} - (18+13i)\,GG_{2n-1} + (3+2i)GG_{2n+1}$.

(l): $4iGH_{2n-3} + (8+5i)\,GH_{2n-1} + GH_{2n+1} = (32+24i)\,GG_{2n-3} - (28+18i)\,GG_{2n-1} + (5+3i)GG_{2n+1}$.

(m): $-(24+22i)C(2n-3) + \left(16+\frac{29}{2}i\right)C(2n-1) - (1+\frac{3}{2}i)C(2n+1) = (8+8i)\,R(2n-4) - 2R(2n-2) + (3+i)R(2n)$.

(n): $-(24+24i)\,C(2n-4)+(14+16i)\,C(2n-2)+(1-i)C(2n)=(8+10i)\,R(2n-3)-\frac{3}{2}iR(2n-1)+(1+\frac{1}{2}i)R(2n+1)$.

Proof. From (3.4) we obtain

$$((4+8i)\,x^2+(5+i)\,x)f_{GH_{2n}}=((24+20i)\,x^2-(18+13i)\,x+3+2i)f_{GG_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((5+i)\,x+(4+8i)\,x^2)\sum_{n=0}^{\infty}GH_{2n}x^n \\
&= (5+i)\,x\sum_{n=0}^{\infty}GH_{2n}x^n+(4+8i)\,x^2\sum_{n=0}^{\infty}GH_{2n}x^n \\
&= (5+i)\sum_{n=0}^{\infty}GH_{2n}x^{n+1}+(4+8i)\sum_{n=0}^{\infty}GH_{2n}x^{n+2} \\
&= (5+i)\sum_{n=1}^{\infty}GH_{2n-2}x^n+(4+8i)\sum_{n=2}^{\infty}GH_{2n-4}x^n \\
&= (5+i)\,GH_0x\sum_{n=2}^{\infty}GH_{2n-2}x^n+(4+8i)\sum_{n=2}^{\infty}GH_{2n-4}x^n \\
&= (5+i)\,(3+2i)x+\sum_{n=2}^{\infty}((4+8i)\,GH_{2n-4}+(5+i)\,GH_{2n-2})\,x^n
\end{aligned}$$

whereas the RHS is

$$\begin{aligned}
RHS &= (3+2i-(18+13i)\,x+(24+20i)\,x^2)\sum_{n=0}^{\infty}GG_{2n}x^n \\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\,x\sum_{n=0}^{\infty}GG_{2n}x^n+(24+20i)\,x^2\sum_{n=0}^{\infty}GG_{2n}x^n \\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\sum_{n=0}^{\infty}GG_{2n}x^{n+1}+(24+20i)\sum_{n=0}^{\infty}GG_{2n}x^{n+2} \\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\sum_{n=1}^{\infty}GG_{2n-2}x^n+(24+20i)\sum_{n=2}^{\infty}GG_{2n-4}x^n \\
&= (3+2i)(GG_0+GG_2x)\sum_{n=2}^{\infty}GG_{2n}x^n-(18+13i)\,(GG_0x)\sum_{n=2}^{\infty}GG_{2n-2}x^n \\
&\quad +(24+20i)\sum_{n=2}^{\infty}GG_{2n-4}x^n \\
&= (3+2i)\,(5+i)\,x+\sum_{n=2}^{\infty}((24+20i)GG_{2n-4}-(18+13i)\,GG_{2n-2}+(3+2i)GG_{2n})\,x^n.
\end{aligned}$$

Compare the coefficients and the proof of the first identity (a) is done. The other identities can be proved similarly. □

We present an identity related with Gaussian general Woodall numbers and Woodall numbers.

THEOREM 3.6. *For all $n, m \in \mathbb{Z}$, the following identity holds:*

$$GW_{m+n} = G_{m+1}GW_n + (-8G_m + 4G_{m-1})GW_{n-1} + 4G_mGW_{n-2}. \tag{3.7}$$

Proof. First, we assume that $m, n \geq 0$. The case $m, n < 0$ can be proved similarly. We prove the identity (3.7) by induction on m. If $m = 0$ then

$$GW_n = G_1GW_n + (-8G_0 + 4G_{-1})GW_{n-1} + 4G_0GW_{n-2}$$

which is true because $G_{-1} = 0$, $G_0 = 0$, $G_1 = 1$. Assume that the equaliy holds for $m \leq k$. For $m = k + 1$, we have

$$\begin{aligned}
GW_{(k+1)+n} &= 5GW_{n+k} - 8GW_{n+k-1} + 4GW_{n+k-2} \\
&= 5(G_{k+1}GW_n + (-8G_k + 4G_{k-1})GW_{n-1} + 4G_kGW_{n-2}) \\
&\quad -8(G_kGW_n + (-8G_{k-1} + 4G_{k-2})GW_{n-1} + 4G_{k-1}GW_{n-2}) \\
&\quad +4(G_{k-1}GW_n + (-8G_{k-2} + 4G_{k-3})GW_{n-1} + 4G_{k-2}GW_{n-2}) \\
&= (5G_{k+1} - 8G_k + 4G_{k-1})GW_n + (-8(G_k + G_{k-1} + G_{k-2}) \\
&\quad +4(G_{k-1} + G_{k-2} + G_{k-3}))GW_{n-1} + 4(G_k + G_{k-1} + G_{k-2})GW_{n-2} \\
&= G_{k+2}GW_n + (-8G_{k+1} + 4G_k)GW_{n-1} + 4G_{k+1}GW_{n-2} \\
&= G_{(k+1)+1}GW_n + (-8G_{k+1} + 4G_{(k+1)-1})GW_{n-1} + 4G_{k+1}GW_{n-2}.
\end{aligned}$$

By mathematical induction on m, this proves (3.6). The case $m, n < 0$ can be shown similarly. □

The previous Theorem gives the following results as particular examples:

For all $n, m \in \mathbb{Z}$, we have (taking $GW_n = GG_n$ or $GW_n = GH_n$ or $GW_n = GR_n$ or $GW_n = GC_n$)

$$\begin{aligned}
GG_{m+n} &= G_{m+1}GG_n + (-8G_m + 4G_{m-1})GG_{n-1} + 4G_mGG_{n-2}, \\
GH_{m+n} &= G_{m+1}GH_n + (-8G_m + 4G_{m-1})GH_{n-1} + 4G_mGH_{n-2}, \\
GR_{m+n} &= G_{m+1}GR_n + (-8G_m + 4G_{m-1})GR_{n-1} + 4G_mGR_{n-2}, \\
GC_{m+n} &= G_{m+1}GC_n + (-8G_m + 4G_{m-1})GC_{n-1} + 4G_mGC_{n-2}.
\end{aligned}$$

4. Simpson's Formula

In this chapter, we present Simpson's formula of generalized Gaussian Woodall numbers.

THEOREM 4.1. *(Simpson's formula of generalized Gaussian Woodall numbers). For all integers n, we have*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix}$$
$$= 4^n(\frac{1}{800}+\frac{7}{800}i)(W_0-W_1+\frac{1}{4}W_2)((2-14i)W_0-(3-21i)W_1 +(1-7i)W_2)^2.$$

Proof. Use [16,Theorem 3.1]. □

From the Theorem (4.1) we get the following corollary.

COROLLARY 4.2. *For all integer n, we get the following identities.*

(a): $\begin{vmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{vmatrix} = (1-7i)\,2^{2n-4}.$

(b): $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 0.$

(c): $\begin{vmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{vmatrix} = -(1-7i)\,2^{2n-4}.$

(d): $\begin{vmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{vmatrix} = (1-7i)\,2^{2n-4}.$

5. SUM FORMULAS

In this chapter, we give some sum formulas of generalized Gaussian Woodall numbers.

THEOREM 5.1. *For all integers $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} GW_k = \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9)+i(\frac{1}{4}(28+16n-5\times 2^{n+2}+2^{n+2}n)W_0+(-33-16n+7\times 2^{n+2}-3\times 2^{n+1}n)W_1+(9+4n-2^{n+3}+2^{n+1}n)W_2).$

(b): $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)+i(\frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)).$

(c): $\sum_{k=0}^{n} GW_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) + i((\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2((18n - 2^{2n+1}(2n-1) + 2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2}))$.

Proof.

(a): When we use (2.2),

$$\sum_{k=0}^{n} GW_k = \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1}.$$

So, then we obtain

$$\sum_{k=0}^{n} W_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9))$$

and

$$\sum_{k=0}^{n} W_{k-1} = (\frac{1}{4}((28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2))$$

from sum formulas on the Generalized Woodall Sequence article. We get

$$\sum_{k=0}^{n} GW_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9)) + i(\frac{1}{4}((28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2)).$$

(b): When we use (2.1), we obtain the following equalities: If we rearrange the above equalities, we obtain. Now, if we add the above equations by side by, we get

$$\sum_{k=0}^{n} GW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + i\sum_{k=0}^{n} W_{2k}$$

and so we know

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)$$

and

$$\sum_{k=0}^{n} W_{2k} = ((\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)).$$

We get

$$\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1)$$
$$+2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)$$
$$+i((\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2)$$
$$+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)).$$

(c): We know

$$\sum_{k=0}^{n} W_{2k} = ((\frac{1}{9}W_0(36n - 2^{2n+2}((2n-1) + 2^{2n+4}((2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2)$$
$$+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32))$$

and

$$\sum_{k=0}^{n} W_{2k-1} = (\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2(18n - 2^{2n+1}(2n-1)$$
$$+2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).$$

So we know

$$\sum_{k=0}^{n} GW_{2k} = \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1}.$$

We get

$$\sum_{k=0}^{n} GW_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2)$$
$$+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)$$
$$+i(\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2((18n - 2^{2n+1}(2n-1)$$
$$+2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).$$

This completes the proof. □

As special cases of above Theorem, we have the following four Corollary, we get the following corollary: First, taking $GW_n = GG_n$ with $GG_0 = 0, GG_1 = 1, GG_2 = 5 + i$.

COROLLARY 5.2. *(Sum of the Gaussian modified Woodall numbers). For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GG_k = (1+i)n + (2+i)2^n n - (4+3i)2^n + (4+3i)$.
(b): $\sum_{k=0}^{n} GG_{2k+1} = \frac{4}{9}((\frac{9}{4} + \frac{9}{4}i)n - (4+5i)2^{2n} + (12+6i)2^{2n}n + (\frac{25}{4} + 5i))$.
(c): $\sum_{k=0}^{n} GG_{2k} = \frac{4}{9}((\frac{9}{4} + \frac{9}{4}i)n - (5+4i)2^{2n} + (6+3i)2^{2n}n + (5+4i))$.

Second, taking $GW_n = GH_n$ with $GH_0 = 3 + 2i, GH_1 = 5 + 3i, GH_2 = 9 + 5i$. We have the following corollary:

COROLLARY 5.3. *(Sum of the Gaussian modified Cullen numbers). For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GH_k = 2^{n+2} + n - 1 + i(n + 2^{n+1})$.
(b): $\sum_{k=0}^{n} GH_{2k+1} = \frac{1}{3}(2^{2n+4} + 3n - 1) + i(\frac{1}{3}(2^{2n+3} + 3n + 1))$.
(c): $\sum_{k=0}^{n} GH_{2k} = \frac{1}{3}(2^{2n+3} + 3n + 1) + i(n + \frac{1}{3}2^{2n+2} + \frac{2}{3})$.

Third, taking $GW_n = GR_n$ with $GR_0 = -1 - \frac{3}{2}i, GR_1 = 1 - i, GR_2 = 7 + i$. We get the following corollary:

COROLLARY 5.4. *(Sum of the Gaussian Woodall numbers). For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GR_k = (n-1)(2^{n+1} - 1) + i(2^{n+1}(n-1) - n - 2^n n + \frac{1}{2})$.
(b): $\sum_{k=0}^{n} GR_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} - 9n + 1) + i(\frac{1}{9}((3n-1)2^{2n+3} - 9n - 1))$.
(c): $\sum_{k=0}^{n} GR_{2k} = \frac{1}{9}((3n-1)2^{2n+3} - 9n - 1) + i(\frac{1}{9}2^{2n+3}(2n-1) - \frac{1}{9}2^{2n+1}(2n+1) - n - \frac{7}{18})$.

Fourth, taking $GW_n = GC_n$ with $GC_0 = 1 + \frac{1}{2}i, GC_1 = 3 + i, GC_2 = 9 + 3i$. We have the following corollary:

COROLLARY 5.5. *(Sum of the Gaussian Cullen numbers). For $n \geq 1$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GC_k = (n-1)2^{n+1} + n + 3 + i(n + 2^{n+1}(n-1) - 2^n n + \frac{5}{2})$.
(b): $\sum_{k=0}^{n} GC_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} + 9n + 19) + i(\frac{1}{9}((3n-1)2^{2n+3} + 9n + 17))$.
(c): $\sum_{k=0}^{n} GC_{2k} = \frac{2}{9}((\frac{9}{2} + \frac{9}{2}i)n - (4+5i)2^{2n} + (12+6i)2^{2n}n + (\frac{17}{2} + \frac{29}{4}i))$.

6. Matrix Formulation of GW_n

Consider the sequence $\{G_n\}$ which is defined by the third-order recurrence relation

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}$$

with the initial conditions

$$G_0 = 0, G_1 = 1, G_2 = 5.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. We give the following Lemma.

LEMMA 6.1. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. The Lemma (6.1) equality can be proved by strong induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n \leq k$. So that the following identity is true.

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} \\ &= \begin{pmatrix} 5GW_{k+2} - 8GW_{k+1} + 4GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by induction on n, the proof is finished. □

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix}.$$

For the proof see [17].

THEOREM 6.2. *We assume that the matrices N_{GW} and E_{GW} are defined as follows*

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix},$$

$$E_{GW} = \begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}.$$

The following identity is true between A^n, N_{GW} and E_{GW}.

$$A^n N_{GW} = E_{GW}.$$

Proof. Note that one gets

$$\begin{aligned} A^n N_{GW} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

such that

$$\begin{aligned} a_{11} &= GW_2 G_{n+1} + GW_1 (4G_{n-1} - 8G_n) + GW_0 G_n, \\ a_{12} &= GW_1 G_{n+1} + GW_0 (4G_{n-1} - 8G_n) + GW_{-1} G_n, \\ a_{13} &= GW_0 G_{n+1} + GW_{-1} (4G_{n-1} - 8G_n) + GW_{-2} G_n, \\ a_{21} &= GW_2 G_n + GW_1 (4G_{n-2} - 8G_{n-1}) + GW_0 G_{n-1}, \\ a_{22} &= GW_1 G_n + GW_0 (4G_{n-2} - 8G_{n-1}) + GW_{-1} G_{n-1}, \\ a_{23} &= GW_0 G_n + GW_{-1} (4G_{n-2} - 8G_{n-1}) + GW_{-2} G_{n-1}, \\ a_{31} &= GW_2 G_{n-1} + GW_1 (4G_{n-3} - 8G_{n-2}) + GW_0 G_{n-2}, \\ a_{32} &= GW_1 G_{n-1} + GW_0 (4G_{n-3} - 8G_{n-2}) + GW_{-1} G_{n-2}, \\ a_{33} &= GW_0 G_{n-1} + GW_{-1} (4G_{n-3} - 8G_{n-2}) + GW_{-2} G_{n-2}. \end{aligned}$$

Using the Theorem (3.6) the proof is completed. □

We have the following identities for N_{GW}, E_{GW} :

$$N_{GG}=\begin{pmatrix} 5+i & 1 & 0 \\ 1 & 0 & \frac{1}{4}i \\ 0 & \frac{1}{4}i & \frac{1}{4}+\frac{1}{2}i \end{pmatrix}, \quad N_{GH}=\begin{pmatrix} 9+5i & 5+3i & 3+2i \\ 5+3i & 3+2i & 2+\frac{3}{2}i \\ 3+2i & 2+\frac{3}{2}i & \frac{3}{2}+\frac{5}{4}i \end{pmatrix},$$

$$N_{GR}=\begin{pmatrix} 7+i & 1-i & -1-\frac{3}{2}i \\ 1-i & -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i \\ -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i & -\frac{3}{2}-\frac{11}{8}i \end{pmatrix}, \quad N_{GC}=\begin{pmatrix} 9+3i & 3+i & 1+\frac{1}{2}i \\ 3+i & 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}+\frac{5}{8}i \end{pmatrix}.$$

and

$$E_{GG}=\begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}, \quad E_{GH}=\begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix},$$

$$E_{GR}=\begin{pmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{pmatrix}, \quad E_{GC}=\begin{pmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{pmatrix}.$$

From the previous theorem presents, we have the following corollary.

Corollary 6.3. *The following identities are true:*

(a): $A^n N_{GG} = E_{GG}$.

(b): $A^n N_{GH} = E_{GH}$.

(c): $A^n N_{GR} = E_{GR}$.

(d): $A^n N_{GC} = E_{GC}$.

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