Original Research Article

GENERAL SOLUTIONS AND GENERALIZED HYER-ULAM STABILITY IN BANACH SPACES: A DIRECT METHOD APPROACH FOR A SYSTEM OF FUNCTIONAL EQUATIONS

Abstract. In this paper, we derived general solutions and established the generalized Hyer-Ulam stability of the following system of functional equations

(i) $h(u_1+u_2+u_3)+h(u_1+u_2-u_3)+h(u_1-u_2+u_3)+h(u_1-u_2-u_3)$
$= 4h(u_1)$,

(ii) $h(3u_1+2u_2+u_3)+h(3u_1+2u_2-u_3)+h(3u_1-2u_2+u_3)+h(3u_1-2u_2-u_3)$
$= 12h(u_1)$,

(iii) $h(u_1+2u_2+3u_3)+h(u_1+2u_2-3u_3)+h(u_1-2u_2+3u_3)+h(u_1-2u_2-3u_3)$
$= 4h(u_1)$

in Banach spaces using direct method.

1. Introduction

Upon raising the fundamental question, “Under what circumstances can we confidently state that a solution to a marginally altered equation remains in close proximity to the solution of the original equation?” The investigation into the stability of functional equations was initiated by S.M. Ulam [24] in the year 1940. This inquiry led to a surge of new research in the field. In 1941, D. H. Hyers [10] presented an affirmative response to the inquiry about the stability of Banach spaces,

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building upon the pioneering investigations conducted by Ulam. In the year 1950, T. Aoki [2] made significant contributions to the advancement of our comprehension of additive mappings.

Th.M. Rassias [22] made a significant contribution by reducing the prerequisites for the Cauchy difference through an extension of Hyers' Theorem. The contributions of Ulam, Hyers, and Rassias to the field of stability notions for functional equations have been extensively recorded in the literature [1, 7, 15, 16, 17, 18]. Their work has played a pivotal role in the advancement of this area, ultimately resulting in the establishment of Hyers-Ulam-Rassias stability.

In the year 1994, P. Gavruta [8] made progress in the field by substituting the unbounded Cauchy difference with a control function that has broader applicability, drawing inspiration from the approach developed by Rassias. The aforementioned references offer a historical framework for examining the stability of functional equations and a comprehensive evaluation of the existing research. Throughout the annals of functional equations history, mathematicians have ushered in a realm of inquiry, delving into the discovery and examination of solutions and stability across diverse categories of functional equations, as documented in various references [4, 5, 6, 9, 11, 12, 19, 20, 21, 23].

In this segment, we delve into the analysis of the robustness of equations related to functional additivity within Banach spaces. Our main emphasis is directed towards scrutinizing the Cauchy functional equation:

$$K(g_1 + g_2) = K(g_1) + K(g_2). \tag{1.1}$$

It's worth noting that the equation $h(s) = cs$ stands as the solution to the aforementioned problem, representing what can be characterized as an additive functional equation. Several authors have conducted analyses and published results related to Ulam-type stability for various additive functional equations using this framework.

Following are some additive functional equations that have been explored in [3, 13, 14] along with their solutions and stability:

$$h(2g_1 - g_2) + h(g_1 - 2g_2) = 3h(g_1) - 3h(g_2), \tag{1.2}$$

$$h(2g_1 \pm g_2 \pm g_3) = h(g_1 \pm g_2) + h(g_1 \pm g_3), \tag{1.3}$$

$$h(g_1 + g_2 - 2g_3) + h(2g_1 + 2g_2 - g_3) = 3h(g_1) + 3h(g_2) - 3h(g_3), \tag{1.4}$$

$$\begin{aligned} &dh(g_1 + dg_2) - h(dg_1 + g_2) \\ &= \frac{d(d^2-1)}{2}[h(g_1 + g_2) + h(g_1 - g_2)] + (d - d^3)h(g_1) + (d^2 - 1)h(g_2). \end{aligned} \tag{1.5}$$

Recently, V. Govindan et al. [9] introduced a novel symmetric additive functional equation, which arises from a characteristic polynomial of degree three in the following form:

$$\begin{aligned}
&h[(a^3+11a)g_1-6(a^2+1)g_2]+h[(11a-6a^2)g_2+(a^3-6)g_3]\\
&-h[(a^3-6a^2)g_3+(11a-6)g_1]\\
&=(a^3+6)h(g_1)+(11a-12a^2-6)h(g_2)+(6a^2-6)h(g_3), \qquad (1.6)
\end{aligned}$$

where $a \neq 0$. They explored the whole response and expanded their study to test Hyers-Ulam stability via fixed-point and direct methods in Banach spaces.

The primary objective of this research is to propose an innovative form of functional equation, which is presented as follows:

(i) $h(u_1+u_2+u_3)+h(u_1+u_2-u_3)+h(u_1-u_2+u_3)+h(u_1-u_2-u_3)$
$= 4h(u_1),$ (1.7)

(ii) $h(3u_1+2u_2+u_3)+h(3u_1+2u_2-u_3)+h(3u_1-2u_2+u_3)+h(3u_1-2u_2-u_3)$
$= 12h(u_1),$ (1.8)

(iii) $h(u_1+2u_2+3u_3)+h(u_1+2u_2-3u_3)+h(u_1-2u_2+3u_3)+h(u_1-2u_2-3u_3)$
$= 4h(u_1)$ (1.9)

In this research article, we employ a direct method to determine the solution of the given equation and to investigate its generalized Hyers-Ulam stability within the framework of Banach spaces.

In Section 2, we demonstrated that there are general solutions to Equations (1.7), (1.8), and (1.9).

In Section 3, we use the Hyers direct approach to examine if the functional equation (1.8) is stable under the generalized Hyers-Ulam conditions.

2. GENERAL SOLUTIONS

Equations (1.7), (1.8), and (1.9) will have their general solutions given in this section. In the following, we will refer to J and L as real vector spaces.

Theorem 2.1 ([5]). *If a mapping $h : J \to L$ satisfies the functional equation (1.1), $\forall u_1, u_2 \in J$, then the following properties hold:*

- (i) $h(0) = 0$;
- (ii) h *is an odd function;*
- (iii) $h(\lambda u_1) = \lambda h(u_1)$ *for every number* $\lambda \in \mathbb{Q}$ *and for every* $u_1 \in J$.

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1, u_2 \in J$.
Setting (u_1, u_2) to $(0, 0)$ in (1.1), we get $h(0) = 0$.

Furthermore, setting $(u_1,\, u_2)$ to $(0,\, u_1)$ in (1.1), we arrive at
$h(-u_1) = -h(u_1), \quad \forall\ u_1 \in J.$
Therefore, h is an odd function.
Replacing u_2 with u_1 and u_2 by $2u_1$ in (1.1), we respectively deduce
$h(2u_1) = 2h(u_1)$ and $h(3u_1) = 3h(u_1), \quad \forall\ u_1 \in J.$
For each positive integer a, it is generally true that
$h(au_1) = ah(u_1), \quad \forall\ u_1 \in J.$
Consequently we see that $h(\lambda u_1) = \lambda h(u_1)$, $\forall \lambda \in \mathbb{Z}$. Let $\lambda = \dfrac{m}{n}$, where $m \in \mathbb{Z}, n \in \mathbb{N}$. Hence $mu_1 = n(\lambda u_1)$ and by what has already been proved,
$h(mu_1) = mh(u_1) = mh(n(\lambda u_1)) = nh(\lambda u_1)$,
which establishes the equality (iii), $\forall \lambda \in \mathbb{Q}$ ☐

Theorem 2.2. *A mapping $h : J \to L$ satisfies the functional equation (1.1), $\forall u_1,\, u_2 \in J$ if and only if $h : J \to L$ satisfies the functional equation (1.7), $\forall u_1,\, u_2,\, u_3 \in\ J$.*

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1, u_2 \in J$. Replacing u_2 by $u_2 + u_3$ in (1.1) and using Theorem 2.1, we obtain

$$h(u_1 + u_2 + u_3) = h(u_1) + h(u_2) + h(u_3).$$

Similarly, replacing u_2 by $u_2 - u_3$, $-u_2 + u_3$, and $-u_2 - u_3$ respectively in (1.1), we get the following equations:

$$\begin{aligned} h(u_1 + u_2 - u_3) &= h(u_1) + h(u_2) + h(-u_3) \\ h(u_1 - u_2 + u_3) &= h(u_1) + h(-u_2) + h(u_3) \\ h(u_1 - u_2 - u_3) &= h(u_1) + h(-u_2) + h(-u_3) \end{aligned}$$

Adding these equations while considering the oddness of h results in equation (1.7). Thus, if h satisfies (1.1), it also satisfies (1.7).

Conversely, let $h : J \to L$ satisfy the functional equation (1.7), $\forall u_1, u_2, u_3 \in J$. By substituting $(u_1, u_2, u_3) = (0, 0, 0)$ into (1.7), we obtain $h(0) = 0$. Furthermore, by substituting $(u_1, u_2, u_3) = (0, 0, u_1)$ into (1.7), we deduce

$$h(-u_1) = -h(u_1), \quad \forall u_1 \in J.$$

Therefore, h is an odd function. Finally, by setting $u_3 = 0$ in (1.7), we derive

$$h(u_1 + u_2) + h(u_1 - u_2) = 2h(u_1), \quad \forall u_1, u_2 \in J.$$

Interchanging u_1 and u_2 in the previous equation and utilizing the oddness of h, we obtain

$$h(u_1 + u_2) - h(u_1 - u_2) = 2h(u_2), \quad \forall u_1, u_2 \in J.$$

Adding the last two equations together, we arrive at equation (1.1). ☐

Theorem 2.3. *A mapping $h : J \to L$ satisfies the functional equation (1.1) , $\forall u_1,\ u_2 \in J$ if and only if $h : J \to L$ satisfies the functional equation (1.8), $\forall u_1,\ u_2,\ u_3 \in J$.*

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1, u_2 \in J$. Replacing (u_1, u_2) by $(3u_1, 2u_2 + u_3)$ in (1.1) and using Theorem 2.1, we obtain

$$h(3u_1 + 2u_2 + u_3) = 3h(u_1) + 2h(u_2) + h(u_3).$$

Similarly, replacing u_2 by $u_2 - u_3$, $-u_2 + u_3$, and $-u_2 - u_3$ respectively in (1.1), we get the following equations:

$$\begin{aligned} h(3u_1 + 2u_2 - u_3) &= 3h(u_1) + 2h(u_2) + h(-u_3) \\ h(3u_1 - 2u_2 + u_3) &= 3h(u_1) + 2h(-u_2) + h(u_3) \\ h(3u_1 - 2u_2 - u_3) &= 3h(u_1) + 2h(-u_2) + h(-u_3) \end{aligned}$$

Adding these equations while considering the oddness of h results in equation (1.8). Thus, if h satisfies (1.1), it also satisfies (1.8).

Conversely, let $h : J \to L$ satisfy the functional equation (1.8), $\forall u_1, u_2, u_3 \in J$. By substituting $(u_1, u_2, u_3) = (0, 0, 0)$ into (1.8), we obtain $h(0) = 0$. Furthermore, by substituting $(u_1, u_2, u_3) = (0, 0, u_1)$ into (1.8), we deduce

$$h(-u_1) = -h(u_1), \quad \forall u_1 \in J.$$

Therefore, h is an odd function. By setting $(u_1, u_2, u_3) = (u_1, 0, 0)$ in (1.8), we derive

$$h(3u_1) = 3h(u_1), \quad \forall u_1, u_2 \in J.$$

Again setting u_1 into $\dfrac{u_1}{3}$ in the previous equation, we obtain

$$h\left(\frac{u_1}{3}\right) = \frac{1}{3}h(u_1), \quad \forall u_1 \in J.$$

By setting $(u_1, u_2, u_3) = \left(\frac{u_1}{3}, \frac{u_2}{2}, 0\right)$ in (1.8) and using the above equation, we derive

$$h(u_1 + u_2) + h(u_1 - u_2) = 2h(u_1), \quad \forall u_1, u_2 \in J.$$

The remaining argument is quite similar to that presented for Theorem 2.2. □

We can establish the following theorem in a manner analogous to the proof of the preceding two theorems. For the sake of brevity, we shall skip the proof.

Theorem 2.4. *A mapping $h : J \to L$ satisfies the functional equation (1.1) $\forall u_1,\ u_2 \in J$ if and only if $h : J \to L$ satisfies the functional equation (1.9) $\forall u_1,\ u_2,\ u_3 \in J$.*

Theorems 2.2, 2.3, and 2.4 establish that a mapping $h : J \to L$ satisfies the functional equation (1.1) $\forall u_1,\ u_2,\ u_3 \in J$ if and only if it satisfies the functional equations (1.7), (1.8), and (1.9) $\forall u_1,\ u_2,\ u_3 \in J$, respectively.

Based on the aforementioned theorems, we can state the following theorem without presenting the proof:

Theorem 2.5. *Let $h : J \to L$ be a mapping. Then the following are equivalent:*

(i) *h fulfills the functional equation (1.1), $\forall u_1,\, u_2,\, u_3 \in\ J$.*
(ii) *h fulfills the functional equation (1.7), $\forall u_1,\, u_2,\, u_3 \in\ J$.*
(iii) *h fulfills the functional equation (1.8), $\forall u_1,\, u_2,\, u_3 \in\ J$.*
(iv) *h fulfills the functional equation (1.9), $\forall u_1,\, u_2,\, u_3 \in\ J$.*

In this article, hereafter we will use the following notation: Let J represents a normed space, and L and K represent Banach spaces. We define a mapping $Dh : J^3 \to L$ as follows:

$$\begin{aligned} Dh(u_1, u_2, u_3) =& h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) \\ & + h(3u_1 - 2u_2 - u_3) - 12h(u_1), \quad \forall u_1, u_2, u_3 \in J. \end{aligned} \tag{2.10}$$

3. Stability Results

The stability of the additive functional equation (1.8) is shown under generalized Hyers-Ulam conditions.

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\psi : J^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(6^{nj}u_1,\, 6^{nj}u_2,\, 6^{nj}u_3\right)}{6^{nj}} = 0, \quad \forall u_1,\, u_2,\, u_3 \in J. \tag{3.11}$$

Let $h : J \to L$ be a function that meets the inequality

$$\|Dh(u_1,\, u_2,\, u_3)\| \le \psi\left(u_1,\, u_2,\, u_3\right), \quad \forall u_1,\, u_2,\, u_3 \in J. \tag{3.12}$$

Then there exists a unique additive mapping $A : J \to L$ which satisfies (1.8) and

$$\|h(u_1) - A(u_1)\| \le \frac{1}{6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(6^{kj}u_1\right)}{6^{kj}}, \quad \forall\ u_1 \in J, \tag{3.13}$$

where $\xi\left(u_1\right)$ and $A(u_1)$ are defined by $\xi\left(u_1\right) = \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right)$ and $A(u_1) = \lim\limits_{n\to\infty} \frac{h(6^{nj}u_1)}{6^{nj}}, \quad \forall\ u_1 \in J$.

Proof. Replacing $(u_1,\, u_2,\, u_3)$ by $(u_1,\, u_1,\, u_1)$ in (3.12), we get

$$\|h(6u_1) + h(2u_1) - 12h(u_1)\| \le \psi\left(u_1,\, u_1,\, u_1\right), \quad \forall\ u_1 \in J. \tag{3.14}$$

Again, replacing $(u_1,\, u_2,\, u_3)$ by $(u_1,\, 0,\, u_1)$ in (3.12), we obtain

$$\|h(4u_1) + h(2u_1) - 6h(u_1)\| \le \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned}
&\|h(6u_1) - 6h(u_1)\|\\
&= \|h(6u_1) + h(4u_1) + h(2u_1) - 12h(u_1) - h(4u_1) - h(2u_1) + 6h(u_1)\|\\
&\leq \|h(6u_1) + h(2u_1) - 12h(u_1)\| + \|h(4u_1) + h(2u_1) - 6h(u_1)\|\\
&\leq \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J. \qquad (3.16)
\end{aligned}$$

By dividing the aforementioned inequality by 6, we get

$$\left\|\frac{h(6u_1)}{6} - h(u_1)\right\| \leq \frac{\xi(u_1)}{6} \qquad (3.17)$$

where $\xi(u_1) = \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J$. Now replacing u_1 by $6u_1$ and dividing by 6 in (3.17), we get

$$\left\|\frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6}\right\| \leq \frac{\xi(6u_1)}{6^2}, \quad \forall\ u_1 \in J. \qquad (3.18)$$

From (3.17) and (3.18), we have

$$\begin{aligned}
\left\|\frac{h(6^2u_1)}{6^2} - h(u_1)\right\| &\leq \left\|\frac{h(6u_1)}{6} - h(u_1)\right\| + \left\|\frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6}\right\|\\
&\leq \frac{1}{6}\left[\xi(u_1) + \frac{\xi(6u_1)}{6}\right], \quad \forall\ u_1 \in J. \qquad (3.19)
\end{aligned}$$

Mathematical induction on a positive integer n allows us to continue the investigation and yields

$$\left\|\frac{h(6^nu_1)}{6^n} - h(u_1)\right\| \leq \frac{1}{6}\sum_{k=0}^{n-1}\frac{\xi(6^ku_1)}{6^k} \leq \frac{1}{6}\sum_{k=0}^{\infty}\frac{\xi(6^ku_1)}{6^k}, \quad \forall\ u_1 \in J. \qquad (3.20)$$

Replace u_1 with 6^mu_1 and divide by 6^m in (3.20) to demonstrate the convergent behavior of the sequence $\left\{\frac{h(6^nu_1)}{6^n}\right\}$; for every $m, n > 0$, we get

$$\begin{aligned}
\left\|\frac{h(6^{n+m}u_1)}{6^{(n+m)}} - \frac{h(6^mu_1)}{6^m}\right\| &= \frac{1}{6^m}\ \left\|\frac{h(6^n\cdot 6^mu_1)}{6^n} - h(6^mu_1)\right\|\\
&\leq \frac{1}{6}\sum_{k=0}^{n-1}\frac{\xi(6^{k+m}u_1)}{6^{(k+m)}}\\
&\leq \frac{1}{6}\sum_{k=0}^{\infty}\frac{\xi(6^{k+m}u_1)}{6^{(k+m)}}\\
&\to 0 \quad as\ \ m \to \infty, \quad \forall\ u_1 \in J.
\end{aligned}$$

Hence $\left\{\dfrac{h(6^n u_1)}{6^n}\right\}$ is Cauchy. Since L is complete, there exists a mapping $A: J \to L$ such that $A(u_1) = \lim\limits_{n\to\infty} \frac{h(6^n u_1)}{6^n}, \quad \forall\, u_1 \in J.$
We can show that (3.13) holds $\forall u_1 \in J$ when we extend n in (3.20) to infinity.

To prove that A satisfies (1.8), replacing $(u_1,\, u_2,\, u_3)$ by $(6^n u_1,\, 6^n u_2,\, 6^n u_3)$ and dividing by 6^n in (3.12), we obtain

$$\begin{aligned}\frac{1}{6^n}\Big\| h(6^n(3u_1+2u_2+u_3)) + h(6^n(3u_1+2u_2-u_3)) + h(6^n(3u_1-2u_2+u_3)) \\ + h(6^n(3u_1-2u_2-u_3)) - 12h(6^n u_1)\Big\| \\ \le \frac{1}{6^n}\psi(6^n u_1,\, 6^n u_2,\, 6^n u_3), \quad \forall u_1,\, u_2,\, u_3 \in J.\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the definition of $A(u_1)$, we see that

$$A(3u_1+2u_2+u_3) + A(3u_1+2u_2-u_3) + A(3u_1-2u_2+u_3) + A(3u_1-2u_2-u_3) - 12A(u_1).$$

Hence A satisfies (1.8) $\forall u_1,\, u_2,\, u_3 \in J$. To prove that A is unique, let $B(u_1)$ be another additive mapping satisfying (1.8) and (3.13), then

$$\begin{aligned}\|A(u_1) - B(u_1)\| &= \frac{1}{6^n}\,\|A(6^n u_1) - B(6^n u_1)\| \\ &\le \frac{1}{6^n}\,\{\|A(6^n u_1) - h(6^n u_1)\| + \|h(6^n u_1) - B(6^n u_1)\|\} \\ &\le \frac{2}{6}\sum_{k=0}^{\infty}\frac{\xi(6^{k+n}u_1)}{6^{(k+n)}} \\ &\to 0 \quad as \;\; n \to \infty, \quad \forall\, u_1 \in J.\end{aligned}$$

Thus A is unique. Hence for $j = 1$ the theorem holds.

Now, replacing u_1 by $\dfrac{u_1}{6}$ in (3.16), we reach

$$\left\| h(u_1) - 1296h\left(\frac{u_1}{6}\right)\right\| \le \psi\left(\frac{u_1}{6},\, \frac{u_1}{6},\, \frac{u_1}{6}\right) + \frac{1}{2}\psi\left(\frac{u_1}{6},\, 0,\, \frac{u_1}{6}\right),\ \forall\, u_1 \in J. \tag{3.21}$$

The remaining steps of the proof are the same as in the $j = 1$ case. Therefore, the theory also holds when $j = -1$. The proof of the theorem is now complete. □

Corollary 3.2. *Let λ and s be non-negative real numbers. Let $h : J \to L$ be a function satisfying the inequality*

$$\|Dh(u_1,\, u_2,\, u_3)\| \leq \begin{cases} \lambda, & \\ \lambda \left\|u_1\right\|^s + \left\|u_2\right\|^s + \left\|u_3\right\|^s, & s \neq 1; \\ \lambda \left\|u_1\right\|^s \left\|u_2\right\|^s \left\|u_3\right\|^s, & 3s \neq 1; \\ \lambda \left\|u_1\right\|^s \left\|u_2\right\|^s \left\|u_3\right\|^s + \left\|u_1\right\|^{3s} + \left\|u_2\right\|^{3s} + \left\|u_3\right\|^{3s}, & 3s \neq 1; \end{cases} \tag{3.22}$$

$\forall u_1,\, u_2,\, u_3 \in J$. Then there exists a unique additive function $A : J \to L$ such that

$$\|h(u_1) - A(u_1)\| \leq \begin{cases} \dfrac{3\lambda}{10}, & \\ \dfrac{4\lambda \left\|u_1\right\|^s}{|6 - 6^s|}, & s \neq 1; \\ \dfrac{\lambda \left\|u_1\right\|^{3s}}{|6 - 6^{3s}|}, & 3s \neq 1; \\ \dfrac{5\lambda \left\|u_1\right\|^{3s}}{|6 - 6^{3s}|}, & 3s \neq 1; \end{cases} \tag{3.23}$$

$\forall u_1 \in J$.

4. CONCLUSIONS

In conclusion, this research article has employed a direct method to ascertain the solution of the specified equation and to explore its generalized Hyers-Ulam stability within the context of Banach spaces.

Through our findings in Section 2, we have demonstrated the existence of general solutions to Equations (1.7), (1.8), and (1.9). Furthermore, in Section 3, we have utilized the Hyers direct approach to assess the stability of the functional equation (1.8) under the generalized Hyers-Ulam conditions. These results contribute to a deeper understanding of the stability properties of the given equation and provide valuable insights into its solutions in the realm of Banach spaces.

References

[1] Aczel, J., Dhombres, J. Functional Equations in Several Variables. Cambridge Univ. Press. 1989.

[2] Aoki, T. On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan. 1950;2:64-66.

[3] Arunkumar, M. Three-dimensional Quartic Functional Equation in Fuzzy Normed spaces. Far East J. Appl. Math. 2010;41(2):83–94.

[4] Arunkumar, M. Solution and stability of Arun-additive functional equations. Int. J. Math. Sci. Eng. Appl. 2010;4(3):33-46.

[5] Czerwik, S. Functional Equations and Inequalities in Several Variables. World Scientific, River Edge, NJ. 2002.

[6] Eskandani, GZ., Găvruţă, P. On the stability problem in quasi-Banach spaces. Nonlinear Funct. Anal. Appl. 2012;5:459-465.

[7] Gajda, Z. On stability of additive mappings. Int. J. Math. Sci. 1991;14:431-434.

[8] Găvruţă, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. 1994;184:431-436.

[9] Govindan, V., Alb Lupaş, A., Noeiaghdam, S. Stability of Additive Functional Equation Originating from Characteristic Polynomial of Degree Three. Symmetry. 2022;14(4):700.

[10] Hyers, DH. On the stability of the linear functional equation. Proc. Nat. Acad. Sci., U.S.A. 1941;27:222-224.

[11] Hyers, DH., Isac, G., Rassias, Th.M. Stability of functional equations in several variables. Birkhauser, Basel. 1998.

[12] Jung, SM. Hyers-Ulam-Rassias Stability for Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor. 2001.

[13] Lee, DO. Hyers-Ulam stability of an additive type functional equation. J. Appl. Math. Comput. 2003;13(1-2):471-477.

[14] Narasimmon, P. Solution and stability of a generalized k–additive functional equation. Journal of Interdisciplinary Mathematics. 2018;21(1):171-184.

[15] Rassias, JM. On approximately of approximately linear mappings by linear mappings. J. Funct. Anal. USA. 1982;46:126-130.

[16] Rassias, JM. Solution of a problem of Ulam. J. Approx. Theory. 1989;57:268-273.

[17] Rassias, JM. Solution of a stability problem of Ulam. Discuss. Math. 1992;12:95-103.

[18] Rassias, JM. On the stability of the Euler-Lagrange functional equation. Chin. J. Math. 1992;20:185-190.

[19] Rassias, JM, Rassias, MJ. Asymptotic behavior of alternative Jensen and Jensen type functional equations. Bull. Sci. Math. 2005;129:545-558.

[20] Rassias, JM. Refined Hyers Ulam approximation of approximately Jensen type mappings. Bull. Sci. Math. 2007;131:89-98.

[21] Rassias, JM, Kim, HM. Generalized Hyers-Ulam stability for general additive functional equations in quasi$-\beta-$normed spaces. J. Math. Anal. Appl. 2009;356(1):302-309.

[22] Rassias, ThM. On the stability of the linear mapping in Banach spaces. Proc. Amer. Math Soc. 1978;72:297-300.

[23] Tober, J. Stability of Cauchy functional equation in quasi-Banach spaces. Ann. Polon. Math. 2004;83:243-255.

[24] Ulam, SM. Problems in Modern Mathematics. Science Editions, Wiley, New York. 1964.