Original Research Article

On Power Chris-Jerry Distribution: Properties and Parameter Estimation Methods

Abstract

In this study, we introduce the "Power Chris-Jerry" distribution, conducting a comprehensive analysis of its fundamental mathematical characteristics and an extensive exploration of various crucial aspects. These encompass investigations into its mode, quantile function, moments, coefficient of skewness, kurtosis, moment generating function, stochastic ordering, distribution of order statistics, reliability analysis, and mean past lifetime.Furthermore, we provide an in-depth assessment of four distinct parameter estimation methodologies: maximum likelihood estimation (MLE), Least Squares (LS), maximum product spacing method (MPS), and the Method of Cram`er-von-Mises (CVM). Our investigation uncovers a consistent pattern wherein the MLE, LS, and CVM approaches consistently yield underestimated parameter values. Intriguingly, we observe a consistent trend of decreasing Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and BIAS across all estimation techniques as sample sizes increase.Remarkably, our simulation results consistently favor the Maximum Product Spacing (MPS) method, highlighting its superiority in generating estimates with smaller MSE values across a broad spectrum of parameter values and sample sizes. These findings emphasize the robustness and dependability of the MPS estimator, offering valuable insights and practical guidance for both practitioners and researchers engaged in probability distribution modeling.

Keyword: Chris-Jerry Distribution, Power Chris-Jerry distribution, Maximum likelihood method,Least squares estimation method, Method of maximum product of spacing

1 Introduction

Ezeilo*et al* [1] introduced a new distribution known as “Power Chris-Jerry distribution (PCD)” with probability density function (pdf) and cumulative density function (cdf)defined as follows:

$$f(x,\alpha,\theta)=\frac{\alpha\theta^2}{\theta+2}(1+\theta x^{2\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}};x>0,\alpha>0,\theta>0 \quad (1)$$

and the corresponding cumulative density function is

$$F(x,\alpha,\theta)=1-\left(1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta+2}\right)e^{-\theta x^{\alpha}};x>0,\alpha>0,\theta>0 \quad (2)$$

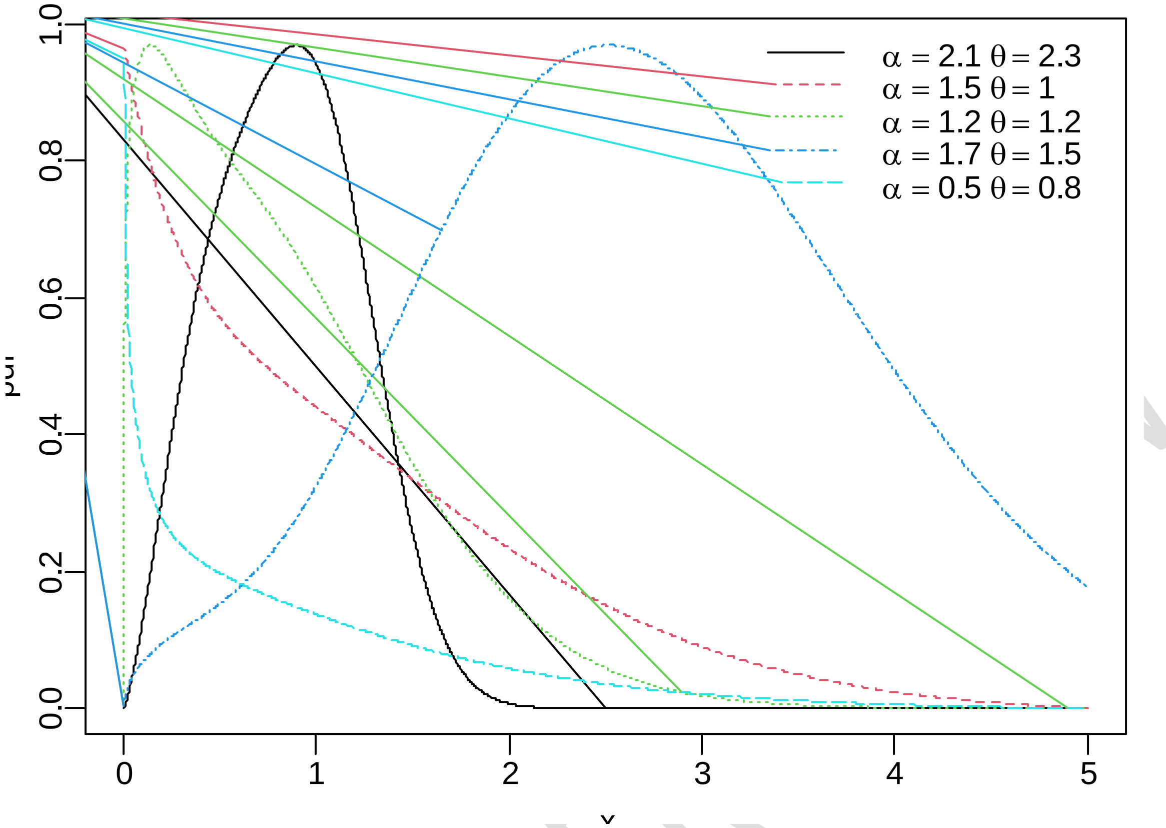


Figure 1: PDF of Power Chris-Jerry distribution

Figure 1 and 2 show the shape of the pdf and cdfof the Power Chris-Jerry distribution for different values of the parameters. The shape of the pdf is an indication that Power Chris-Jerry distribution is unimodal.

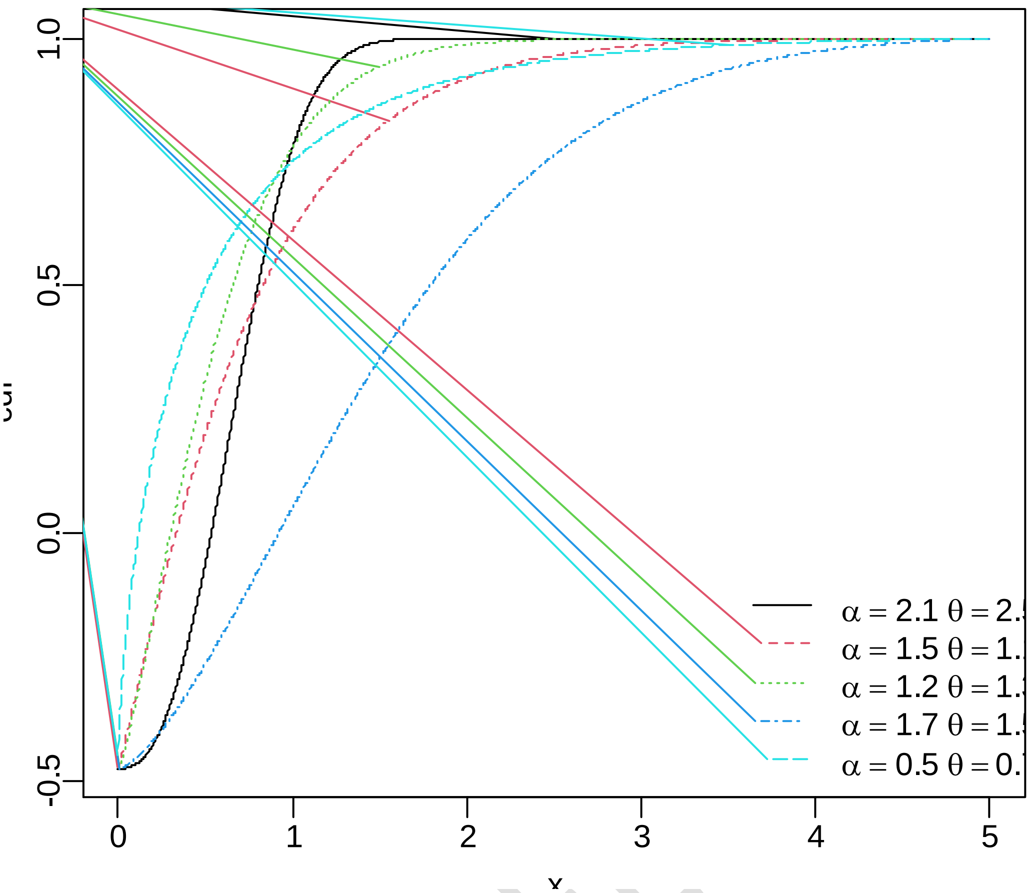


Figure 2: CDF of Power Chris-Jerry distribution

The survival and hazard functions are respectively given by

$$R(x,\alpha,\theta)=S(x,\alpha,\theta)=1-\left(1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right) \quad (3)$$

$$h(x,\alpha,\theta)=\frac{\dfrac{\alpha\theta^{2}}{\theta+2}\left(1+\theta x^{2\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}}{1-\left(1-\left(1+\dfrac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right)} \quad (4)$$

The estimation of the survival and hazard function are significant for reliability measures in engineering.

The new distribution is an extension of the Chris-Jerry distribution by Onyekwere and Obulezi[2]. The authors applied the developed distribution to CD4 count of patients suffering from HIV/AIDS. See Udofia *et al* [3]. Aside from the Power Chris-Jerry distribution, other literature on the extensions of Chris-Jerry distribution. Check out the following articlesObulezi*et al* [4], Innocent *et al*[5], Obulezi*et al*[6], Ahmad, P. B., & Wani, M. K [7], others. All these extensions are geared towards demonstrating the robustness, flexibility and field of applications of Chris-Jerry distribution. This article aims to provide the significant properties of Power Chris-Jerry distribution and study and compare five famous estimation methods for the parameters.

2. StatisticalProperties ofPower Chris-Jerry Distribution

2.1. Mode ofpower Chris-Jerry distribution: In a continuous distribution, the mode represents the value or values at which the probability density function (PDF) reaches its peak. Unlike discrete distributions, where specific values have probabilities associated with them, continuous distributions have an infinite number of possible values within a range. A continuous distribution can have multiple modes. In this article, we show that power Chris-Jerry distribution is unimodal

The mode is derived from the first derivative of $f(x,\alpha,\theta)$ in (1) as follows

$$f'(x,\alpha,\beta)=\frac{\alpha\theta^2}{\theta+2}x^{\alpha-2}\left((\alpha-1)x^{\alpha-2}+(3\alpha-1)\theta x^{2\alpha}-\alpha\theta x^{\alpha}-\theta x^{3\alpha}\right) \tag{5}$$

Letting $m=x^{\alpha}$ in (9), we have

$$f'(x,\alpha,\theta)=\frac{\alpha\theta^2}{\theta+2}x^{\alpha-2}k(m) \tag{6}$$

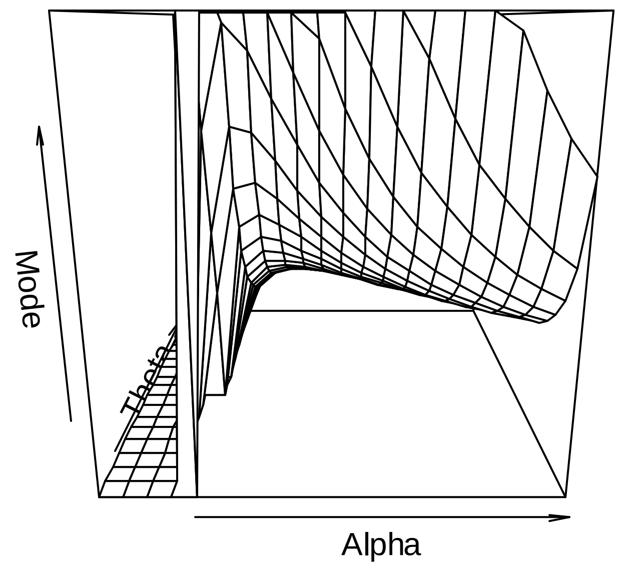
where $\kappa(m)=\left((\alpha-1)+(3\alpha-1)\theta m^2-\alpha\theta m-\theta m^3\right)$

Letting $\kappa(m)=0$, and numerically solve the nonlinear equation, the positive root of the equation gives the mode of the Power Chris-Jerry distribution. To observe the asymptotic behaviour, the limit of $f(x,\alpha,\theta)$ is evaluated at $x=0$ and $x=\infty$ respectively

$$\lim_{x\to 0} f(x;\alpha,\theta)=\lim_{x\to 0}\left\{\frac{\alpha\theta^2}{\theta+2}\left(1+\theta x^{2\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\right\}=0$$

$$\lim_{x\to +\infty} f(x;\alpha,\theta)=\lim_{x\to \infty}\left\{\frac{\alpha\theta^2}{\theta+2}\left(1+\theta x^{2\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\right\}=0$$

The Power Chris-Jerry distribution is said to be unimodal since $\lim_{x\to 0} f'(x;\alpha,\theta)=0$ and $\lim_{x\to +\infty} f'(x;\alpha,\theta)=0$. Figure 1 below is used to show the shape of the mode of Power Chris-Jerry distribution. The shape of the mode equally indicates that the new distribution is unimodal.



Picture 1 : Mode of Power Chris-Jerry Distribution

2.2 Moment and Related Measures

Somecaptivatingproperties of a distribution can be studied via the moments. For instance, measure of central tendency, dispersion, coefficient of skewness and coefficient of kurtosis. Consequently, it is essential to derive the moments for any new distribution proposed.

Proposition 1: Given a random variable X from APCJ distribution, the rth crude moment $E\left(X^r\right)$ is given by

$$E\left(X^r\right)=\mu_r^{'}=\frac{\left(\theta\Gamma\left(\frac{r}{\alpha}+1\right)+\Gamma\left(\frac{r}{\alpha}+3\right)\right)}{\left(\theta+2\right)\theta^{\frac{r}{\alpha}}}$$

Proof. By definition, the rth moment about the origin is given by

$$E\left(X^r\right)=\int_0^\infty x^r f_{APCJ}\left(x;\alpha,\theta\right)dx \quad (7)$$

Substituting (3) into (14) gives

$$E\left(X^r\right)=\frac{\alpha\theta^2}{\theta+2}\int_0^\infty x^r\left(1+\theta x^{2\alpha}\right)x^{\alpha-1}e^{-\theta x^\alpha}dx \quad (8)$$

By letting $y=x^\alpha$, and taking note of the fact that $\int_0^\infty x^n e^{-\theta x}dx=\frac{\Gamma(n+1)}{\theta^{n+1}}$, making good use of algebra in simplifying (15) gives the rth moment of PCJ distribution. Thus,

$$E\left(X^r\right)=\mu_r^{'}=\frac{\left(\theta\Gamma\left(\frac{r}{\alpha}+1\right)+\Gamma\left(\frac{r}{\alpha}+3\right)\right)}{(\theta+2)\theta^{\frac{r}{\alpha}}},\ r>0 \tag{9}$$

For $r=1,2,3,\text{and } 4$, we obtain the first four crude moments of the PCJ distribution as follows

$$\mu_1^{'}=\frac{\theta\Gamma\left(\frac{1}{\alpha}+1\right)+\Gamma\left(\frac{1}{\alpha}+3\right)}{(\theta+2)\theta^{\frac{1}{\alpha}}},\ \mu_2^{'}=\frac{\theta\Gamma\left(\frac{2}{\alpha}+1\right)+\Gamma\left(\frac{2}{\alpha}+3\right)}{(\theta+2)\theta^{\frac{2}{\alpha}}},$$

$$\mu_3^{'}=\frac{\theta\Gamma\left(\frac{3}{\alpha}+1\right)+\Gamma\left(\frac{3}{\alpha}+3\right)}{(\theta+2)\theta^{\frac{3}{\alpha}}}\ \text{and}\ \mu_4^{'}=\frac{\theta\Gamma\left(\frac{4}{\alpha}+1\right)+\Gamma\left(\frac{4}{\alpha}+3\right)}{(\theta+2)\theta^{\frac{4}{\alpha}}}$$

The first crude moment is the mean of the distribution. The central moments can be obtained by using the associationbetween crude moments and central moments. The first central moment μ_1 is zero. The second, also the variance, third and fourth central moments are respectively given by;

$$\mu_2=\mu_2^{'}-\left(\mu_1^{'}\right)^2$$

$$\mu_3=\mu_3^{'}-3\mu_2^{'}\mu_1^{'}+2\left(\mu_1^{'}\right)^3$$

$$\mu_4=\mu_4^{'}-4\mu_3^{'}\mu_1^{'}+6\mu_2^{'}\left(\mu_1^{'}\right)^2-3\left(\mu_1^{'}\right)^4$$

The moments are helpful when estimating the coefficient of Kurtosis and Skewness of a distribution

2.3. Moment generating function

Proposition 2: Let X be a non-negative random variable from PCJ distribution. Then, the moment generating function of X is given by:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(\frac{\left(\theta\Gamma\left(\frac{r}{\alpha}+1\right)+\Gamma\left(\frac{r}{\alpha}+3\right)\right)}{(\theta+2)\theta^{\frac{r}{\alpha}}}\right)$$

Proof: The moment generating function of a random variable X is given by

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}f(x,\alpha,\theta)\,dx \tag{10}$$

$$=\int_0^{\infty}\left[1+tx+\frac{(tx)^2}{2!}+\ldots\right]f(x,\alpha,\theta)\,dx$$

$$= \int_0^\infty \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f(x,\alpha,\theta)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r f(x,\alpha,\theta)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right) \quad (11)$$

Substituting for $E\left(X^r\right)$, we obtain the moment generating function of PCJ distribution. Thus

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\left(\theta\Gamma\left(\frac{r}{\alpha}+1\right)+\Gamma\left(\frac{r}{\alpha}+3\right)\right)}{(\theta+2)\theta^{\frac{r}{\alpha}}} \right) \quad (12)$$

2.4 Stochastic Ordering

Let X and Y be distributed according to (5). Let $f_x(x,\alpha,\theta), f_y(x,\alpha,\theta)$,and $F_X(x,\alpha,\theta), F_Y(x,\alpha,\theta)$denote the probability density function and distribution function of X and Y, respectively. The random variable X is said to be smaller than the random variable Y, if the following holds;

a) Stochastic order $(X \leq_{st} Y)$if $F_X(x) \geq F_Y(x); \forall x$
b) Hazard rate order $(X \leq_{hr} Y)$if $h_X(x) \geq h_Y(x); \forall x$
c) Mean residual life order $(X \leq_{mrl} Y)$if $m_X(x) \geq m_Y(x); \forall x$
d) Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(y)}$ decreases in x.

These results were established by Shaked and Shanthikumar[8]. The order of the distributions is as follows

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$$

The PCJ distribution is ordered based on the distribution with the strongest likelihood ratio, as showing in theorem 5 below

Proposition 3: Suppose $X \sim PCJ(\alpha_1,\theta_1)$ and $Y \sim PCJ(\alpha_2,\theta_2)$. If $\alpha_1 = \alpha_2$ and $\theta_2 \geq \theta_1$, then $X \leq_{lr} Y$. Hence, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

Proof. The likelihood ratio is

$$\frac{f_X(x;\alpha_1,\theta_1)}{f_Y(x;\alpha_2,\theta_2)} = \frac{\frac{\alpha_1\theta_1^2}{\theta_1+2}\left(1+\theta_1 x^{2\alpha_1}\right)x^{\alpha_1-1}e^{-\theta_1 x^{\alpha_1}}}{\frac{\alpha_2\theta_2^2}{\theta_2+2}\left(1+\theta_1 x^{2\alpha_1}\right)x^{\alpha_2-1}e^{-\theta_2 x^{\alpha_2}}} \quad (13)$$

Taking the natural logarithm of (25), one obtains

$$\ln\frac{f_X(x;\alpha_1,\theta_1)}{f_Y(x;\alpha_2,\theta_2)}=\ln\left(\frac{\alpha_1\theta_1^2(\theta_2+2)}{\alpha_2\theta_2^2(\theta_1+2)}\right)+\ln\left(\frac{1+\theta_1 x^{2\alpha_1}}{1+\theta_2 x^{2\alpha_2}}\right)+(\alpha_1-\alpha_2)\ln x-\left(\theta_1 x^{\alpha_1}-\theta_2 x^{\alpha_2}\right)$$

Taking the derivative with respect to x, the following is obtained

$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)}=\frac{2\left(\alpha_1\theta_1 x^{2\alpha_1-1}-\alpha_2\theta_2 x^{2\alpha_2-1}\right)+2\theta_1\theta_2(\alpha_1-\alpha_2)x^{2(\alpha_1+\alpha_2)-1}}{\left(1+\theta_2 x^{2\alpha_2}\right)\left(1+\theta_1 x^{2\alpha_1}\right)}+\frac{(\alpha_1-\alpha_2)}{x}-\left(\alpha_1\theta_1 x^{\alpha_1-1}-\alpha_2\theta_2 x^{\alpha_2-1}\right)$$

If $\theta_1>\theta_2$ and $\alpha_1=\alpha_2$ or $\alpha_1<\alpha_2$ and $\theta_1=\theta_2$ then, $\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)}<0$. Thus, $X\le_{lr}Y$ and consequently, $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$. Hence, it can be concluded that PCJ distribution follows the strongest likelihood ratio ordering.

2.5. The Quantile function and distribution of order statistics

The quantile function is significant for random number generation. It can also be used in finding percentiles of a distribution. If $F(x)$ is continuous and strictly increasing, the quantile function x_p is defined by

$$x_p=F_{(p)}^{-1};\, p\in(0,1) \tag{14}$$

where p is distributed as uniform distribution, $p\sim[0,1]$, and $F(p)$ is the CDF.

Proposition 4 Let X be a random variable having the pdf of an PCJ distribution, then the quantile x_p function is

$$\left\{(\theta+2)+\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)\right\}e^{-\theta x^{\alpha}}=(1-p)(\theta+2)$$

Proof:

Substituting (2) in (14), we obtain

$$1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}=p \tag{15}$$

$$\left(1+\frac{\theta x_p^{\alpha}\left(\theta x_p^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x_p^{\alpha}}=1-p$$

$$(\theta+2)e^{-\theta x_p^{\alpha}}+\theta x_p^{\alpha}\left(\theta x_p^{\alpha}+2\right)e^{-\theta x_p^{\alpha}}=(1-p)(\theta+2)$$

$$\left\{(\theta+2)+\theta x_p^{\alpha}\left(\theta x_p^{\alpha}+2\right)\right\}e^{-\theta x_p^{\alpha}}=(1-p)(\theta+2) \tag{16}$$

By solving the nonlinear equation in (16), for x_p^{α}, the solution gives the complete proof of the quantile function of PCJ distribution

Proposition 5: Supposethere exist a system having two components, and are independent and identical. Each of these systemsare assumed to follow Power Chris-Jerry distribution. Also, if the components are connected in series, the complete system will have aPower Chris-Jerry distribution. If the system has parallel connection, the system will also have alpha power Chris-Jerry distribution.

For a series with n components to work efficiently and not fail, all the components of the system work. Also, for a system with n components and are connected in parallel to work efficiently and not fail, at least one of the components of the system works. Suppose $X_1, X_2, ..., X_n$ are random samples of size n from the Power Chris-Jerry distribution with cumulative distribution function (cdf) $F_{X_{(\omega)}}(x;\alpha,\theta)$ and probability density function $f_{X_{(\omega)}}(x;\alpha,\theta)$.Then, $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote corresponding order statistics, where $X_{(1)} \le X_{(2)}, ... \le X_{(n)}$, $X_{(1)} = \min(X_1, X_2, ..., X_n)$ and $X_{(n)} = \max(X_1, X_2, ..., X_n)$. The probability density function (pdf) of the ωth order statistics defined by Hogg and Craig [9] is

$$f_X(x)=\frac{n!}{(\omega-1)!(n-\omega)!}(x)\sum_{j=0}^{n-\omega}\binom{n-\omega}{j}(-1)^j F^{\omega+j-1}(x)f(x) \tag{17}$$

Substituting for $f(x)$and$F(x)$ gives

$$f_{X_{(\omega)}}(x)=\frac{n!\alpha\theta^2\left(1+\theta x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{(\omega-1)!(n-\omega)!(\theta+2)}\sum_{j=0}^{n-\omega}\binom{n-\omega}{j}(-1)^j\left(1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right)^{\omega+j-1} \tag{18}$$

The pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x)=\frac{n!\alpha\theta^2\left(1+\theta x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{(n-1)!(\theta+2)}\sum_{j=0}^{n-\omega}\binom{n-1}{j}(-1)^j\left(1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right)^{j} \tag{19}$$

Also, the pdf of the nth order statistics $X_{(n)}$ can be obtained by setting $\omega = n$ .Thus, we have

$$f_{X_{(n)}}(x)=\frac{n!\alpha\theta^2\left(1+\theta x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{(n-1)(\theta+2)}\left(1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right)^{n+j-1} \tag{20}$$

The corresponding cdf of the ωth order statistics denoted by $F_{X_{(\omega)}}(x)$ is given by

$$F_{X_{(\omega)}}(x)=\sum_{j=\omega}^{n}\binom{n-j}{j}G^j(x,\theta)\left(1-G(x,\theta)\right)^{n-j}$$

$$=\sum_{j=\omega}^{n}\binom{n}{j}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^{i}\,G^{j+i}(x,\theta) \tag{21}$$

Where $G(x,\theta)$ denote the cdf of the Power Chris-Jerry distribution. Substituting for $G(x,\theta)$, gives the cdf of the ωth order statistics. Thus, we have

$$F_{x}(x)=\sum_{j=\omega}^{n}\binom{n}{j}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^{i}\left(1-\left(1+\frac{\theta x^{\alpha}\left(\theta x^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta x^{\alpha}}\right)^{j+i} \tag{22}$$

2.6. Mean Past Lifetime (MPL): In practical scenarios, systems are seldom under continuous surveillance, making it imperative to gain deeper insights into their operational history. This becomes especially crucial when individual components within the system experience failures. Let's suppose that a component with a lifetime of X has failed either at or before time t, where $t\geq 0$. Now, consider the conditional random variable $\left(t-T\,|\,T\leq t\right)$. This conditional random variable captures the elapsed time since the component's failure, given that its lifetime is less than or equal to t. Consequently, we define the Mean Past Lifetime (MPL) of the component as follows:

$$\zeta(t,\theta)=E\left(t-T\,|\,T\leq t\right)=t-\frac{\int_{0}^{t}F(x,\theta)\,dx}{F(t,\theta)}=t-\frac{\int_{0}^{t}xf(x,\theta)\,dx}{F(t,\theta)} \tag{23}$$

Where $f(x,\theta)$ and $F(t,\theta)$ respectively denote the pdf and cdf of Power Chris-Jerry distribution

$$\int_{0}^{t}xf(x)\,dx=\frac{\alpha\theta^{2}}{\theta+2}\int_{0}^{t}t^{\alpha}\left(1+\theta t^{2\alpha}\right)e^{-\theta t^{\alpha}}\,dt$$

$$=\frac{\theta\gamma\left(\frac{1}{\alpha}+1,\theta t^{\alpha}\right)+\gamma\left(\frac{1}{\alpha}+3,\theta t^{\alpha}\right)}{(\theta+2)\theta^{\frac{1}{\alpha}}}$$

Substituting $f(x,\theta)$ and $F(t,\theta)$, in (23) and simplifying, we obtain the mean past life of the PCJ distribution. Hence, we have

$$\zeta(t,\theta)=t-\frac{\dfrac{\theta\gamma\left(\frac{1}{\alpha}+1,\theta t^{\alpha}\right)+\gamma\left(\frac{1}{\alpha}+3,\theta t^{\alpha}\right)}{(\theta+2)\theta^{\frac{1}{\alpha}}}}{1-\left(1+\dfrac{\theta t^{\alpha}\left(\theta t^{\alpha}+2\right)}{\theta+2}\right)e^{-\theta t^{\alpha}}} \tag{24}$$

3.Methods of Estimation

In this section, four methods of estimation shall be adopted to derive the estimators $(\hat{\alpha}, \hat{\theta})$ for the parameters (α, θ) of Power Chris-Jerry distribution. The four methods to be adopted are the maximum likelihood estimation method, least squares method, the maximum product spacing method and Method of Cram`er-von-Mises

3.1 Maximum likelihood method

The maximum likelihood method stands as the most commonly employed technique for parameter estimation. See Casella and Berger [10]. The triumph of this method, no doubt stems from itsmany desirable properties such as consistency, asymptotic efficiency, invariance and simply its intuitive appeal. Let $x_1, x_2, ..., x_n$ constitute a random sample of size n from PCJ(α, θ) distribution. Then, the likelihood function is defined as

$$L = L(\alpha, \theta | x) = \prod_{i=1}^{n} \frac{\alpha\theta^2}{\theta+2}\left(1+\theta x^{2\alpha}\right) x^{\alpha-1} e^{-\theta x^{\alpha}} \quad (25)$$

$$= \left(\frac{\alpha\theta^2}{\theta+2}\right)^n \sum_{i=1}^{n}\left(1+\theta x^{2\alpha}\right) \sum_{i=1}^{n} x^{\alpha-1} e^{-\theta \sum_{i=1}^{n} x^{\alpha}} \quad (26)$$

Taking the natural logarithm of (26) gives the following

$$\ln L = n\ln\alpha + 2n\ln\theta - n\ln(\theta+2) + (\alpha-1)\sum_{i=1}^{n}\ln x + \sum_{i=1}^{n}\ln\left(1+\theta x^{2\alpha}\right) - \theta\sum_{i=1}^{n} x^{\alpha} \quad (27)$$

The maximum likelihood estimators of $\hat{\alpha}_{MLE}$ and $\hat{\theta}_{MLE}$ for the parameters α and θ can be obtained numerically by maximizing, with respect to α and θ the log-likelihood function. In this case, the log-likelihood function is maximized by solving in α and θ, the non-linear equations are:

$$\frac{\partial LL}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+2} - \sum_{i=1}^{n} x^{\alpha} + \sum_{i=1}^{n}\frac{x^{2\alpha}}{1+\theta x^{2\alpha}} \quad (28)$$

$$\frac{\partial LL}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x + \sum_{i=1}^{n}\frac{2\theta x^{2\alpha}\ln x}{1+\theta x^{2\alpha}} - \theta\sum_{i=1}^{n} x^{\alpha}\ln x \quad (29)$$

The solution of $\frac{\partial \ln L}{\partial\theta} = 0$ and $\frac{\partial \ln L}{\partial\alpha} = 0$, provides the maximum likelihood estimates of the parameters (α, θ) However, it is practically impossible to obtain the solution analytically dues to the complexity involved. The best approach that can be adopted is to solve it numerically using software like R

3.2 Least squares estimation method

Let $x_1, x_2, x_3, x_4, x_5, ..., x_n$ be the order statistics from the Power Chris-Jerry distribution. By minimizing the following equations, the least squares estimation of the parameters α and θ would be obtained. Thus, Swain *et al*[11] defined the least square equation as

$$LS_{PCJ} = \sum_{i=1}^{n}\left(F\left(x_{i:n} \left| \alpha,\theta \right. \right) - \frac{i}{n+1} \right)^2 \tag{30}$$

$$= \sum_{i=1}^{n}\left(1 - \left(1 + \frac{\theta x_{i:n}^{\alpha}\left(\theta x_{i:n}^{\alpha} + 2 \right)}{\theta + 2} \right) e^{-\theta x_{i:n}^{\alpha}} - \frac{i}{n+1} \right)^2 \tag{31}$$

The estimates can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left(F\left(x_{i:n} \left| \alpha,\theta \right. \right) - \frac{i}{n+1} \right)^2 \Delta_1\left(x_{i:n} \left| \alpha,\theta \right. \right) = 0 \tag{32}$$

$$\sum_{i=1}^{n}\left(F\left(x_{i:n} \left| \alpha,\theta \right. \right) - \frac{i}{n+1} \right)^2 \Delta_2\left(x_{i:n} \left| \alpha,\theta \right. \right) = 0 \tag{33}$$

Where

$$\begin{aligned}\Delta_1\left(x_{i:n} \left| \alpha,\theta \right. \right) = & \left\{ 2x_{i:n}^{\alpha} e^{-\theta x_{i:n}^{\alpha}} \left(\left(\theta + 2 \right)\left(\theta x_{i:n}^{\alpha} + 1 \right) \right) - \theta x_{i:n}^{\alpha}\left(\theta x_{i:n}^{\alpha} + 2 \right) \right\}\left(\theta + 2 \right)^{-2} \\ & - \left(1 + \left(\theta + 2 \right)^{-1} \theta x_{i:n}^{\alpha}\left(\theta x_{i:n}^{\alpha} + 2 \right) \right) x_{i:n}^{\alpha} e^{-\theta x_{i:n}^{\alpha}} \end{aligned} \tag{34}$$

$$\begin{aligned}\Delta_2\left(x_{i:n} \left| \alpha,\theta \right. \right) = & \, 2\alpha\theta x_{i:n}^{\alpha-1} e^{-\theta x_{i:n}^{\alpha}} \left(\theta + 2 \right)^{-1}\left(x_{i:n}^{\alpha} + 1 \right) \\ & - \left(1 + \left(\theta + 2 \right)\theta x_{i:n}^{\alpha}\left(\theta x_{i:n}^{\alpha} + 2 \right) \right)\theta x_{i:n}^{\alpha} \, ln \, x_{i:n}^{\alpha} e^{-\theta x_{i:n}^{\alpha}} \end{aligned} \tag{35}$$

3.3 Maximum product of spacing Method

Cheng and Amin [12-13] introduced the maximum product of spacing method as an alternative approach to estimating parameters of continuous univariate distributions, distinct from the conventional maximum likelihood estimation.To set the stage, we begin by defining the uniform spacings of a random sample drawn from $PCJ\left(\alpha,\theta \right)$ distribution as follows:

$$\mathrm{K}_i\left(\alpha,\theta \right) = F\left(x_{i:n} \left| \alpha,\theta \right. \right) - F\left(x_{i-1:n} \left| \alpha,\theta \right. \right) \tag{36}$$

Where $F\left(x_{0:n} \left| \alpha,\theta \right. \right) = 0, F\left(x_{n+1:n} \left| \alpha,\theta \right. \right) = 1 \text{and} \sum_{i=1}^{n+1} \mathrm{K}_i\left(\alpha,\theta \right) = 1$

The maximum product of spacings estimators $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ of the parameters α and θ are obtained by maximizing the geometric mean of the spacings with respect to α and θ. The function is defined by

$$G\left(\alpha,\theta \right) = \left[\prod_{i=1}^{n+1} \mathrm{K}_i\left(\alpha,\theta \right) \right]^{\frac{1}{n+1}} \tag{37}$$

In a similar manner, same result can be achieved by maximizing the function

$$H\left(\alpha,\theta \right) = \frac{1}{n+1}\sum_{i=1}^{n+1} ln \, \mathrm{K}_i\left(\alpha,\theta \right) \tag{38}$$

The estimators $\hat{\Psi}_{MPS} = \left(\hat{\alpha}_{MPS}, \hat{\theta}_{MPS} \right)$ of the parameters $\Psi = \left(\alpha,\theta \right)$ can be obtained by solving the nonlinear equations

$$\frac{\partial}{\partial\alpha} H\left(\Psi \right) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{\mathrm{K}_i\left(\Psi \right)} \left\{ \Delta_1\left(x_{i:n} \left| \Psi \right. \right) - \Delta_1\left(x_{i-1:n} \left| \Psi \right. \right) \right\} = 0 \tag{39}$$

$$\frac{\partial}{\partial\theta} H\left(\Psi \right) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{\mathrm{K}_i\left(\Psi \right)} \left\{ \Delta_2\left(x_{i:n} \left| \Psi \right. \right) - \Delta_2\left(x_{i-1:n} \left| \Psi \right. \right) \right\} = 0 \tag{40}$$

maximizing H as a method of parameter estimation is as efficient as MLE estimation and the MPS estimators are consistent under more general conditions than the MLE estimators.

3.4 Method of Cram`er-von-Mises

To inspire our choice of Cram ́er-von-Mise type minimum distance estimators, MacDonald [14] provided pragmatic evidence that the bias of the estimator is smaller than the other minimum distance estimators . Thus, the Cram ́er -von Mises estimates $\hat{\alpha}_{CVM}$ and $\hat{\theta}_{CVM}$ of the parameters α and θ are obtained by minimizing $\vartheta(\alpha,\theta)$ with respect to α and θ

$$\vartheta(\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{i:n}|\alpha,\theta\right)-\frac{2i-1}{2n}\right)^2 \tag{41}$$

The estimators can be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left(F\left(x_{i:n}|\alpha,\theta\right)-\frac{2i-1}{2n}\right)\Delta_1\left(x_{i:n}|\alpha,\theta\right)=0 \tag{42}$$

$$\sum_{i=1}^{n}\left(F\left(x_{i:n}|\alpha,\theta\right)-\frac{2i-1}{2n}\right)\Delta_2\left(x_{i:n}|\alpha,\theta\right)=0 \tag{43}$$

Where $\Delta_1\left(x_{i:n}|\alpha,\theta\right)$ and $\Delta_2\left(x_{i:n}|\alpha,\theta\right)$ are given in equations (34) and (35)

4 Simulation study

A simulation study was carried out to evaluate the performance of the parameter estimates obtained by MLE, LS, MPS and CVM. To do this, we generated 1000 random samples of sizes $n=5,10,20,25,50,100,150,200,500$ and 1000 for different values of the parameters and took thevalues at $\alpha=1.1$ and $\theta=1.2$, $\alpha=1.3$ and $\theta=1.5$, and $\alpha=1.7$ and $\theta=1.8$. The resultof the estimates

are shown in tables 1, 2, 3, 4 and 5, respectively.

Table 1: Average estimated MSEs, RMSEs and BIASs of different estimation methods for PCJ distribution at different sample sizes n and different values of $\alpha=1.1$ and $\theta=1.2$.

Methods		n = 5			n = 10		
		MSE	RMSE	BIAS	MSE	RMSE	BIAS
MLE	α	0.65365	0.80848	0.39843	0.11487	0.33892	0.15157
	Ɵ	0.40681	0.63782	0.02076	0.13551	0.36812	0.024432
LSE	α	0.39202	0.62612	0.01516	0.12066	0.34736	0.01005
	Ɵ	0.29365	0.5419	0.09856	0.12663	0.35585	0.04963
MPS	α	0.26284	0.51268	0.04806	0.07152	0.26743	0.08697
	Ɵ	0.25852	0.50845	0.12181	0.11482	0.33886	0.08485
CVM	α	1.5642	1.25068	0.52698	0.2022	0.44967	0.16304
	Ɵ	9.68299	3.11175	0.16361	0.1547	0.39332	0.00816

Table 2: Average estimated MSEs, RMSEs and BIASs of different estimation methods for PCJ distribution at different sample sizes n and different values of $\alpha = 1.1$ and $\theta = 1.2$.

Methods		n = 20			n = 25		
		MSE	RMSE	BIAS	MSE	RMSE	BIAS
MLE	α	0.03901	0.19751	0.06154	0.03596	0.18965	0.0576
	Θ	0.05696	0.23866	0.00639	0.05305	0.23034	0.00553
LSE	α	0.05054	0.22482	0.01148	0.04527	0.21278	0.00576
	Θ	0.06127	0.24752	0.02997	0.0553	0.23517	0.02662
MPS	α	0.03443	0.18555	0.07831	0.03098	0.17601	0.06159
	Θ	0.05469	0.2338	0.06586	0.05113	0.22614	0.05677
CVM	α	0.06463	0.25422	0.06921	0.05567	0.23595	0.05916
	Θ	0.06634	0.25758	0.0017	0.0587	0.24228	0.00111

Table 3: Average estimated MSEs, RMSEs and BIASs for PCJ distribution at $\alpha = 1.3$ and $\theta = 1.5$.

Methods		n = 50			n = 100		
		MSE	RMSE	BIAS	MSE	RMSE	BIAS
MLE	α	0.0223	0.14934	0.03231	0.01017	0.10089	0.01749
	Θ	0.03258	0.1805	0.0041	0.01648	0.1284	0.00294
LSE	α	0.03027	0.17399	0.00379	0.01525	0.1235	0.00216
	Θ	0.03311	0.18198	0.01453	0.01733	0.13164	0.0044
MPS	α	0.02193	0.1481	0.05471	0.01036	0.10182	0.03357
	Θ	0.03079	0.17549	0.03226	0.01584	0.12587	0.01476
CVM	α	0.03355	0.18318	0.03474	0.01605	0.12671	0.01687
	Θ	0.03467	0.18621	0.0076	0.01774	0.13321	0.00076

Table 4: Average estimated MSEs, RMSEs and BIASs ofestimation methods for PCJ distribution at $\alpha = 1.3$ and $\theta = 1.5$

Methods		n = 150			n = 200		
		MSE	RMSE	BIAS	MSE	RMSE	BIAS
MLE	α	0.00617	0.07857	0.00732	0.00484	0.06957	0.00921
	Θ	0.0094	0.09719	0.00186	0.00781	0.08839	0.0059

LSE	α	0.00896	0.09468	0.00485	0.00707	0.08413	0.00181
	Θ	0.01017	0.10087	0.00608	0.00834	0.09135	0.00371
MPS	α	0.00675	0.08219	0.02999	0.005	0.07073	0.02063
	Θ	0.00932	0.09655	0.01507	0.00757	0.08704	0.00496
CVM	α	0.00921	0.09598	0.00773	0.00732	0.08555	0.0113
	Θ	0.01032	0.10162	0.00363	0.00846	0.09203	0.00562

Table 5: Average estimated MSEs, RMSEs and BIASsestimation methods for PCJ distribution at $\alpha = 1.7$ and $\theta = 1.8$.

Methods		**n = 500**			**n = 1000**		
		MSE	**RMSE**	**BIAS**	**MSE**	**RMSE**	**BIAS**
MLE	α	0.00349	0.05915	0.00481	0.00158	0.03979	0.00012
	Θ	0.0039	0.06247	0.00211	0.00199	0.04471	0.00086
LSE	α	0.00527	0.07264	0.00022	0.00275	0.05245	0.00162
	Θ	0.00418	0.06471	0.00217	0.00202	0.04498	0.00067
MPS	α	0.00361	0.06012	0.01447	0.00167	0.04095	0.01076
	Θ	0.00384	0.06203	0.00501	0.00198	0.04456	0.00257
CVM	α	0.00533	0.07306	0.00485	0.00275	0.05253	0.00091
	Θ	0.00421	0.06492	0.00224	0.00202	0.04505	0.0007

As can be seen from table 1, 2, 3, 4 and 5, the maximum likelihood method (MLE), least squares estimation method (LS), andCram`er-von-Mises method (CVM) tend to underestimate the value of the parameters. The MSE, RMSE and BIAS decrease as n increases for all the estimation methods studied in this research. From the simulation results, it must be emphasized that, in general, the performance of Maximum product spacing (MPS) was better than that of the other methods, because the MPS method produced estimates with smaller MSE for different values of the parameters and sample sizes. This also provided evidences on the consistency of the estimators.

5 Conclusion

This paper introduced a novel probability distribution called the "Power Chris-Jerry" distribution and comprehensively explored its mathematical properties, including mode, quantile function, moments, coefficient of skewness, kurtosis, moment generating function, stochastic ordering, distribution of order statistics, reliability analysis, and mean past lifetime. Additionally, the study presented four distinct parameter estimation techniques: maximum likelihood, Least Squares, maximum product spacing method, and Method of Cram`er-von-Mises.

Our findings, as illustrated in Table 1, 2, 3, 4, and 5, indicated that the maximum likelihood method (MLE), least squares estimation method (LS), and Cram`er-von-Mises method (CVM) exhibited a tendency to underestimate parameter values. However, it was observed that the Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and BIAS decreased as the sample size increased for all the estimation methods investigated in this research.

Notably, our simulation results emphasized the superior performance of the Maximum Product Spacing (MPS) method compared to the other techniques. The MPS method consistently produced estimates with smaller MSE across various parameter values and sample sizes. This outcome provides strong evidence for the reliability and consistency of the MPS estimator.

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