0.4pt1pt

Counting spanning trees in the ladders

Original Research Article

Abstract

Aims: In order to study the rule of entropy in different graph transformations, this paper introduces and further generalizes the ladder graph from different perspectives, including pinwheel ladders and 3-dimensional ladders. Then the problem of entropy comes down to the calculation of spanning trees in these graphs.

Place and Duration of Study: School of Mathematical Sciences, Jiangsu University, between January 2023 and October 2023.

Methodology: A method by seeking similar terms was generalized to help study the essential principle in these graphs. By similarity diagonalization and computer aided calculation, the exact expressions for the number of spanning trees in these graphs are obtained.

Keywords: spanning tree; recursive system of equations; family of Ladders; pinwheel ladders; 3-dimensional ladders; entropy.

2010 Mathematics Subject Classification:05C05,05C63

1 Introduction

Spanning tree is an important concept in many fields, including Computer scienceand Fractal geometry[1]. In the meantime, it has a deep relationship with many aspects of network, such as transportation[2], reliability[3], random walks[4], and optimal Synchronization[5]. Furthermore, the problem of counting spanning trees has been associated with other interesting problems, including origin of fractality in complex networks[6], Potts model[7] and dimer coverings[8].

The problem of counting spanning trees was first proposed by Kirchhoff[9] in his analysis of electric circuits. His matrix-tree theorem can calculate the number of spanning trees in any connected graph. However, due to the diversity and complexity of some graphs, the work of counting spanning trees is still a tremendous challenge. Till now, many other efficient methods have been studied in the calculation of spanning trees, such as taxonomic discussion of tree species[10] and deleting edges[11].

Abundant literature exists on the property of the Ladder graph, such as subdivision[12], split domination[13] and planarity[14]. The number of spanning trees in this graph has been solved by Haghighi[11]. Some generalized forms of Ladder graph have been posed by kwun et al.[15], and they studied the Tutte Polynomial in these graphs.

In this paper, the Ladder graph is introduced firstly and a method of seeking similar terms is generalized to count the number of spanning trees in Ladder type graphs. Then, some Ladder-type graphs including pinwheel ladders and 3-dimensional ladders are introduced. Finally, the entropy of these graphs are discussed based on their spanning tree counting formulas.

2 Method to count spanning trees

In this section, a well-known formula will be introduced firstly, which can be found in Ref.[16]. By taking the ladder graph as an example, a method of seeking similar terms is posed. Furthermore, this method is compared with that of Haghighi[11], finding that they have different forms but same essence of order reduction.

The graph obtained from G by deleting the edge e is denoted by $G-e$. If an edge e of a graph G is deleted and its ends are identified, it is said to be contracted and the resulting graph is defined as $G.e$. $t(G)$ denotes the number of spanning trees in G. Then we have

Proposition 2.1. *Let G be a graph, then for any edge e*

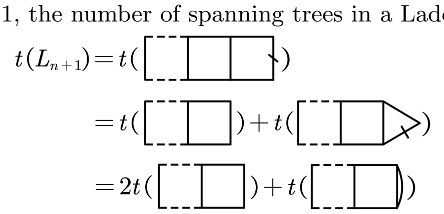
$$t(G) = t(G-e) + t(G.e) \tag{2.1}$$

$G\square H$ denotes the cartesian product of graphs G and H, whose vertex set is $V(G)\times V(H)$ and whose edge set contains all pairs $(u_1,v_1)(u_2,v_2)$, such that either $v_1v_2 \in E(H)$ and $u_1 = u_2$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$. P_n represents a path with n edges.

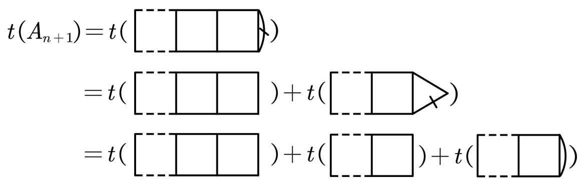
The ladder graph L_n is the cartesian product $P_2\square P_n$.

In Ref.[11], Haghighi et al. obtained the recursive relation of the spanning tree number in the ladders: $t(L_n) = 4t(L_{n-1}) - t(L_{n-2})$, thereby deriving the exact expression for the spanning trees. However, this expression can also be obtained by seeking similar terms and solving recursive system of equations.

By applying Theorem 2.1, the number of spanning trees in a Ladder can be writen by:



Define the graph which has one more edge than L_n on the right as A_n. Then the number of spanning trees in A_{n+1} can be given by:



Therefore, the recursive system of L_n and A_n are found:

$$\begin{cases} t(L_{n+1}) = 2t(L_n) + t(A_n) \\ t(A_{n+1}) = 3t(L_n) + 2t(A_n) \end{cases} \tag{2.2}$$

This system can be writen by the form of matrix:

$$\begin{pmatrix} t(L_{n+1}) \\ t(A_{n+1}) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} t(L_n) \\ t(A_n) \end{pmatrix} \tag{2.3}$$

The digital matrix in Eq.2.3 is called the transition matrix T. By a finite number of recursions, $t(L_n)$ can be calculated by the following equation:

$$\begin{pmatrix} t(L_n) \\ t(A_n) \end{pmatrix} = T^n \begin{pmatrix} t(L_0) \\ t(A_0) \end{pmatrix} \tag{2.4}$$

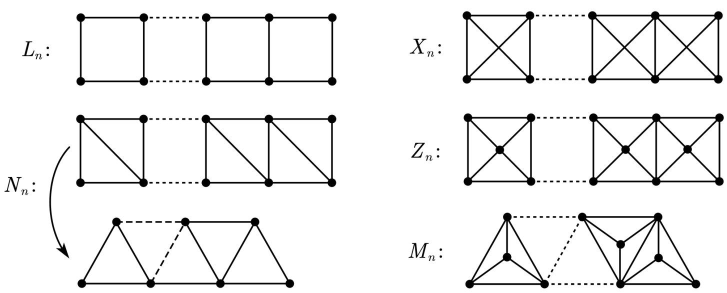


Figure 1: The family of ladder-type graphs.

Then, the problem is reduced to calculating T^n, and this can be solved by similarity diagonalization. Denote the eigenvalues and eigenvectors of a matrix by λ and η, respectively. The matrix T can be converted similarity diagonally to a diagonal matrix H:

$$T = P^{-1}HP \tag{2.5}$$

where matrix $P = (\eta_1, \eta_2)$, matrix $H = diag(\lambda_1, \lambda_2)$ and P^{-1} is the inverse of P.

Expand the matrix T^n and apply the Eq.2.5, we have:

$$T^n = P^{-1}HPP^{-1}HP \dots P^{-1}HP = P^{-1}H^nP \tag{2.6}$$

Finally, Eq.2.4 can be written as the following equation:

$$\begin{pmatrix} t(L_n) \\ t(A_n) \end{pmatrix} = P^{-1} \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} P \begin{pmatrix} t(L_0) \\ t(A_0) \end{pmatrix} \tag{2.7}$$

Since the eigenvalues of T are

$$\lambda_1 = 2 + \sqrt{3}, \lambda_2 = 2 - \sqrt{3} \tag{2.8}$$

and the eigenvectors of T are $\eta_1 = (\frac{1}{\sqrt{3}}, 1)^T$ and $\eta_2 = (-\frac{1}{\sqrt{3}}, 1)^T$, with the initial condition $t(L_0) = 1$ and $t(A_0) = 2$, the exact expression of spanning trees can be derived:

$$t(L_n) = \frac{\sqrt{3}}{6}((2 + \sqrt{3})^n - (2 - \sqrt{3})^n) \tag{2.9}$$

which is identical with the result in Ref.[11].

Although two methods are all related to order reduction, they have different forms of equations. Our method is the generalized form and the core is to seek similar terms during the process of order reduction. This generalized method can help solve a class of graph where the recursive relation of $t(G)$ can't be derived in an equation.

3 Some Ladder-type graphs

In this section, the family of ladder-type graphs will be discussed. Among them, some members have been posed by Kwun et al.[15] when they studied the Tutte Polynomial. By using the method of seeking similar terms, the numbers of spanning trees in these ladder-type graphs are obtained. Let us first introduce this large family, as illustrated in Fig.1.

N_n denotes the n-order ladder whose lattice has a diagonal. It is not difficult to find that N_n can be broken down into triangles. If L_n is called a square ladder, then N_n is a triangular ladder. X_n is a ladder which has n lattices and each of its lattices has two diagonals. Z_n is the n-order ladder whose lattice has one more vertice than that of X_n. The construction of M_n is similar to X_n, for they all have a vertice in each lattice and the internal vertex is connected with lattices.

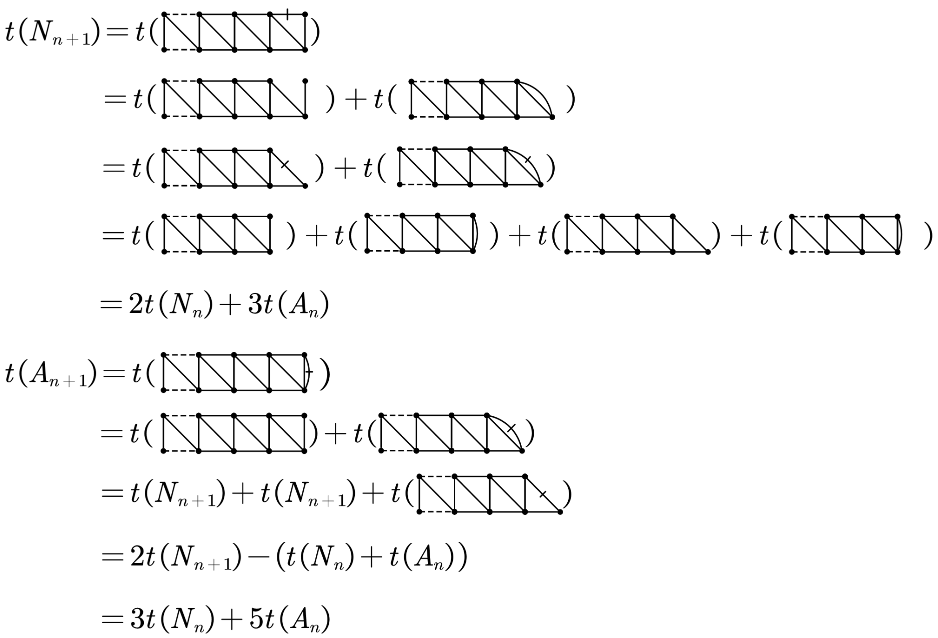
3.1 Spanning trees in N_n

Similarly, A_n denotes the graph which has one more edge than N_n on the right. By applying Theorem 2.1, the recursive system in N_n can be obtained:

Proposition 3.1. *In the ladder-type graph N, the number of spanning trees $t(N_n)$ satisfies the recursive system of equations:*

$$\begin{pmatrix} t(N_{n+1}) \\ t(A_{n+1}) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} t(N_n) \\ t(A_n) \end{pmatrix} \tag{3.1}$$

Proof. The process of reducing order and seeking base entries is given below:



□

By computation, the eigenvalues of the transition matrix in Theorem 3.1 are $\lambda_1 = \frac{7-3\sqrt{5}}{2}$ and $\lambda_2 = \frac{7+3\sqrt{5}}{2}$, the eigenvectors are $\eta_1 = (1+\sqrt{5}, 2)^T$ and $\eta_2 = (\sqrt{5}-1, 2)^T$, and the initial condition is $t(L_0) = 1$ and $t(A_0) = 2$. Applying the method of similarity diagonalization in Section 2, the recursive system can be given by:

$$\begin{pmatrix} t(N_n) \\ t(A_n) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1-\sqrt{5}}{4} \\ -\frac{1}{2} & \frac{1+\sqrt{5}}{4} \end{pmatrix} \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1+\sqrt{5} & \sqrt{5}-1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.2}$$

Finally, the exact expression for spanning trees in N_n is derived:

$$t(N_n) = \frac{-1+3\sqrt{5}}{2} \left(\frac{7-3\sqrt{5}}{2}\right)^n + \frac{3-3\sqrt{5}}{2} \left(\frac{7+3\sqrt{5}}{2}\right)^n \tag{3.3}$$

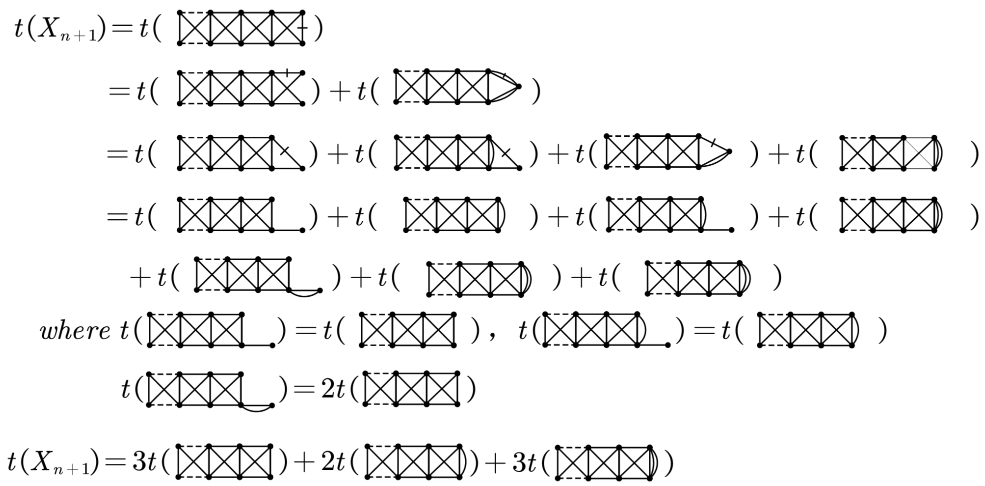
3.2 Spanning trees in X_n

A_n denotes the graph which has one more edge than X_n on the right. B_n denotes the graph which has two more edges than X_n on the right. By deleting the edges of X, the recursive system in X_n can be obtained:

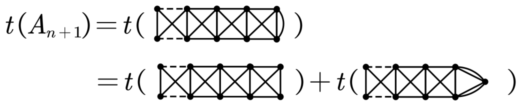
Proposition 3.2. *In the ladder-type graph X, the number of spanning trees $t(X_n)$ satisfies the recursive system of equations:*

$$\begin{pmatrix} t(X_{n+1}) \\ t(A_{n+1}) \\ t(B_{n+1}) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 \\ 5 & 2 & 5 \\ 7 & 2 & 7 \end{pmatrix} \begin{pmatrix} t(X_n) \\ t(A_n) \\ t(B_n) \end{pmatrix} \tag{3.4}$$

Proof. The detail of the proof is given below:

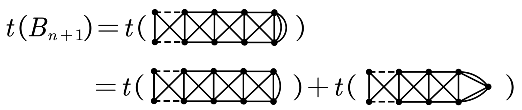


$$= 3t(X_n) + 2t(A_n) + 3t(B_n)$$



$$= t(X_{n+1}) + 2t(X_n) + 2t(B_n)$$

$$= 5t(X_n) + 2t(A_n) + 5t(B_n)$$



$$= t(A_{n+1}) + 2t(X_n) + 2t(B_n)$$

$$= 7t(X_n) + 2t(A_n) + 7t(B_n)$$

□

The characteristic polynomial of the transition matrix T in Theorem 3.2 is

$$\lambda^3 - 12\lambda^2 = 0 \tag{3.5}$$

According to Hamilton-Cayley Theorem, the matrix T will satisfy the equation as follows:

$$T^3 - 12T^2 = 0 \tag{3.6}$$

Therefore,

$$T^n = 12T^{n-1} = \cdots = 12^{n-2}T^2 = 12^{n-2}\begin{pmatrix} 40 & 16 & 40 \\ 60 & 24 & 60 \\ 80 & 32 & 80 \end{pmatrix} \tag{3.7}$$

Then, Eq.(3.4) can be simplified into:

$$\begin{pmatrix} t(X_n) \\ t(A_n) \\ t(B_n) \end{pmatrix} = T^n \begin{pmatrix} t(X_0) \\ t(A_0) \\ t(B_0) \end{pmatrix} = 12^{n-2}\begin{pmatrix} 40 & 16 & 40 \\ 60 & 24 & 60 \\ 80 & 32 & 80 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{3.8}$$

Finally, the exact expression for spanning trees in X_n is derived:

$$t(X_n) = 16 \cdot 12^{n-1} \tag{3.9}$$

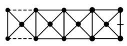
3.3 Spanning trees in Z_n

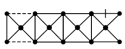
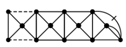
A_n denotes the graph which has one more edge than Z_n on the right. B_n denotes the graph which has two more edges than Z_n on the right. C_n denotes the graph which has four more edges than Z_n on the right. Through the operation of edge deletion and edge contraction, the recursive system in Z_n can be obtained:

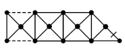
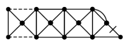
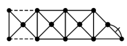
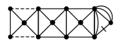
Proposition 3.3. *In the ladder-type graph Z, the number of spanning trees* $t(Z_n)$ *satisfies the recursive system of equations:*

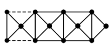
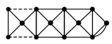
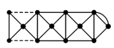
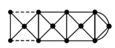
$$\begin{pmatrix} t(Z_{n+1}) \\ t(A_{n+1}) \\ t(B_{n+1}) \\ t(C_{n+1}) \end{pmatrix} = \begin{pmatrix} 12 & 5 & 6 & 1 \\ 17 & 9 & 8 & 2 \\ 22 & 13 & 10 & 3 \\ 32 & 21 & 14 & 5 \end{pmatrix}\begin{pmatrix} t(Z_n) \\ t(A_n) \\ t(B_n) \\ t(C_n) \end{pmatrix} \tag{3.10}$$

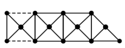
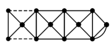
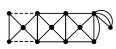
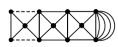
Proof. Similarly, the process of reducing order and seeking base entries is given below:

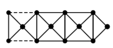
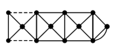
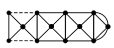
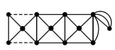
$t(Z_{n+1}) = t($  $)$

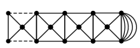
$= t($  $) + t($  $)$

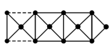
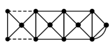
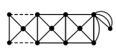
$= t($  $) + t($  $) + t($  $) + t($  $)$

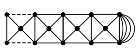
$= t($  $) + t($  $) + t($  $) + t($  $)$

$+ t($  $) + t($  $) + t($  $) + t($  $)$

$= 2t($  $) + 4t($  $) + t($  $) + t($  $)$

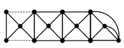
$+ t($  $)$

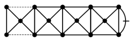
$= 2t($  $) + 5t($  $) + t(B_n\ ) + t($  $)$

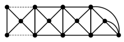
$+ t($  $)$

$= 2(t(Z_n + A_n)) + 5(2t(Z_n) + t(B_n)) + t(B_n) + 3t(A_n) + t(C_n)$

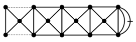
$= 12t(Z_n) + 5t(A_n) + 6t(B_n) + t(C_n)$

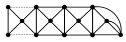
where $t($  $) = 5t(Z_n) + 4t(A_n) + 2t(B_n) + t(C_n)$

$t(A_{n+1}) = t($  $)$

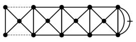
$= t(Z_{n+1}) + t($  $)$

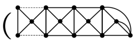
$= 17t(Z_n) + 9t(A_n) + 8t(B_n) + 2t(C_n)$

$t(B_{n+1}) = t($  $)$

$= t(A_{n+1}) + t($  $)$

$= 22t(Z_n) + 13t(A_{n-1}) + 10t(B_n) + 3t(C_n)$

$t(C_{n+1}) = t($  $)$

$= t(B_{n+1}) + 2t($  $)$

$= 32t(Z_n) + 21t(A_{n-1}) + 14t(B_n) + 5t(C_n)$

□

By computer aided calculation, the eigenvalues of the transition matrix T in Theorem 3.3 are

$$\lambda_1 = 6(3+2\sqrt{2}), \lambda_2 = 6(3-2\sqrt{2}), \lambda_3 = 0, \lambda_4 = 0 \tag{3.11}$$

The eigenvectors are

$$\eta_1 = \begin{pmatrix} \frac{369+262\sqrt{2}}{1145+806\sqrt{2}} \\ \frac{563+389\sqrt{2}}{1145+806\sqrt{2}} \\ \frac{757+534\sqrt{2}}{1145+806\sqrt{2}} \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} \frac{369-262\sqrt{2}}{1145-806\sqrt{2}} \\ \frac{563-389\sqrt{2}}{1145-806\sqrt{2}} \\ \frac{757-534\sqrt{2}}{1145-806\sqrt{2}} \\ 1 \end{pmatrix}, \eta_3 = \begin{pmatrix} 1 \\ -7 \\ 0 \\ 23 \end{pmatrix}, \eta_4 = \begin{pmatrix} -14 \\ 6 \\ 23 \\ 0 \end{pmatrix}$$

With the initial condition: $t(Z_0) = 1$, $t(A_0) = 2$, $t(B_0) = 3$ and $t(C_0) = 5$, by similarity diagonalization again, the recursive system can be given by:

$$\begin{pmatrix} t(Z_n) \\ t(A_n) \\ t(B_n) \\ t(C_n) \end{pmatrix} = P^{-1} \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & 0 & \\ & & & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} \tag{3.12}$$

Finally, the exact expression for spanning trees in Z_n can be derived:

$$t(Z_n) = \frac{91904 - 147558\sqrt{2}}{1752}(6(3+2\sqrt{2}))^n + \frac{3(9676 - 4801\sqrt{2})}{1752}(6(3-2\sqrt{2}))^n \tag{3.13}$$

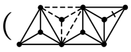
3.4 Spanning trees in M_n

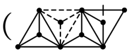
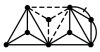
Similarly, A_n denotes the graph which has one more edge than M_n, B_n denotes the graph which has two more edges than M_n. By order reduction, the recursive system in M_n can be obtained:

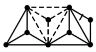
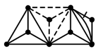
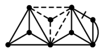
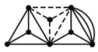
Proposition 3.4. *In the ladder-type graph M, the number of spanning trees $t(M_n)$ satisfies the recursive system of equations:*

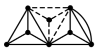
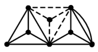
$$\begin{pmatrix} t(M_n) \\ t(A_n) \\ t(B_n) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix} \begin{pmatrix} t(M_{n-1}) \\ t(A_{n-1}) \\ t(B_{n-1}) \end{pmatrix} \tag{3.14}$$

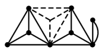
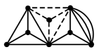
Proof. The process of edge deletion and edge contraction is given below:

$t(M_n) = t($  $)$

$= t($  $) + t($  $)$

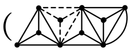
$= t($  $) + t($  $) + t($  $) + t($  $)$

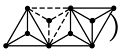
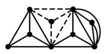
$= t($  $) + t($  $) + t($  $) + t($  $)$

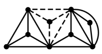
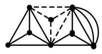
$+ t($  $) + t($  $) + t($  $)$

$$= t(M_{n-1}) + 2t(A_{n-1}) + t(B_{n-1}) + 2t(M_{n-1}) + 2t(B_{n-1})$$

$$= 3t(M_{n-1}) + 2t(A_{n-1}) + 3t(B_{n-1})$$

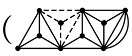
$t(A_n) = t($  $)$

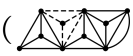
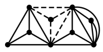
$= t($  $) + t($  $)$

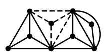
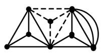
$= t(M_n) + t($  $) + t($  $)$

$$= t(M_n) + t(A_{n-1}) + 2t(B_{n-1})$$

$$= 3t(M_{n-1}) + 3t(A_{n-1}) + 5t(B_{n-1})$$

$t(B_n) = t($  $)$

$= t($  $) + t($  $)$

$= t(A_n) + t($  $) + t($  $)$

$$= t(A_n) + t(A_{n-1}) + 2t(B_{n-1})$$

$$= 3t(M_{n-1}) + 4t(A_{n-1}) + 7t(B_{n-1})$$

□

By the calculation on computer, the eigenvalues of the digital matrix T in Theorem 3.4 are

$$\lambda_1 = \frac{13+\sqrt{105}}{2}, \lambda_2 = \frac{13-\sqrt{105}}{2}, \lambda_3 = 0$$

The eigenvectors are

$$\eta_1 = \begin{pmatrix} \frac{3+\sqrt{105}}{15+\sqrt{105}} \\ \frac{9+\sqrt{105}}{15+\sqrt{105}} \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} \frac{3+\sqrt{105}}{-15+\sqrt{105}} \\ \frac{9+\sqrt{105}}{-15+\sqrt{105}} \\ 1 \end{pmatrix}, \eta_3 = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$$

With the initial condition: $t(M_0) = 1$, $t(A_0) = 2$ and $t(B_0) = 3$, the recursive system can be given by:

$$\begin{pmatrix} t(M_n) \\ t(A_n) \\ t(B_n) \end{pmatrix} = P^{-1} \begin{pmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{3.15}$$

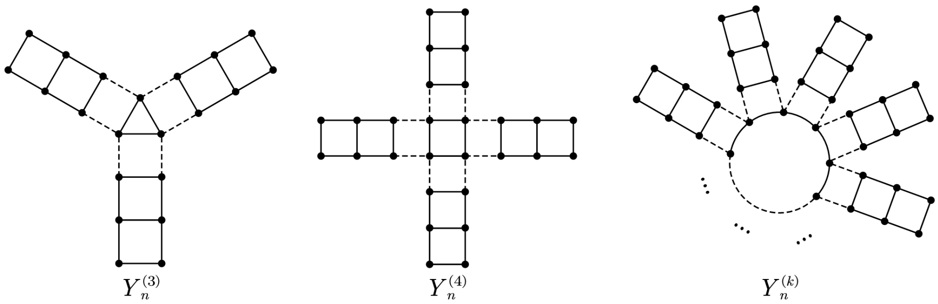


Figure 2: Pinwheel ladders.

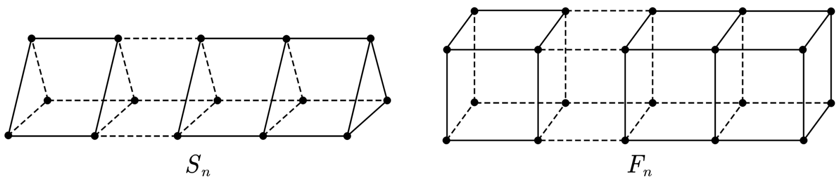


Figure 3: 3-dimensional ladders.

Therefore, the exact expression for spanning trees in M_n can be obtained:

$$t(M_n) = \frac{-315+39\sqrt{105}}{560}(\frac{13+\sqrt{105}}{2})^n - \frac{-1855+149\sqrt{105}}{560}(\frac{13-\sqrt{105}}{2})^n \tag{3.16}$$

4 Further generalized Ladders

In this section, the ladder graphs are further generalized from different angles. By enclosing a certain amount of ordinary ladders, a pinwheel ladder is created, as illustrated in Fig.2. $Y_n^{(k)}$ denotes the n-order pinwheel graph which is made up of number k of ladders. Otherwise, 3-dimensional ladders can be constructed by connecting ladders of different planes, such as triangular prism S_n and quadrilateral prism F_n, as illustrated in Fig.3. Based on the method in Section 2, the spanning trees in these graphs are discussed.

4.1 pinwheel ladders

By using permutation and combination, the spanning trees in Y_n^3 are studied and its exact counting formula is derived. Based on the work in Y_n^3, the law of $Y_n^{(k)}$ is then summerized.

Let us first introduce two special similar terms of the ladders, which is called Bald and Hat by us. Hat has one more outmost edge than Bald. The recursive relations between Bald and Hat are illustrated in Fig.4.

A_n denotes the n-order pinwheel ladder which has one Hat and two Balds. B_n denotes the n-order pinwheel ladder which has two Hats and one Bald. C_n denotes the n-order pinwheel ladder which has three Hats.

Based on the formula of edge deleting, it is not hard to obtain the recursive relation of $t(Y_n)$:

$$t(Y_n^{(3)}) = 8t(Y_{n-1}^{(3)}) + 12t(A_{n-1}) + 6t(B_{n-1}) + t(C_{n-1}) \tag{4.1}$$

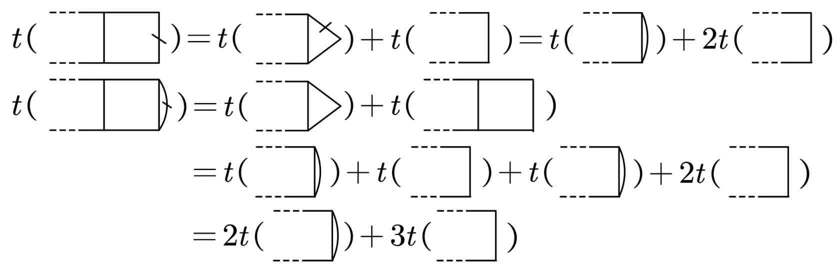
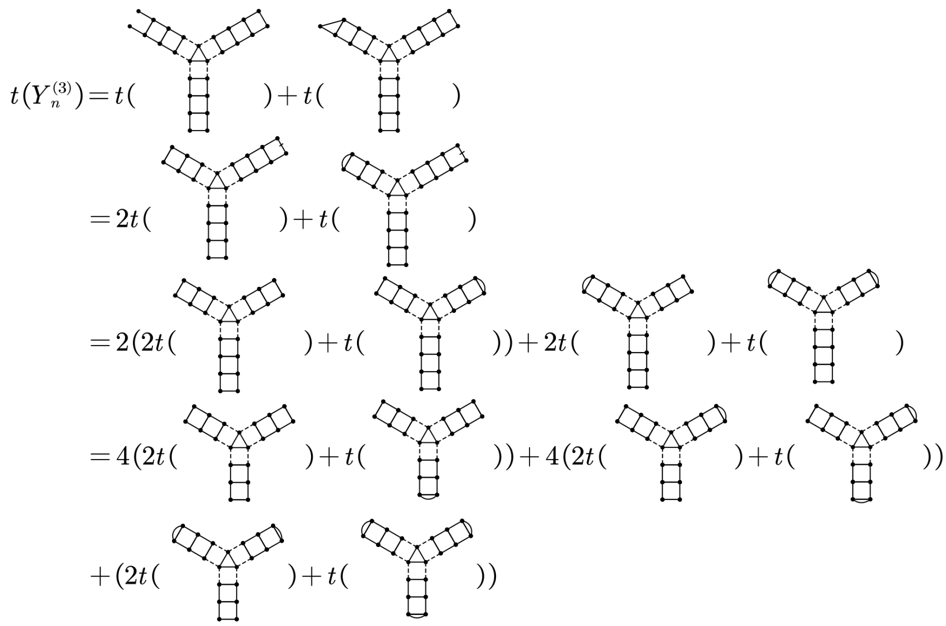


Figure 4: Recursive relations between Bald and Hat.

Detailed procedure is given below:



$$= 8t(Y^{(3)}_{n-1}) + 12t(A_{n-1}) + 6t(B_{n-1}) + t(C_{n-1})$$

Eq.(4.1) can be writen by the form as below:

$$t(Y_n^{(3)}) = 2^3 t(Y_{n-1}^{(3)}) + C_3^2 2^2 t(A_{n-1}) + C_3^1 2t(B_{n-1}) + t(C_{n-1}) \tag{4.2}$$

All the coefficients in this recursive relation can be described by permutation and combination theory. Just take the first coefficient 8 for example. There are three n-order Balds, and all of them can be reduced to two Balds of order $(n-1)$ and one Hat of order $(n-1)$. Recombination of these Balds and Hats results in an pinwheel ladder of order $(n-1)$. In order for the combination to be Y_{n-1}, the Bald part in each reduction should be elected, leading to the coefficient of Y_{n-1} being 2^3.

By applying permutation and combination theory, the following recursive relation can also be obtained:

$$
\begin{aligned}
t(A_n) =& C_2^0 1^0 2^2 C_1^0 2^0 3^1 t(Y_{n-1}^{(3)}) + \left(C_2^1 1^1 2^1 C_1^0 2^0 3^1 + C_2^0 1^0 2^2 C_1^1 2^1 3^0\right) t(A_{n-1}) \\
&+ \left(C_2^1 1^1 2^1 C_1^1 2^1 3^0 + C_2^2 1^2 2^0 C_1^0 2^0 3^1\right) t(B_{n-1}) + C_2^2 1^2 2^0 C_1^1 2^1 3^0 t(C_{n-1}) \\
=& 12t(Y_{n-1}^{(3)}) + 20t(A_{n-1}) + 11t(B_{n-1}) + 2t(C_{n-1})
\end{aligned} \tag{4.3}
$$

As in former recursive relation, the coefficients can be analyzed by permutation and combination. We mainly analyze the second coefficient and omit the details of other coefficients. In order for the combination to be A_{n-1}, we should choose two Balds and one Hat during the process of order reduction. There are two cases: the Hat of order $n-1$ comes from Bald of order n or comes from Hat of order n. If it is the former, the Hat of order $(n-1)$ has to choose one of two Balds of order n. If it is the latter, the Hat of order $(n-1)$ should choose the Hat of order n.

Similarly, the recursive relations of $t(B_n)$ and $t(C_n)$ can be calculated:

$$
\begin{aligned}
t(B_n) =& C_1^0 1^0 2^1 C_2^0 2^0 3^2 t(Y_{n-1}^{(3)}) + \left(C_1^0 1^0 2^1 C_2^1 2^1 3^1 + C_1^1 1^1 2^0 C_2^0 2^0 3^2\right) t(A_{n-1}) \\
&+ \left(C_1^0 1^0 2^1 C_2^2 2^2 3^0 + C_1^1 1^1 2^0 C_2^1 2^1 3^1\right) t(B_{n-1}) + C_1^1 1^1 2^0 C_2^2 2^2 3^0 t(C_{n-1}) \\
=& 18t(Y_{n-1}^{(3)}) + 33t(A_{n-1}) + 20t(B_{n-1}) + 4t(C_{n-1})
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
t(C_n) =& C_3^0 2^0 3^3 t(Y_{n-1}^{(3)}) + C_3^1 2^1 3^2 t(A_{n-1}) + C_3^2 2^2 3 t(B_{n-1}) + C_3^3 2^3 3^0 t(C_{n-1}) \\
=& 27t(Y_{n-1}^{(3)}) + 54t(A_{n-1}) + 36t(B_{n-1}) + 8t(C_{n-1})
\end{aligned} \tag{4.5}
$$

Combining these recursive equations, we have:

Proposition 4.1. *In the pinwheel ladder $Y^{(3)}$, the number of spanning trees $t(Y_n^{(3)})$ satisfies the recursive system of equations:*

$$
\begin{pmatrix} t(Y_n^{(3)}) \\ t(A_n) \\ t(B_n) \\ t(C_n) \end{pmatrix} = \begin{pmatrix} 8 & 12 & 6 & 1 \\ 12 & 20 & 11 & 2 \\ 18 & 33 & 20 & 4 \\ 27 & 54 & 36 & 8 \end{pmatrix} \begin{pmatrix} t(Y_{n-1}^{(3)}) \\ t(A_{n-1}) \\ t(B_{n-1}) \\ t(C_{n-1}) \end{pmatrix} \tag{4.6}
$$

The eigenvalues of the digital matrix T in Theorem 4.1 are

$$
\lambda_1 = 26 + 15\sqrt{3}, \lambda_2 = 2 + \sqrt{3}, \lambda_3 = 2 - \sqrt{3}, \lambda_4 = \frac{1}{26 + 15\sqrt{3}}
$$

Applying the method in Section 2, the exact expression for spanning trees in $Y_n^{(3)}$ can be calculated on the computer:

$$
t(Y_n^{(3)}) = \frac{5\lambda_1^n - 26\lambda_2^n - 9\lambda_3^n + 42\lambda_4^n}{4} \tag{4.7}
$$

This method of permutaion and combination can also be applied in the further generalized pinwheel ladder Y^k. Ax_n^k denotes the n-order pinwheel graph which has number x of Hats. Then, the recursive system can be obtained:

Proposition 4.2. *In the pinwheel ladder $Y^{(k)}$, the recursive system of equations is:*

$$
t(Ax_{n+1}^{(k)}) = \sum_{i=0}^{x} \left(\sum_{0 \le j \le i} C_{k-x}^{i-j} 1^{i-j} 2^{k-x-i+j} C_x^j 2^j 3^{x-j} \right) t(Ai_n^{(k)}) \tag{4.8}
$$

Solve this system and one can get the exact expression when k is given arbitrarily.

4.2 3-dimensional ladders

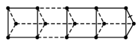
In this section, we mainly discuss the Triangular Prism. We denote A and B the other two base terms, which is shown in the proof of Theorem 4.3.

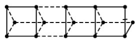
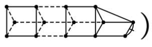
Through edge deletion and edge contraction, the recursive system of equations is obtained:

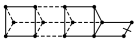
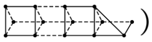
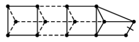
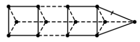
Proposition 4.3. *In the triangular prism S, the number of spanning trees $t(S_n)$ satisfies the recursive system of equations:*

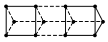
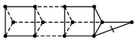
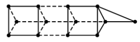
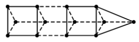
$$\begin{pmatrix} t(S_n) \\ t(A_n) \\ t(B_n) \end{pmatrix} = \begin{pmatrix} 7 & 6 & 3 \\ 10 & 8 & 5 \\ 14 & 13 & 8 \end{pmatrix} \begin{pmatrix} t(S_{n-1}) \\ t(A_{n-1}) \\ t(B_{n-1}) \end{pmatrix} \tag{4.9}$$

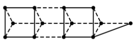
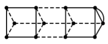
Proof. The process of edge deletion and edge contraction is given below:

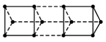
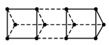
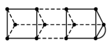
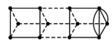
$t(S_n) = t($  $)$

$= t($  $) + t($  $)$

$= t($  $) + t($  $) + t($  $) + t($  $)$

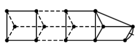
$= (t($  $) + t($  $)) + 2(t($  $) + t($  $))$

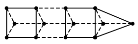
$+ (t($  $) + t($  $))$

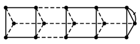
$= t($  $) + 6(t($  $) + t($  $)) + 3t($  $)$

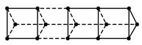
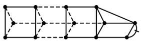
$= 7t(S_{n-1}) + 6t(A_{n-1}) + 3t(B_{n-1})$

where

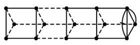
$t($  $) = 3t(S_{n-1}) + 2t(A_{n-1}) + 2t(B_{n-1})$

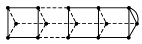
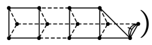
$t($  $) = t(S_{n-1}) + t(A_{n-1}) + t(B_{n-1})$

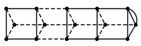
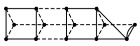
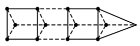
$t(A_n) = t($  $)$

$= t($  $) + t($  $)$

$= 10t(S_{n-1}) + 8t(A_{n-1}) + 5t(B_{n-1})$

$t(B_n) = t($  $)$

$= t($  $) + t($  $)$

$= t($  $) + t($  $) + t($  $)$

$= 14t(S_{n-1}) + 13t(A_{n-1}) + 8t(B_{n-1})$ □

Eigenvalues of the transition matrix T in Theorem 4.3 are calculated:

$$\lambda_1 = 11 + \sqrt{134}, \lambda_2 = 1, \lambda_3 = 11 - \sqrt{134}$$

With the initial condition: $t(S_0) = 3$, $t(A_0) = 5$ and $t(B_0) = 8$, the exact expression for the number of spanning trees in Triangular Prism can be calculated on the computer:

$$t(S_n) = \frac{(7850 - 1040\sqrt{134})\lambda_1^n + (676 + 983\sqrt{134})\lambda_2^n - 8715(-10 + \sqrt{134})\lambda_3^n}{68(469 - 43\sqrt{134})} \tag{4.10}$$

where $\lambda_1 = 11 + \sqrt{134}$, $\lambda_2 = 1$, $\lambda_3 = 11 - \sqrt{134}$.

Table 1: Entropy in the family of ladders.

Graph	Entropy
L_n	$\frac{\ln{(2+\sqrt{3})}}{2} \approx 0.6585$
N_n	$\frac{\ln{(7+3\sqrt{5})}-\ln 2}{2} \approx 0.9624$
X_n	$\frac{\ln 12}{2} \approx 1.2425$
Z_n	$\frac{\ln{(18+12\sqrt{2})}}{3} \approx 1.1848$
M_n	$\frac{\ln{(13+\sqrt{105})}-\ln 2}{2} \approx 1.2265$
Y_n	$\frac{\ln{(26+15\sqrt{3})}}{6} \approx 0.6585$
S_n	$\frac{\ln{(11+\sqrt{134})}}{3} \approx 1.0389$

5 Entropy

Entropy is an important theoretical parameter of Geometry, for it can describe the complexity of a graph(network)[19]. Entropy of a graph has deep relationship with the number of spanning trees[20] and the problem of studying entropy comes down to the problem of counting spanning trees.

In Geometry and Network Sciences, the entropy can be derived by the following expression:

$$h = \lim_{n\to\infty} \frac{\ln t(X_n)}{V_n} \tag{5.1}$$

where X_n is an increasing sequence of graphs approaching an infinite graph. Entropy is a useful descriptor to describe the relative complexity of a graph. In the previous calculation, one can find that if the transition matrix of the graph is of k order, then the number of spanning trees in Ladder-type graph can be written by:

$$t(X_n) = C_1\lambda_1^n + C_2\lambda_2^n + ... + C_k\lambda_k^n \tag{5.2}$$

where C_1, C_2 ... C_n are appropriate constants, and V_n can be written by:

$$V_n = kn \tag{5.3}$$

where k is also an appropriate constant.

Denote the largest number among the eigenvalues λ_1, λ_2 ... λ_n by λ_i. By using the Squeeze Theorem,

$$\frac{\ln C_i\lambda_i^n}{kn} \leq \frac{\ln{(C_1\lambda_1^n + C_2\lambda_2^n + ... + C_k\lambda_k^n)}}{kn} \leq \frac{\ln{[(C_1 + C_2 + ... + C_k)\lambda_i^n]}}{kn} \tag{5.4}$$

Since the limits on both sides are the same, the limit in the middle is:

$$h = \frac{\ln \lambda_i}{k} \tag{5.5}$$

Therefore, the entropy of these graphs can be calculated, and the results are reported in Table 1.

From the entropy of L_n, N_n and X_n, one can find that adding edges based on the original graph will increase entropy. However, adding both edges and vertices can't draw similar conclusion, for X_n and Z_n are special cases. Otherwise, Y_n has the same entropy as that of L_n, and we surmise that the graph obtained by the same growth rule in different directions may not be substantially different from the original one.

6 Conclusions

In this paper, the method of seeking similar terms is generalized to derive the recursive system of spanning trees. Then, the ladders are further generalized, including further generalized pinwheel ladders and 3-dimensional ladders. Finally, the exact expressions for the number of spanning trees in these graphs are derived, thereby obtaining the entropy of them.

Acknowledgement

The authors are greatly indebted to the referees for their valuable comments and suggestions, which were very helpful for improving the presentation of the paper.

AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between all authors. 'Gujun Wang' designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. 'Min Xiao' and 'Feng Zhu' managed the analyses of the study. All authors read and approved the final manuscript.

Competing Interests

Authors have declared that no competing interests exist

References

[1] A. Abd El Fattah, A. Aboutahoun, A. Elsaid, Correct proof of the main result in "the number of spanning trees of a class of self-similar fractal models" by ma and yao, Information Processing Letters 170 (2021) 106117.

[2] R. Mirjalili, H. Barati, A. Yazici, Resilience analysis of new york city transportation network after snow storms, Transportation research record 2677 (1) (2023) 694–707.

[3] J. I. Brown, C. J. Colbourn, Network reliability, in: Handbook of the Tutte Polynomial and Related Topics, Chapman and Hall/CRC, 2022, pp. 307–327.

[4] F. Zhu, M. Dai, Y. Dong, J. Liu, Random walk and first passage time on a weighted hierarchical network, International Journal of Modern Physics C 25 (09) (2014) 1450037.

[5] T. Nishikawa, A. E. Motter, Synchronization is optimal in nondiagonalizable networks, Physical Review E 73 (6) (2006) 065106.

[6] J. Kim, K.-I. Goh, G. Salvi, E. Oh, B. Kahng, D. Kim, Fractality in complex networks: Critical and supercritical skeletons, Physical Review E 75 (1) (2007) 016110.

[7] J. Böhm, J. L. Jacobsen, Y. Jiang, Y. Zhang, Geometric algebra and algebraic geometry of loop and potts models, Journal of High Energy Physics 2022 (5) (2022) 1–56.

[8] S. Oh, Dimer coverings of 1-slab cubic lattices, Graphs and Combinatorics 38 (4) (2022) 117.

[9] G. Kirchhoff, Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird, Annalen der Physik 148 (12) (1847) 497–508.

[10] Z. Zhang, S. Wu, M. Li, F. Comellas, The number and degree distribution of spanning trees in the tower of hanoi graph, Theoretical Computer Science 609 (2016) 443–455.

[11] M. H. S. Haghighi, K. Bibak, Recursive relations for the number of spanning trees, Applied Mathematical Sciences 3 (46) (2009) 2263–2269.

[12] S. Ahmad, On the evaluation of a subdivision of the ladder graph, Punjab University Journal of Mathematics 47 (1) (2020).

[13] K. Suja, K. Vishwanathan, Split domination of some special graph, in: AIP Conference Proceedings, Vol. 2516, AIP Publishing LLC, 2022, p. 210003.

[14] H. Chu, S.-R. Kim, H. Ryu, Planarity of generalized ladder graphs, arXiv preprint arXiv:2208.13637 (2022).

[15] Y. C. Kwun, M. Munir, A. R. Nizami, W. Nazeer, S. M. Kang, On t-equivalence of some families of ladder-type graphs, Global J. Pure Appl. Math 12 (2016) 4273–4284.

[16] J. A. Bondy, U. S. R. Murty, et al., Graph theory with applications, Vol. 290, Macmillan London, 1976.

[17] L. Lei, X. Geng, S. Li, Y. Peng, Y. Yu, On the normalized laplacian of möbius phenylene chain and its applications, International Journal of Quantum Chemistry 119 (24) (2019) e26044.

[18] X. Geng, Y. Lei, On the kirchhoff index and the number of spanning trees of linear phenylenes chain, Polycyclic Aromatic Compounds 42 (8) (2022) 4984–4993.

[19] E. Teufl, S. Wagner, The number of spanning trees in self-similar graphs, Annals of Combinatorics 15 (2) (2011) 355–380.

[20] R. Lyons, Asymptotic enumeration of spanning trees, Combinatorics, Probability and Computing 14 (4) (2005) 491–522.