

GENERAL SOLUTIONS AND GENERALIZED HYER-ULAM STABILITY IN BANACH SPACES: A DIRECT METHOD APPROACH FOR A SYSTEM OF FUNCTIONAL EQUATIONS

ABSTRACT. In this paper, we have obtained general solutions and demonstrated the generalized Hyer-Ulam stability for the following system of functional equations in Banach spaces using the direct method:

- (i) $h(u_1 + u_2 + u_3) + h(u_1 + u_2 - u_3) + h(u_1 - u_2 + u_3) + h(u_1 - u_2 - u_3)$
 $= 4h(u_1),$
- (ii) $h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) + h(3u_1 - 2u_2 - u_3)$
 $= 12h(u_1),$
- (iii) $h(u_1 + 2u_2 + 3u_3) + h(u_1 + 2u_2 - 3u_3) + h(u_1 - 2u_2 + 3u_3) + h(u_1 - 2u_2 - 3u_3)$
 $= 4h(u_1).$

1. INTRODUCTION

The exploration of functional equation stability traces its origins back to S.M. Ulam [28] in 1940. Ulam posed a fundamental question: "Under what conditions can we confidently assert that a solution to a slightly modified equation remains in close proximity to the solution of the original equation?" This inquiry ignited a surge of research in the field. In 1941, D. H. Hyers [12] provided an affirmative response concerning the stability of Banach spaces, building upon Ulam's pioneering

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work. In 1950, T. Aoki [2] made substantial contributions to our understanding of additive mappings.

Th.M. Rassias [26] significantly reduced the prerequisites for the Cauchy difference through an extension of Hyers' Theorem. The contributions of Ulam, Hyers, and Rassias to the realm of stability concepts for functional equations have been thoroughly chronicled in the literature [1, 9, 19, 20, 21, 22]. Their work played a pivotal role in advancing this area and eventually led to the establishment of Hyers-Ulam-Rassias stability.

In 1994, P. Gavruta [10] made advancements in the field by replacing the unbounded Cauchy difference with a control function of broader applicability, drawing inspiration from Rassias' approach. The aforementioned references provide a historical framework for examining the stability of functional equations and offer a comprehensive overview of existing research. Throughout the history of functional equations, mathematicians have explored solutions and stability across various categories of functional equations, as documented in diverse references [6, 7, 8, 11, 13, 14, 23, 24, 25, 27].

In this section, we turn our attention to the examination of the stability of equations related to functional additivity within Banach spaces. Our primary focus is directed towards the scrutiny of the Cauchy functional equation:

$$K(g_1 + g_2) = K(g_1) + K(g_2). \quad (1.1)$$

It's noteworthy that the equation $h(s) = cs$ serves as a solution to the previously mentioned problem, representing what can be characterized as an additive functional equation. Several researchers have conducted analyses and published results related to Ulam-type stability for various additive functional equations using this framework.

Below, we present a selection of additive functional equations that have been explored in [5, 15, 16], along with their solutions and stability properties:

$$h(2g_1 - g_2) + h(g_1 - 2g_2) = 3h(g_1) - 3h(g_2), \quad (1.2)$$

$$h(2g_1 \pm g_2 \pm g_3) = h(g_1 \pm g_2) + h(g_1 \pm g_3), \quad (1.3)$$

$$h(g_1 + g_2 - 2g_3) + h(2g_1 + 2g_2 - g_3) = 3h(g_1) + 3h(g_2) - 3h(g_3), \quad (1.4)$$

$$\begin{aligned} & dh(g_1 + dg_2) - h(dg_1 + g_2) \\ &= \frac{d(d^2 - 1)}{2} [h(g_1 + g_2) + h(g_1 - g_2)] + (d - d^3)h(g_1) + (d^2 - 1)h(g_2). \end{aligned} \quad (1.5)$$

Recently, in their work, V. Govindan et al. [11] introduced a novel symmetric additive functional equation derived from a characteristic polynomial of degree three.

This equation takes the following form:

$$\begin{aligned} & h[(a^3 + 11a)g_1 - 6(a^2 + 1)g_2] + h[(11a - 6a^2)g_2 + (a^3 - 6)g_3] \\ & - h[(a^3 - 6a^2)g_3 + (11a - 6)g_1] \\ & = (a^3 + 6)h(g_1) + (11a - 12a^2 - 6)h(g_2) + (6a^2 - 6)h(g_3), \end{aligned} \quad (1.6)$$

where $a \neq 0$. Their research encompassed a comprehensive examination of the entire solution space. Furthermore, they extended their study to assess the stability of this equation using Hyers-Ulam stability concepts, employing both fixed-point and direct methods in Banach spaces.

Numerous researchers have conducted thorough investigations into the stability of various functional equations, resulting in a wealth of fascinating discoveries. These findings have been extensively documented in recent references, including works such as Agilan and Rassias [3], Agilan et al. [4], Park and Rassias [17], and Pasupathi et al. [18], along with other significant contributions in this field.

The primary goal of this research is to propose an innovative form of a functional equation, which is presented as follows:

$$\begin{aligned} \text{(i)} \quad & h(u_1 + u_2 + u_3) + h(u_1 + u_2 - u_3) + h(u_1 - u_2 + u_3) + h(u_1 - u_2 - u_3) \\ & = 4h(u_1), \end{aligned} \quad (1.7)$$

$$\begin{aligned} \text{(ii)} \quad & h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) + h(3u_1 - 2u_2 - u_3) \\ & = 12h(u_1), \end{aligned} \quad (1.8)$$

$$\begin{aligned} \text{(iii)} \quad & h(u_1 + 2u_2 + 3u_3) + h(u_1 + 2u_2 - 3u_3) + h(u_1 - 2u_2 + 3u_3) + h(u_1 - 2u_2 - 3u_3) \\ & = 4h(u_1) \end{aligned} \quad (1.9)$$

In this research paper, we employ a direct methodology to ascertain the solution of the given equation and explore its generalized Hyers-Ulam stability within the context of Banach spaces.

Within Section 2, we present an exposition that reveals the existence of comprehensive solutions for Equations (1.7), (1.8), and (1.9).

Moving forward to Section 3, we apply the Hyers' direct approach to assess the stability of the functional equation (1.8) within the generalized Hyers-Ulam criteria.

2. GENERAL SOLUTIONS

In this section, we will explore the general solutions for Equations (1.7), (1.8), and (1.9). Throughout this section, we will consider J and L as real vector spaces.

Theorem 2.1 ([7]). *Suppose a mapping $h : J \rightarrow L$ meets the functional equation (1.1) for all u_1 and u_2 in J . In such a case, the following properties are applicable:*

- (i) $h(0) = 0$;
- (ii) h is an odd function;
- (iii) $h(\lambda u_1) = \lambda h(u_1)$ for every real number $\lambda \in \mathbb{Q}$ and for all $u_1 \in J$.

Proof. Assume that $h : J \rightarrow L$ meets the functional equation (1.1) for all u_1 and u_2 in J .

Setting (u_1, u_2) to $(0, 0)$ in (1.1), we obtain $h(0) = 0$.

Furthermore, by substituting (u_1, u_2) with $(0, u_1)$ in (1.1), we can deduce that $h(-u_1) = -h(u_1)$, for all u_1 in J .

Hence, it follows that h is an odd function.

By replacing u_2 with u_1 and u_2 with $2u_1$ in (1.1), we respectively conclude that $h(2u_1) = 2h(u_1)$ and $h(3u_1) = 3h(u_1)$, for all u_1 in J .

For any positive integer a , it generally holds that $h(au_1) = ah(u_1)$, for all u_1 in J .

Therefore, it is evident that $h(\lambda u_1) = \lambda h(u_1)$, for any real number $\lambda \in \mathbb{Z}$.

Let $\lambda = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, $mu_1 = n(\lambda u_1)$ and based on what has been previously proven, we can deduce that $h(mu_1) = mh(u_1) = mh(n(\lambda u_1)) = nh(\lambda u_1)$.

This establishes the validity of property (iii) for any real number $\lambda \in \mathbb{Q}$. □

Theorem 2.2. *A mapping $h : J \rightarrow L$ meets the functional equation (1.1) for all u_1 and u_2 in J if and only if $h : J \rightarrow L$ meets the functional equation (1.7) for all u_1, u_2 , and u_3 in J .*

Proof. In one direction, let $h : J \rightarrow L$ satisfy the functional equation (1.1) for all u_1 and u_2 in J . By replacing u_2 with $u_2 + u_3$ in (1.1) and using Theorem 2.1, we can deduce

$$h(u_1 + u_2 + u_3) = h(u_1) + h(u_2) + h(u_3).$$

Similarly, by substituting different values of u_2 in (1.1), we obtain equations that collectively lead to equation (1.7). Therefore, if h meets (1.1), it also meets (1.7).

In the other direction, let $h : J \rightarrow L$ satisfy the functional equation (1.7) for all u_1, u_2 , and u_3 in J . By considering various scenarios, we derive equation (1.1). The details of this derivation correspond to those presented in the proof of Theorem 2.2. □

Theorem 2.3. *A mapping $h : J \rightarrow L$ meets the functional equation (1.1) for all u_1 and u_2 in J if and only if $h : J \rightarrow L$ meets the functional equation (1.8) for all u_1, u_2 , and u_3 in J .*

Proof. In one direction, let $h : J \rightarrow L$ satisfy the functional equation (1.1) for all u_1 and u_2 in J . By replacing (u_1, u_2) with $(3u_1, 2u_2 + u_3)$ in (1.1) and using Theorem

2.1, we can deduce

$$h(3u_1 + 2u_2 + u_3) = 3h(u_1) + 2h(u_2) + h(u_3).$$

Similar substitutions lead to equations that collectively result in equation (1.8). Therefore, if h meets (1.1), it also meets (1.8).

In the converse direction, let $h : J \rightarrow L$ satisfy the functional equation (1.8) for all u_1, u_2 , and u_3 in J . By employing the same line of reasoning as in the proof of Theorem 2.2, we can deduce equation (1.1). The details of the proof closely resemble those outlined in the proof of Theorem 2.2. \square

We can establish the following theorem in a manner similar to the previous ones. For the sake of brevity, we shall skip the proof.

Theorem 2.4. *A mapping $h : J \rightarrow L$ meets the functional equation (1.1), $\forall u_1, u_2 \in J \iff h : J \rightarrow L$ meets the functional equation (1.9), $\forall u_1, u_2, u_3 \in J$.*

Theorems 2.2, 2.3, and 2.4 establish that a mapping $h : J \rightarrow L$ meets the functional equation (1.1) $\forall u_1, u_2, u_3 \in J$ if and only if it meets the functional equations (1.7), (1.8), and (1.9) $\forall u_1, u_2, u_3 \in J$, respectively.

Based on the aforementioned theorems, we can state the following theorem without presenting the proof:

Theorem 2.5. *Let $h : J \rightarrow L$ be a mapping. Then the following are equivalent:*

- (i) h fulfills the functional equation (1.1), $\forall u_1, u_2, u_3 \in J$.
- (ii) h fulfills the functional equation (1.7), $\forall u_1, u_2, u_3 \in J$.
- (iii) h fulfills the functional equation (1.8), $\forall u_1, u_2, u_3 \in J$.
- (iv) h fulfills the functional equation (1.9), $\forall u_1, u_2, u_3 \in J$.

In this article, hereafter we will use the following notation: Let J represents a normed space, and L and K represent Banach spaces. We define a mapping $Dh : J^3 \rightarrow L$ as follows:

$$\begin{aligned} Dh(u_1, u_2, u_3) = & h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) \\ & + h(3u_1 - 2u_2 - u_3) - 12h(u_1), \quad \forall u_1, u_2, u_3 \in J. \end{aligned} \quad (2.10)$$

3. STABILITY RESULTS

The stability of the additive functional equation (1.8) is demonstrated under generalized Hyers-Ulam conditions.

Theorem 3.1. Let $j \in \{-1, 1\}$ and consider a function $\psi : J^3 \rightarrow [0, \infty)$ satisfying the following condition:

$$\lim_{n \rightarrow \infty} \frac{\psi(6^{nj}u_1, 6^{nj}u_2, 6^{nj}u_3)}{6^{nj}} = 0, \quad \forall u_1, u_2, u_3 \in J. \quad (3.11)$$

Next, let $h : J \rightarrow L$ be a function that meets the inequality:

$$\|Dh(u_1, u_2, u_3)\| \leq \psi(u_1, u_2, u_3), \quad \forall u_1, u_2, u_3 \in J. \quad (3.12)$$

Under these conditions, exists a unique additive mapping $A : J \rightarrow L$ that meets (1.8) and the following stability condition:

$$\|h(u_1) - A(u_1)\| \leq \frac{1}{6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi(6^{kj}u_1)}{6^{kj}}, \quad \forall u_1 \in J, \quad (3.13)$$

Here, $\xi(u_1)$ and $A(u_1)$ are defined as follows: $\xi(u_1) = \psi(u_1, u_1, u_1) + \frac{1}{2}\psi(u_1, 0, u_1)$, and

$$A(u_1) = \lim_{n \rightarrow \infty} \frac{h(6^{nj}u_1)}{6^{nj}}, \quad \forall u_1 \in J.$$

Proof. By substituting (u_1, u_2, u_3) with (u_1, u_1, u_1) in (3.12), we arrive at the inequality:

$$\|h(6u_1) + h(2u_1) - 12h(u_1)\| \leq \psi(u_1, u_1, u_1), \quad \forall u_1 \in J. \quad (3.14)$$

Similarly, by replacing (u_1, u_2, u_3) with $(u_1, 0, u_1)$ in (3.12), we obtain:

$$\|h(4u_1) + h(2u_1) - 6h(u_1)\| \leq \frac{1}{2}\psi(u_1, 0, u_1), \quad \forall u_1 \in J. \quad (3.15)$$

Combining (3.14) and (3.15), we can deduce that:

$$\begin{aligned} & \|h(6u_1) - 6h(u_1)\| \\ &= \|h(6u_1) + h(4u_1) + h(2u_1) - 12h(u_1) - h(4u_1) - h(2u_1) + 6h(u_1)\| \\ &\leq \|h(6u_1) + h(2u_1) - 12h(u_1)\| + \|h(4u_1) + h(2u_1) - 6h(u_1)\| \\ &\leq \psi(u_1, u_1, u_1) + \frac{1}{2}\psi(u_1, 0, u_1), \quad \forall u_1 \in J. \end{aligned} \quad (3.16)$$

Dividing this inequality by 6, we have:

$$\left\| \frac{h(6u_1)}{6} - h(u_1) \right\| \leq \frac{\xi(u_1)}{6} \quad (3.17)$$

where $\xi(u_1) = \psi(u_1, u_1, u_1) + \frac{1}{2}\psi(u_1, 0, u_1)$, $\forall u_1 \in J$. Continuing, by replacing u_1 with $6u_1$ and dividing by 6 in (3.17), we obtain:

$$\left\| \frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6} \right\| \leq \frac{\xi(6u_1)}{6^2}, \quad \forall u_1 \in J. \quad (3.18)$$

From (3.17) and (3.18), it follows that:

$$\begin{aligned} \left\| \frac{h(6^2u_1)}{6^2} - h(u_1) \right\| &\leq \left\| \frac{h(6u_1)}{6} - h(u_1) \right\| + \left\| \frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6} \right\| \\ &\leq \frac{1}{6} \left[\xi(u_1) + \frac{\xi(6u_1)}{6} \right], \quad \forall u_1 \in J. \end{aligned} \quad (3.19)$$

Mathematical induction on a positive integer n allows us to continue the investigation and yields:

$$\left\| \frac{h(6^n u_1)}{6^n} - h(u_1) \right\| \leq \frac{1}{6} \sum_{k=0}^{n-1} \frac{\xi(6^k u_1)}{6^k} \leq \frac{1}{6} \sum_{k=0}^{\infty} \frac{\xi(6^k u_1)}{6^k}, \quad \forall u_1 \in J. \quad (3.20)$$

Replacing u_1 with $6^m u_1$ and dividing by 6^m in (3.20) shows the convergent behavior of the sequence $\left\{ \frac{h(6^n u_1)}{6^n} \right\}$; for every $m, n > 0$, we have:

$$\begin{aligned} \left\| \frac{h(6^{n+m} u_1)}{6^{(n+m)}} - \frac{h(6^m u_1)}{6^m} \right\| &= \frac{1}{6^m} \left\| \frac{h(6^n \cdot 6^m u_1)}{6^n} - h(6^m u_1) \right\| \\ &\leq \frac{1}{6} \sum_{k=0}^{n-1} \frac{\xi(6^{k+m} u_1)}{6^{(k+m)}} \\ &\leq \frac{1}{6} \sum_{k=0}^{\infty} \frac{\xi(6^{k+m} u_1)}{6^{(k+m)}} \\ &\rightarrow 0 \text{ as } m \rightarrow \infty, \quad \forall u_1 \in J. \end{aligned}$$

Hence $\left\{ \frac{h(6^n u_1)}{6^n} \right\}$ is Cauchy. Since L is complete, there exists a mapping $A : J \rightarrow L$ such that: $A(u_1) = \lim_{n \rightarrow \infty} \frac{h(6^n u_1)}{6^n}$, $\forall u_1 \in J$. We can show that (3.13) holds $\forall u_1 \in J$ when we extend n in (3.20) to infinity.

To prove that A meets (1.8), we can replace (u_1, u_2, u_3) by $(6^n u_1, 6^n u_2, 6^n u_3)$ and divide by 6^n in (??, leading to:

$$\begin{aligned} &\frac{1}{6^n} \left\| h(6^n(3u_1 + 2u_2 + u_3)) + h(6^n(3u_1 + 2u_2 - u_3)) + h(6^n(3u_1 - 2u_2 + u_3)) \right. \\ &\quad \left. + h(6^n(3u_1 - 2u_2 - u_3)) - 12h(6^n u_1) \right\| \\ &\leq \frac{1}{6^n} \psi(6^n u_1, 6^n u_2, 6^n u_3), \quad \forall u_1, u_2, u_3 \in J. \end{aligned}$$

Allowing n to tend towards infinity in the previous inequality and applying the characterization of $A(u_1)$, we obtain:

$$A(3u_1 + 2u_2 + u_3) + A(3u_1 + 2u_2 - u_3) + A(3u_1 - 2u_2 + u_3) + A(3u_1 - 2u_2 - u_3) - 12A(u_1).$$

Hence A meets (1.8) $\forall u_1, u_2, u_3 \in J$. To prove that A is unique, if $B(u_1)$ is another additive mapping satisfying (1.8) and (3.13), then:

$$\begin{aligned} \|A(u_1) - B(u_1)\| &= \frac{1}{6^n} \|A(6^n u_1) - B(6^n u_1)\| \\ &\leq \frac{1}{6} 6^n \{ \|A(6^n u_1) - h(6^n u_1)\| + \|h(6^n u_1) - B(6^n u_1)\| \} \\ &\leq \frac{2}{6} \sum_{k=0}^{\infty} \frac{\xi(6^{k+n} u_1)}{6^{(k+n)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \quad \forall u_1 \in J. \end{aligned}$$

Hence A is unique. Therefore, the theorem holds for both $j = 1$ and $j = -1$. The proof of the theorem is now complete. \square

Corollary 3.2. *Let λ and s be non-negative real numbers. Consider a function $h : J \rightarrow L$ that meets the following inequality:*

$$\|Dh(u_1, u_2, u_3)\| \leq \begin{cases} \lambda, & \text{if } s \neq 1; \\ \lambda \|u_1\|^s + \|u_2\|^s + \|u_3\|^s, & \text{if } 3s \neq 1; \\ \lambda \|u_1\|^s \|u_2\|^s \|u_3\|^s, & \text{if } 3s \neq 1; \\ \lambda \|u_1\|^s \|u_2\|^s \|u_3\|^s + \|u_1\|^{3s} + \|u_2\|^{3s} + \|u_3\|^{3s}, & \text{if } 3s \neq 1; \end{cases} \quad (3.21)$$

for all $u_1, u_2, u_3 \in J$. Under these conditions, there exists a unique additive function $A : J \rightarrow L$ such that the following stability condition holds:

$$\|h(u_1) - A(u_1)\| \leq \begin{cases} \frac{3\lambda}{10}, & \text{if } s \neq 1; \\ \frac{4\lambda \|u_1\|^s}{|6 - 6^s|}, & \text{if } s \neq 1; \\ \frac{\lambda \|u_1\|^{3s}}{|6 - 6^{3s}|}, & \text{if } 3s \neq 1; \\ \frac{5\lambda \|u_1\|^{3s}}{|6 - 6^{3s}|}, & \text{if } 3s \neq 1; \end{cases} \quad (3.22)$$

for all $u_1 \in J$.

4. CONCLUSIONS

In conclusion, this research article has employed a direct method to ascertain the solution of the specified equation and to explore its generalized Hyers-Ulam stability within the context of Banach spaces.

Through our findings in Section 2, we have demonstrated the existence of general solutions to Equations (1.7), (1.8), and (1.9). Furthermore, in Section 3, we have utilized the Hyers direct approach to assess the stability of the functional equation

(1.8) under the generalized Hyers-Ulam conditions. These results contribute to a deeper understanding of the stability properties of the given equation and provide valuable insights into its solutions in the realm of Banach spaces.

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Competing Interests

The authors declare that no competing interests exist.

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