An analytic Weighted Lipschitz Algebraic sequence with Closed Ideals

In a series of weighted Lipschitz algebras $(\Lambda_r)_\omega$ of a series of analytic functions on the unit disk, we obtain a comprehensive description of closed ideals that satisfies the following requirement

$$\sum_r \frac{|f_r(z) - f_r(z+\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0).$$

where ω is a continuous modulus meeting certain regularity requirements. The closed ideals of the algebras $(\Lambda_r)_{\chi_{1-\varepsilon}}$, where $\chi_{1-\varepsilon}(2-\varepsilon) := \frac{1}{(|\log(2-\varepsilon)|+1)^{1-\varepsilon}}$, $\varepsilon > 1$, in particular, are standard and this resolves Shirokov's query. Namely the weighted Lipschitz algebra possesses a factorization property, i.e a sequence of analytic functions and an inner functions such that their quotients belong to the essential algebra of a disk of analytic functions, [Closed ideals of algebras by N.A. Shirokov of $B_{pq}^{1-\varepsilon}$.

Keywords:; Banach algebra; Closed ideals; invariant subspaces; Resolvent method ; Weighted Lipschitz algebra; Factorization property; Hardy space; Beurling – Rudin characterization.

1.Introduction and statement of primary finding

Be the complex plane's unit disk $\mathbb{D}$ and boundary for $\mathbb{T}$. By $\mathcal{A}(\mathbb{D})$ we mean the standard disk algebra of all continuous analytic functions f on $\mathbb{D}$. The weighted Lipschitz algebra is defined [10, 11]

$(\Lambda_r)_\omega(\mathbb{D}) = (\Lambda_r)_\omega$ to $(\Lambda_r)_\omega := \left\{f_r \in \mathcal{A}(\mathbb{D}): \sup_{z\in\mathbb{D}} \sum_r \frac{|f_r(z)-f_r(z+\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0)\right\}$, where $\omega(2-\varepsilon)$ is a nondecreasing continuous real-valued function on $[0,2]$ with $\omega(0) = 0$ and $\frac{\omega(2-\varepsilon)}{(2-\varepsilon)}$ is non increasing function such that $\lim_{(2-\varepsilon)\to 0} \frac{\omega(2-\varepsilon)}{(2-\varepsilon)} = \infty$,which is a modulus of continuity. With the norm,

$$\|f_r\|_\omega := \sum_r \|f_r\|_\infty + \sup_{z\in\mathbb{D}} \sum_r \frac{|f_r(z) - f_r(w)|}{\omega(|\varepsilon|)},$$

it is evident that $(\Lambda_r)_\omega$ is a commutative Banach algebra .

With $\|f_r\|_\infty := \sup_{z\in\mathbb{D}} |f_r(z)|$. Similarly the definition of the weighted Lipschitz algebra $(\Lambda_r)_\omega(\mathbb{T})$ is given by

$$(\Lambda_r)_\omega := \left\{f_r \in \mathcal{A}(\mathbb{D}): \sup_{z\in\mathbb{T}} \sum_r \frac{|f_r(z) - f_r(w)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0)\right\}$$

In [7] Shirokov demonstrated that $(\Lambda_r)_\omega$ has the so-called F-property (Factorization property), which states that for every given $f_r \in (\Lambda_r)_\omega$ and inner function U_r such that f_r/U_r is member $\mathcal{H}^\infty(\mathbb{D})$ of the algebra of bounded analytic functions, we have $f_r/U_r \in (\Lambda_r)_\omega$ and $\left\|\sum f_r/U_r\right\|_\omega \le c\sum\|f_r\|_\omega$, for an absolute constant c (see Appendix B).You should be aware that Tamrazov [8] demonstrated that the algebras and coincide for any arbitrary modulus of continuity ω. (see Appendix A).

Beurling and Rudin separately provide the disk algebra's closed ideals structure [2]. They demonstrated $\mathfrak{T}$ the existence of $\mathcal{A}(\mathbb{D})$ an inner function $U_{\mathfrak{T}}$ (the greatest common divisor of the inner portions of the nonzero functions in $\mathfrak{T}$) such that if is a closed ideal of

$\mathfrak{T} = \{f_r \in \mathcal{A}(\mathbb{D}): f_{r|E_{\mathfrak{T}}^r} \equiv 0$ and $f_r/(U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})\}$,the location $E_{\mathfrak{T}}^r := \{\zeta + \varepsilon \in \mathbb{T}: f_r(\zeta+\varepsilon) = 0,\ \forall f_r \in \mathfrak{T}\}$.

In some algebras of analytic functions, the problem of characterization of closed ideal [10, 11] can be reduced to a problem of approximation of outer functions using Beurling–Carleman–Domar resolvent's approach and the F-property. (see for example [1] and references therein). The closed ideals of the algebras H_1^2 of analytic functions f_r that are in the Hardy space have been described by Korenblum [3]. He demonstratwd the universality of these values. (in the sense of Beurling–Rudin characterization of the closed ideals in the disk algebra). This conclusion has since been expanded to include certain more analytic functions Banach algebras [11]. In particular by Matheson [4] and independently by Shamoyan [5], for the algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}$, where $\varphi_{1-\varepsilon}(2-\varepsilon) := (2-\varepsilon)^{1-\varepsilon}$, $\varepsilon > 0$.

The resolvent method is explained as follows: Define $d(\zeta+\varepsilon, E^r)$ to be the distance from $(\zeta+\varepsilon) \in \mathbb{T}$ to the closed subset E^r of $\mathbb{T}$ and let $\mathfrak{I}$ be a closed ideal of the algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}$.

(i). In the first stage, where is the canonical quotient map, we estimate the norm of the resolvent $\sum\|((\zeta+\varepsilon)-\pi(z))^{-1}\|_{(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{I}}$ in the quotient algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{I}$, where $\pi : (\Lambda_r)_{\varphi_{1-\varepsilon}} \to (\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{I}$. We get at $\|((\zeta+\varepsilon)-\pi(z))^{-1}\|_{(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{I}} \leq \sum \frac{c}{d^4(\zeta+\varepsilon, E^r_{\mathfrak{I}})}$, where $1 \leq |\zeta+\varepsilon| \leq 2$ and c is an unchanging constant. Therefore, we infer from the Cauchy formula for the quotient algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{I}$, that all functions in $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ such that $f_r/(U_r)_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})$ and $\|\sum f_r(\zeta+\varepsilon)| \leq \sum d^4(\zeta+\varepsilon, E^r_{\mathfrak{I}})$, $(\zeta+\varepsilon) \in \mathbb{T}$, are in *I*.

(ii) The second stage is to demonstrate how dense the space of all functions in $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ such that $f_r/(U_r)_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})$ and $|\sum f_r(\zeta+\varepsilon)| \leq \sum d^4(\zeta+\varepsilon, E^r_{\mathfrak{I}})$, $(\zeta+\varepsilon) \in \mathbb{T}$, , a way that and the standard ideal $\{f_r \in (\Lambda_r)_{\varphi_{1-\varepsilon}} : f_{r|E^r_{\mathfrak{I}}} \equiv 0 \; f_r/(U_r)_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})\}$.

If the Carleson criterion is met ,i.e., a closed subset $E^r \subset \mathbb{T}$ is a Carleson set,

$$\frac{1}{2\pi}\int_0^{2\pi} \sum \log\left(\frac{1}{d(e^{i(2-\varepsilon)}, E^r)}\right) d(2-\varepsilon) < +\infty .$$

In any Banach algebras H_1^2 , $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ and other algebras where the structure of closed ideals is also explored using the resolvent approach [1,6], the zeros of each given function constitute a Carleson set.

Consider the algebras $(\Lambda_r)_{\chi_{1-\varepsilon}}$, where $\chi_{1-\varepsilon}(2-\varepsilon) := \frac{1}{(|\log(2-\varepsilon)|+1)^{1-\varepsilon}}$, $\varepsilon > 1$ [6, p. 587] as an illustration of how the resolvent technique does not work in the general situation. Infact, let $\mathfrak{I}$ be a closed ideal of $(\Lambda_r)_{\chi_{1-\varepsilon}}$ such that $E^r_{\mathfrak{I}}$ is not a Carleson set. The location

$$\left\|\sum((\zeta+\varepsilon)-\pi(z))^{-1}\right\|_{(\Lambda_r)_{\chi_{1-\varepsilon}}/\mathfrak{I}} \leq \sum \frac{c}{d^4(\zeta+\varepsilon, E^r_{\mathfrak{I}})}, \text{ where } 1 \leq |\zeta+\varepsilon| \leq 2 \text{ and}$$

$\pi : (\Lambda_r)_{\chi_{1-\varepsilon}} \to (\Lambda_r)_{\chi_{1-\varepsilon}}/\mathfrak{I}$ is the canonical quotient map are available. It is obvious that there is no power sufficient that it could cause $\int_{\mathbb{T}} \sum \frac{|f_r^M(e^{i(2-\varepsilon)})|}{d^4(e^{i(2-\varepsilon)}, E^r)}$ all functions $(\Lambda_r)_{\chi_{1-\varepsilon}}$ to cease $E^r_{\mathfrak{I}}$. As a result, the resolvent method's initial step cannot be finished as previously described.

From this point forward, there ω will be a continuity modulus such that for each $0 \leq \varepsilon \leq 1$ of the following conditions

$$\omega\left((2-\varepsilon)^{(1+\varepsilon)}\right) \geq \eta_{(1+\varepsilon)}\omega^{(1+\varepsilon)}(2-\varepsilon) \quad (\varepsilon \geq 0) \tag{1.1}$$

is met, where $\eta_{(1+\varepsilon)} > 0$ is a constant that only depends on $(1+\varepsilon)$.

In this paper, we demonstrate the standardity of closed ideals of the algebras $(\Lambda_r)_\omega$ are standard, and we do so using only a unique approach to approximating outer functions in $(\Lambda_r)_\omega$ and the F-property. More specifically, we obtain the following.

Theorem (1.1): Let be ω a fulfiling modulus of continuity .(1.1).Where and is the largest common divisor of the inner parts of the nonzero functions in $\mathfrak{I}$, if is closed ideal of $(\Lambda_r)_\omega$, then..

$$\mathfrak{I} = \left\{f_r \in (\Lambda_r)_\omega : f_{|E^r_{\mathfrak{I}}} \equiv 0 \text{ and } f_r/(U_r)_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})\right\},$$

where $E^r_{\mathfrak{I}} := \{(\zeta+\varepsilon) \in \mathbb{T} : f_r(\zeta+\varepsilon) = 0, \ \forall f_r \in \mathfrak{I}\}$ and $(U_r)_{\mathfrak{I}}$ is the greatest common divisor of the inner parts of the nonzero functions in $\mathfrak{I}$.

As a result , we are able to determine how the particular algebra's closed ideals $(\Lambda_r)_{\chi_{1-\varepsilon}}$. are structured.

2.Additional finding and the theorem's proof (1.1)

In this section we want to demonstrate that every closed ideals of the particular algebra is standard. Remembering that every function f_r in the disk algebra has a canonical factorization

$f_r = c_{f_r} U_{f_r} O_{f_r}$, where c_{f_r} is a constant of modulus 1, $(U_r)_{f_r}$ is an inner function (that is $|U_r f_r| = 1$ a. e. on $\mathbb{T}$) and O_{f_r} the outer function given by

$$O_{f_r}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_{j=1}^{n} e^{i\theta_j} + z}{\sum_{j=1}^{n} e^{i\theta_j} - z} \log\left| f_r(\sum_{j=1}^{n} e^{i\theta_j})\right| d(\sum_{j=1}^{n}\theta_j)\right\} \quad (z \in \mathbb{D}).$$

It is designed by The closed ideal of all functions in $(\Lambda_r)_\omega$ vanishing on E^r is designed by $\mathcal{J}_{E^r}$. We proved the proof for the following theorem in section 3.2.

Theorem (2.1): Let ω be a continuity modulus that satisfies the following requerment.

$$\omega((2-\varepsilon)^2) \geq \eta_2 \omega^2(2-\varepsilon), \quad \varepsilon \geq 0 \tag{2.1}$$

where $\eta_2 > 0$ is a fixed value. Let $\mathfrak{T}$ be an outer function and let be a closed ideal in $(\Lambda_r)_\omega$ this way, therefore, $(U_r)_{\mathfrak{T}} \equiv 1$ and let $g_r \in \mathcal{J}_{E^r_{\mathfrak{T}}}$ be an outer function. Then g_r^2 belongs to $\mathfrak{T}$.

Remark (2.2): Similar to Theorem (2.1), we can achieve this if the ω aforementioned stronger requerment is also met.

$$\omega((2-\varepsilon)^2) \geq \eta\, \omega\, (2-\varepsilon)\, , \qquad (\varepsilon \geq 0) \tag{2.2}$$

then g_r is owned by $\mathfrak{T}$.

We demonstrate the following theorem in Section 3.3 .

Theorem (2.3): Let ω be a modulus of continuity meeting the requerment.(46). Let $\mathfrak{T}$ be a closed ideal in $(\Lambda_r)_\omega$ and let $g_r \in (\Lambda_r)_\omega$ be a function such that $(U_r)_{g_r} O_{g_r}^2 \in \mathfrak{T}$. Consequently g belongs to $\mathfrak{T}$.

Proving the Theory (1.1). We need to demonstrate the standard nature of every closed ideal of the algebra $(\Lambda_r)_\omega$of this. Let g_r be a function in $\mathcal{J}_{E^r_{\mathfrak{T}}}$. Consequently, using the F-property of $(\Lambda_r)_\omega$, it follows $O_{g_r} \in (\Lambda_r)_\omega$, and thus $O_{g_r} \in \mathcal{J}_{E^r_{\mathfrak{T}}}$. So, in accordance with Theorem (4.3.2), we derive before proceeding $O_{g_r}^2 \in \mathfrak{T}$ and then $(U_r)_{g_r} O_{g_r}^4 \in \mathfrak{T}$. Theorem (4.3.4) is then applied twice, and we draw our conclusion $g_r \in \mathfrak{T}$.

Now suppose that $(U_r)_{\mathfrak{T}} \not\equiv 1$,that we make $g_r \in \mathcal{J}_{E^r_{\mathfrak{T}}}$ such that $g_r / (U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D}$ a decision ithis manner. Consequently, the linked ideal

$$\mathcal{K}_g := \{f_r \in (\Lambda_r)_\omega : (f_r)_{g_r} \in \mathfrak{T}\}$$

is closed, which we can plainly see thanks to its F-property of $(\Lambda_r)_\omega$ and after that $(U_r)_{\mathcal{K}_{g_r}} \not\equiv 1$ and then $\mathcal{K}_g = \mathcal{J}_{(E^r)_{\mathcal{K}_{g_r}}}$. Now, ever since $(E^r)_{\mathcal{K}_{g_r}} \subseteq E^r_{\mathfrak{T}}$, then $O_{g_r} \in \mathcal{K}_{g_r}$.Thus, It follows $(U_r)_{g_r} O_{g_r}^2 \in \mathfrak{T}$.Theorem (4.3.4) , $g_r \in \mathfrak{T}$ dictates as a result. The theorem's proof is now complete.

3. functions approximation in $(\Lambda_r)_\omega$

In this section we attempt to demonstrate that a function is a member of the closed ideal of all vanishing functions. The canonical factorization that results from the disk algebra when the set $E^r_{\mathfrak{T}}$ in an outer function and connected to it also belong to the closed ideal. We proved the Theorems (2.1) and (2.3). The following Tamrazov's Theorem (see Appendix A): If f_r is a disk algebraic function such that $f_r \in \Lambda_\omega(\mathbb{T})$, and $f_r \in (\Lambda_r)_\omega$. The next straightforward lemma is also necesssary.

Lemma(3.1). Assume that there is a chain $(f_r)_n \in (\Lambda_r)_\omega$ of functions that uniformly convergence to on the closed unit disk to $f_r \in (\Lambda_r)_\omega$. If

$$\sum \frac{|(f_r)_n(z) - (f_r)_n(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \qquad (as\ |\varepsilon| \to 0)\, ,$$

Consistently in regard to n , then $\lim_{n\to+\infty} \sum \|(f_r)_n - f_r\|_\omega = 0$.

For $f_r \in (\Lambda_r)_\omega$, the inner function U_{f_r} is uniquely factored in the form $U_{f_r} = B_{f_r} S_{f_r}$, where B_{f_r} the function is related with the typical Blashke product $Z_{f_r} \cap D, Z_{f_r} := \{z \in \mathbb{D}: f_r(z) = 0\}$ and the function

$$S_{f_r}(z) := \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \sum \frac{\sum\ e^{i\theta_j} + z}{\sum\ e^{i\theta_j} - z} d\mu_{f_r}(\sum \theta_j)\right\} \quad (z \in \mathbb{D})\, ,$$

is the unique internal process connected to the unique positive measure μ_{f_r} . Keep in mind that the closed subset of the support(μ_{f_r}) of the singular positive measure.

$E^r_{f_r} := \{(\zeta + \varepsilon) \in \mathbb{T}: f_r(\zeta + \varepsilon) = 0\}$. For $a, a + \varepsilon \in \mathbb{T}$, we create a closed arc or open arc by $(a, a + \varepsilon)$ (resp. $[a, a + \varepsilon]$) joining the points a and $a + \varepsilon$, respectively.

Lemma (3.2). Let ω be a continuity modulus meeting the requirment (2.1). Let g_r be a function in $(\Lambda_r)_\omega$ and let $U_r = B_{U_r}S_{U_r} \in \mathcal{H}^\infty(\mathbb{D})$ be an inner function so that $B_{g_r}/B_{U_r} \in \mathcal{H}^\infty(\mathbb{D})$, $\text{supp}(\mu_{U_r}) \subseteq E^r_{g_r}$ and $(1/2\pi)\int_0^{2\pi} d\mu_{U_r}(\sum_{j=1}^n \theta_j) \leq \widetilde{M}$, where $\widetilde{M}$ is a constant. Then $U_r O^2_{g_r}$ belongs to $(\Lambda_r)_\omega$ and we have

$$\sum \frac{\left|U_r(\zeta + \varepsilon)O^2_{g_r}(\zeta + \varepsilon) - U_r(\zeta)O^2_{g_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.1}$$

uniformly with regards to U_r. Additionally, if ω you meet criterion (2.2), you $U_r O_{g_r}$ belongs to $(\Lambda_r)_\omega$ and we have

$$\sum \frac{\left|U_r(\zeta + \varepsilon)O_{g_r}(\zeta + \varepsilon) - U_r(\zeta)O_{g_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.2}$$

consistently with regard to U_r.

Proof. Consider $\zeta + \varepsilon, \zeta \in \mathbb{T}$ two separate points so that $d(\zeta + \varepsilon, E^r_{g_r}) \leq d(\zeta, E^r_{g_r})$. It is obvious that

$$\sum \frac{\left|U_r(\zeta + \varepsilon)O_{g_r}(\zeta + \varepsilon) - U_r(\zeta)O_{g_r}(\zeta)\right|}{\omega(\varepsilon)} \leq \sum \frac{\left|B_{U_r}(\zeta + \varepsilon)O^2_{g_r}(\zeta + \varepsilon) - B_{U_r}(\zeta)O^2_{g_r}(\zeta)\right|}{\omega(\varepsilon)} + \sum \left|O^2_g(\zeta)\right| \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)}.$$

We have $O_{g_r} \in (\Lambda_r)_\omega$ and $B_{U_r}O^2_{g_r} \in (\Lambda_r)_\omega$ by virtue of the F-property of $(\Lambda_r)_\omega$. Then, it is enough to demonstrate that in order to prove (3.1). It is suffices to demonstrate that

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.3}$$

First, let's assume that $\varepsilon \geq \left(\frac{d(\zeta, E^r_{g_r})}{2}\right)^2$. Then

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} \leq 8\eta_2^{-1} \sum \left(\frac{\left|O_{g_r}(\zeta)\right|}{\omega(d(\zeta, E^r_{g_r}))}\right)^2 = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.4}$$

Now, Let's assume that$\varepsilon \geq \left(\frac{d(\zeta, E^r_{g_r})}{2}\right)^2$. Then $[\zeta + \varepsilon, \zeta] \subset \mathbb{T} \backslash E_{g_r}$ and therefore $d(z, E^r_{g_r}) \geq d(\zeta, E^r_{g_r})$, for each $z \in [\zeta + \varepsilon, \zeta]$. There is $z \in [\zeta + \varepsilon, \zeta]$ such that

$$\sum \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\varepsilon} = \sum \left|S'_{U_r}(z)\right| \leq \frac{1}{\pi}\int_0^{2\pi} \sum \frac{1}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\mu_{U_r}(\sum_{j=1}^n \theta_j) \leq \sum \frac{2\widetilde{M}}{d^2(z, E^r_{g_r})}.$$

Following is

$$\sum \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\varepsilon} \leq \sum \frac{2\widetilde{M}}{d^2(\zeta, E^r_{g_r})}.$$

We discover

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} = \sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta + \varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} \frac{\varepsilon}{\omega(\varepsilon)}$$

$$\leq 2M\eta_2^{-1} \sum \left(\frac{\left|O_{g_r}(\zeta)\right|}{\omega(d(\zeta, E^r_{g_r}))}\right)^2 \frac{\omega(d^2(\zeta, E^r_{g_r}))}{d^2(\zeta, E^r_{g_r})} \frac{\varepsilon}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0). \tag{3.5}$$

In light of (3.3) , (3.4) follows, and (3.5).Thus $U_r O^2_{g_r}$ belongs to $(\Lambda_r)_\omega$. Additionally ω, if criterion (2.2) is met, we can use similar arguments to demonstrate (3.2). This completes the lemma's proof.

Lemma (3.3). Allow a function f_r in to be Λ_ω. Let $\delta > 0$, $N \in \mathbb{N}$ and $\{\text{an}: 0 \le n \le N\}$ be a set of descrete points in $E^r_{f_r}$. Then

$$\lim \sum_{\delta \to 0} \left\| \psi_{\delta,N} f_r - f_r \right\|_\omega = 0 ,$$

where $\psi_{\delta,N}(z) \coloneqq \prod_{n=0}^{n=N} \frac{z\overline{a_n}-1}{z\overline{a_n}-1-\delta}, \; z\in\mathbb{D}$

Proof. We can assume that $N = 0$ and $a_0 = 1$ without losing the generality. Assume for $\psi_\delta(z) \coloneqq \frac{z-1}{z-1-\delta}$, $z \in \mathbb{D}$, a moment that $\{1\} \in E^r_{f_r}$. We must demonstrate that $\lim_{\delta\to 0} \sum \| \psi_\delta f_r - f_r \|_\omega = 0$. Let $\zeta + \varepsilon , \zeta \in \mathbb{T}$ be two separate points so that $\varepsilon \ge 0|$. To date

$$\sum \frac{|\psi_\delta(\zeta+\varepsilon) f_r(\zeta+\varepsilon) - \psi_\delta(\zeta) f_r(\zeta)|}{\omega(\varepsilon)}$$

$$\le |\psi_\delta(\zeta+\varepsilon)| \sum \frac{|f_r(\zeta+\varepsilon) - f_r(\zeta)|}{\omega(\varepsilon)} + \sum |f_r(\zeta)| \frac{|\psi_\delta(\zeta+\varepsilon) - \psi_\delta(\zeta)|}{\omega(\varepsilon)} . \tag{3.6}$$

Let's first assume that $\varepsilon \ge 0$. Then

$$\sum |f_r(\zeta)| \frac{|\psi_\delta(\zeta+\varepsilon) - \psi_\delta(\zeta)|}{\omega(\varepsilon)} \le 2 \sum \frac{|f_r(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) . \tag{3.7}$$

Now if $\varepsilon \ge 0$, then $|z-1| \ge |\zeta - 1|$ for each $z \in [\zeta+\varepsilon, \zeta]$. We discover

$$\sum |f_r(\zeta)| \frac{|\psi_\delta(\zeta+\varepsilon) - \psi_\delta(\zeta)|}{\omega(\varepsilon)} = \sum |f_r(\zeta)| \frac{|\psi_\delta(\zeta+\varepsilon) - \psi_\delta(\zeta)|}{(\varepsilon)} \frac{|f_r(\zeta)|}{\omega(\varepsilon)}$$

$$= \sum |f_r(\zeta)| |\psi'_\delta(Z)| \frac{\varepsilon}{\omega(\varepsilon)} \quad (Z \in [\varepsilon]) \le \sum \frac{|f_r(\zeta)|}{\omega(|\zeta-1|)} \frac{\omega(|\zeta-1|)}{|\zeta-1|} \frac{\varepsilon}{\omega(\varepsilon)}$$

$$= o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.8}$$

From (3.6), (3.7) and (3.8) we make

$$\sum \frac{|\psi_\delta(\zeta+\varepsilon) f_r(\zeta+\varepsilon) - \psi_\delta(\zeta) f_r(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0),$$

Consistently with regard to $\delta > 0$. Applying Lemma (3.1) to the family of functions $\psi_\delta f_r, \delta > 0$ yields the desired result. The lemma's proof is now complete.

We indicate by K^C the conclusion of $\mathbb{T}$ within the subset K of $\mathbb{T}$. Regarging a closed subset E^r of $\mathbb{T}$, to date, $(E^r)^C = \cup_{n\in\mathbb{N}} (a_n, a_n + \varepsilon_n)$, where $(a_n, a_n + \varepsilon_n) \subset (E^r)^C$ and $a_n, a_n + \varepsilon_n \in E^r$. We specify Ω_{E^r} being the family of all arc unions $(a_n, a_n + \varepsilon_n)$, where $(a_n, a_n + \varepsilon_n) \subset (E^r)^C$ and a_n , $a_n + \varepsilon_n \in E^r$. Define the outer function f_r using algebra and $\Gamma \in \Omega_E$ (E^r is a closed subset of $\mathbb{T}$), defining the external function $(f_r)_\Gamma \in \mathcal{H}^\infty(\mathbb{D})$ linked to the external element of f_r by

$$(f_r)_\Gamma(z) \coloneqq \exp\left\{ \frac{1}{2\pi} \int_\Gamma \sum \frac{\sum_{j=1}^n e^{i\theta_j} + z}{\sum_{j=1}^n e^{i\theta_j} - z} \log \left| f_r(\sum_{j=1}^n e^{i\theta_j}) \right| d(\sum_{j=1}^n \theta_j) \right\} \quad (z \in \mathbb{D}).$$

Then, we claim.

Lemma (3.4). Assume that $\mathfrak{I}$ is a closed ideal of $(\Lambda_r)_\omega, f_r \in (\Lambda_r)_\omega$ an outer function, and consider that $h \in \mathcal{J}_{E^r_{\mathfrak{I}}}$ a function like that $h_{f_r} \in \mathfrak{I}$. Let $\Gamma \in \Omega_{E^r_{\mathfrak{I}}}$ be such that $\mathbb{T}\backslash\overline{\Gamma}$ is the combination of a limited number of arcs $(a, a+\varepsilon) \subseteq \mathbb{T}\backslash E^r_{\mathfrak{I}}$ $(a, a+\varepsilon \in E^r_{\mathfrak{I}})$. If $h(f_r)_\Gamma \in (\Lambda_r)_\omega$, then $h(f_r)_\Gamma \in \mathfrak{I}$.

Proof. Simply put, we'll assume that $\mathbb{T}\backslash\overline{\Gamma} = (a, a+\varepsilon) \coloneqq \gamma$, where $a, a+\varepsilon \in E^r_{\mathfrak{I}}$ and $(a, a+\varepsilon) \subseteq \mathbb{T}\backslash E^r_{\mathfrak{I}}$. Let $\varepsilon > 0$ make sure that $\gamma_\varepsilon \coloneqq (ae^{i\varepsilon}, (a+\varepsilon)e^{-i\varepsilon}) \subset \gamma$. For $\delta > 0$, we set

$$\phi_{\delta,\varepsilon}(z) \coloneqq \left(\frac{z\bar{a}e^{-i\varepsilon} - 1}{z\bar{a}e^{-i\varepsilon} - 1 - \delta} \right) \left(\frac{z\overline{a+\varepsilon}e^{i\varepsilon} - 1}{z\overline{a+\varepsilon}e^{i\varepsilon} - 1 - \delta} \right) \quad (z \in \mathbb{D}).$$

It is obvious that $\phi_{\delta,\varepsilon} \in (\Lambda_r)_\omega$ plus that $\phi_{\delta,\varepsilon}(ae^{i\varepsilon}) = \phi_{\delta,\varepsilon}((a+\varepsilon)e^{-i\varepsilon}) = 0$. So, in accordance with Proposition 3.6.1 (below), we can see that the function $(f_r)_{\gamma_\varepsilon}$ is multiplied by the square of $\phi_{\delta,\varepsilon}$ belongs to $(\Lambda_r)_\omega$, i.e., $\phi^2_{\delta,\varepsilon}(f_r)_{\gamma_\varepsilon} \in (\Lambda_r)_\omega$. i.e. we also obtain $\phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon} \in (\Lambda_r)_\omega$. Then, for $\pi : (\Lambda_r)_\omega \to (\Lambda_r)_\omega / \mathfrak{T}$ because it is the standard quotient map, it follows

$$0 = \sum \pi\left(\phi^4_{\delta,\varepsilon} h_{f_r}\right) = \sum \pi\left(h\phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon}\right)\pi\left(\phi^2_{\delta,\varepsilon}(f_r)_{\gamma_\varepsilon}\right).$$

Because, the function $\phi^2_{\delta,\varepsilon}(f_r)_{\gamma_\varepsilon}$ may be inverted via quotient algebra $(\Lambda_r)_\omega / \mathfrak{T}$, then

$$h\phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon} \in \mathfrak{T} \qquad (\delta, \varepsilon > 0).$$

using that reality

$$\lim_{\varepsilon\to 0} \sum \left\| \phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon} - \phi^2_{\delta,0}(f_r)_\Gamma \right\|_\omega = 0\,,$$

we can exmine

$$\phi^2_{\delta,0} h f_\Gamma \in \mathfrak{T} \qquad (\delta > 0).$$

Now that $h(f_r)_\Gamma \in (\Lambda_r)_\omega$ and $h(f_r)_\Gamma\,(a) = h(f_r)_\Gamma\,(a+\varepsilon) = 0$, Lemma (4.3.6) reveals to us that

$$\lim_{\delta\to 0} \left\| \phi^2_{\delta,0} h(f_r)_\Gamma - h(f_r)_\Gamma \right\|_\omega = 0\,.$$

So $h(f_r)_\Gamma \in \mathfrak{T}$. The lemma's proof is finished at this step.

Assume that $\mathfrak{T}$ is a closed ideal of the algebra $(\Lambda_r)_\omega$. To date $U_\mathfrak{T} = B_\mathfrak{T} S_\mathfrak{T}$. When applied to the zero set, the inner function $B_\mathfrak{T}$ is the standard Blashke product $Z_\mathfrak{T} \cap \mathbb{D}$, where $Z_\mathfrak{T} := \{z \in \overline{\mathbb{D}} : f_r(z) = 0 \text{ in all } f_r \in \mathfrak{T}\}$. The greatest common denominator among them is the positive single measure $\mu_\mathfrak{T}$ related to the unique inner function $S_\mathfrak{T}$ of all $\mu_{f_r}, f_r \in \mathfrak{T}$. Take note that $\operatorname{supp}(\mu_\mathfrak{T})$ is contained in $E^r_\mathfrak{T}$. For a portion K of T, we set

$$(S_{f_r})_K(z) := \exp\left\{-\frac{1}{2\pi}\int_K \sum \frac{\sum_{j=1}^n e^{i\theta_j} + z}{\sum_{j=1}^n e^{i\theta_j} - z}\, d\mu_{f_r}\Big(\sum_{j=1}^n \theta_j\Big)\right\} \qquad (f_r \in \mathcal{A}(\mathbb{D}))$$

Lemma (3.5) Assume that f_r is a function in a closed ideal $\mathfrak{T}$ of the algebra $(\Lambda_r)_\omega$. Then $B_\mathfrak{T}(S_{f_r})_{E_\mathfrak{T}} O_{f_r}$ relates to $\mathfrak{T}$.

Proof. Define $f_r \in \mathfrak{T}$, and let $B_{f_r,n}$ and $B_{\mathfrak{T},n}$ be respective Blashke product with zeros $Z_{f_r} \cap \mathbb{D}_n$ and $Z_\mathfrak{T} \cap \mathbb{D}_n$, where $\mathbb{D}_n := \{z \in \mathbb{D} : |z| < \frac{n-1}{n}\,,\ n \in \mathbb{N}\}$. Fix $n \in \mathbb{N}$. The function $B_{f_r,n}/B_{\mathfrak{T},n}$ is reversible in the algebra of quotients $(\Lambda_r)_\omega / \mathcal{J}_n$, where $\mathcal{J}_n := \{g \in (\Lambda_r)_\omega : gB_{\mathfrak{T},n} \in \mathfrak{T}\}$. Then $f_r/B_{\mathfrak{T},n} \in \mathcal{J}_n$ Thus, it follows $B_{\mathfrak{T},n}(f_r/B_{\mathfrak{T},n}) \in \mathfrak{T}$. It is obvious that

$$\lim_{n\to+\infty} \sum \left\| B_{\mathfrak{T},n}(f_r/B_{f_r,n}) - B_\mathfrak{T} S_\mathfrak{T} O_f \right\|_\infty = 0\,.$$

In Appendix B (F-property of Λ_ω), corollary B.2 we obtain

$$\lim_{n\to+\infty} \sum \left\| B_{\mathfrak{T},n}(f_r/B_{f_r,n}) - B_\mathfrak{T} S_\mathfrak{T} O_{f_r} \right\|_\omega = 0\,.$$

So, $B_\mathfrak{T} S_\mathfrak{T} O_{f_r} \in \mathfrak{T}$. Let $\varepsilon > 0$ in such a way $\gamma_\varepsilon := (ae^{i\varepsilon}, (a+\varepsilon)e^{-i\varepsilon}) \subset \gamma := (a, a+\varepsilon)$, where $a, a+\varepsilon \in E^r_\mathfrak{T}$ and $(a, a+\varepsilon) \subseteq \mathbb{T} \backslash E^r_\mathfrak{T}$. By utilizing the F-property of $(\Lambda_r)_\omega$, the function $B_\mathfrak{T}(S_{f_r})\gamma^c_\varepsilon O_{f_r}$ relates to $(\Lambda_r)_\omega$. We set

$$\phi_\varepsilon(z) := \left(z\bar{a}e^{-i\varepsilon} - 1\right)\left(z\overline{a+\varepsilon}e^{-i\varepsilon} - 1\right) \quad (z \in \mathbb{D}).$$

The function $(g_r)_\varepsilon := \phi^2_\varepsilon(f_r)_{\gamma_\varepsilon}$ is derived from Proposition 3.6.1 below, and belongs to $(\Lambda_r)_\omega$. As a result $(\mu_{(S_{f_r})_{\gamma_\varepsilon}}) \subseteq E^r_{(g_r)_\varepsilon}$, we infer from Lemma (4.3.5) that $(S_{f_r})_{\gamma_\varepsilon}(g_r)^2_\varepsilon \in (\Lambda_r)_\omega$. To date,

$$0 = \sum \pi\left((g_r)^2_\varepsilon B_\mathfrak{T} S_\mathfrak{T} O_{f_r}\right) = \sum \pi\left(\left(S_{f_r}\right)_{\gamma_\varepsilon}(g_r)^2_\varepsilon\right) \times \pi\left(B_\mathfrak{T}(S_{f_r})_{\gamma^c_\varepsilon} O_{f_r}\right)$$

where $\pi : (\Lambda_r)_\omega \to (\Lambda_r)_\omega/\mathfrak{I}$, is the traditional quotient map. In the quotient algebra $(\Lambda_r)_\omega/\mathfrak{I}$,the function $(S_{f_r})_{\gamma_\varepsilon} g_\varepsilon^2$ is then invertible $B_\mathfrak{I}(S_{f_r})_{\gamma_\varepsilon^c} O_{f_r} \in \mathfrak{I}$. It ‘s obvious that

$$\lim_{\varepsilon\to 0} \sum \left\| B_\mathfrak{I}(S_{f_r})_{\gamma_\varepsilon^c} O_{f_r} - B_\mathfrak{I}(S_{f_r})_{\gamma^c} O_{f_r} \right\|_\infty = 0 .$$

Afterward, with Corollary B.2 in Appendix B, we arrive at

$$\lim_{\varepsilon\to 0} \left\| B_\mathfrak{I}(S_f)_{\gamma_\varepsilon^c} O_f - B_\mathfrak{I}(S_f)_{\gamma^c} O_f \right\|_\omega = 0 .$$

So $B_\mathfrak{I}(S_f)_{\gamma^c} O_f \in \mathfrak{I}$. Likewise, we can demonstrate that $B_\mathfrak{I}(S_f)_{\Gamma_N^c} O_f \in \mathfrak{I}$, where $\Gamma_N \coloneqq \bigcup_{n\leq N}(a_n, a_n + \varepsilon_n) \in \Omega_{E_\mathfrak{I}}$. To date

$$\lim_{N\to+\infty} \left\| B_\mathfrak{I}(S_f)_{\Gamma_N^c} O_f - B_\mathfrak{I}(S_f)_{E_\mathfrak{I}} O_f \right\|_\infty = 0 .$$

Using Corollary B.2 once more we arrive at

$$\lim_{N\to+\infty} \left\| B_\mathfrak{I}(S_f)_{\Gamma_N^c} O_f - B_\mathfrak{I}(S_f)_{E_\mathfrak{I}} O_f \right\|_\omega = 0 .$$

Then $B_\mathfrak{I}(S_f)_{E_\mathfrak{I}} O_f \in \mathfrak{I}$. This demonstrate the lemma.

3.2. Proof of Theorem (2.1)

The following statement regarding the approximation of functions in Λ_ω is necessary for the proof of Theorem (2.1).

Proposition (3.6) Assume that ω is a continuity modulus that satisfies the requirement (2.1). Suppose there exists $f \in \Lambda_\omega$ a function such that $\|f\|_\omega \leq 1$ and E a closed subset of $\mathbb{T}$. Let there $g \in \mathcal{J}_E$ be an outer function and S single inner function that is such that $\operatorname{supp}(\mu_S) \subseteq E_g$. Then

(i) the roles that each play Sg^2 and $Sg^2 f_{\Gamma_N^c}$ who they belong to Λ_ω, for each $N \in \mathbb{N}$,

(ii) There are $\lim_{N\to+\infty}\left\|Sg^2 f_{\Gamma_N^c} - Sg^2\right\|_\omega = 0$, where $\Gamma_N \coloneqq \bigcup_{n\leq N}(a_n, a_n + \varepsilon_n) \in \Omega_E$. Additionally, if ω the stricter requirement (2.2) is met , then

(i)′. functions Sg and $Sgf_{\Gamma_N^c}$ belonging to Λ_ω, for every $N \in \mathbb{N}$,

(ii)′. There are $\lim_{N\to+\infty}\left\|Sgf_{\Gamma_N^c} - Sg\right\|_\omega = 0$.

Proof. We have from Lemma (3.2) we have $Sg^2 \in \Lambda_\omega$. It is obvious that the disk algebra contains a series of functions $\{Sgf_{\Gamma_N^c}\}_{N\in\mathbb{N}}$. Observe that if

$$\frac{|S(z)g^2(z)f_\Gamma(z) - S(z-\varepsilon)g^2(z-\varepsilon)f_\Gamma(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0) \tag{3.9}$$

consistently with respect to $\Gamma \in \Omega_E$, then assertion (i) immediately follows, Furthermore, by using Lemma (3.1), claim (ii) can be inferred.

So, merely demonstrating (3.9) is sufficient.. In order to achieve this, we fix $\Gamma \in \Omega_E$ and we let $\zeta + \varepsilon$ and ζ be two separate points in $\mathbb{T}$ such that $d(\zeta+\varepsilon, E) \geq d(\zeta, E)$. It is obvious that

$$\frac{|S(\zeta+\varepsilon)g^2(\zeta+\varepsilon)f_\Gamma(\zeta+\varepsilon) - S(\zeta)g^2(\zeta)f_\Gamma(\zeta)|}{\omega(\varepsilon)}$$
$$\leq |f_\Gamma(\zeta+\varepsilon)| \frac{|S(\zeta+\varepsilon)g^2(\zeta+\varepsilon) - S(\zeta)g^2(\zeta)|}{\omega(\varepsilon)} + |g^2(\zeta)| \frac{|f_\Gamma(\zeta+\varepsilon) - f_\Gamma(\zeta)|}{\omega(\varepsilon)}.$$

Given that $Sg^2 \in \Lambda_\omega$ (by Lemma 3.2), the proof of (3.9) becomes

$$|g^2(\zeta)| \frac{|f_\Gamma(\zeta+\varepsilon) - f_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (as\ |\varepsilon| \to 0) \tag{3.10}$$

Case (i). For $\varepsilon \geq (\frac{d(\zeta,E)}{2})^2$, ours has

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le 2\frac{|g^2(\zeta)|}{\omega(\varepsilon)} \le 8\eta_2^{-1}\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^2$$
$$= o(1) \qquad (as\ |\varepsilon| \to 0)\,. \qquad (3.11)$$

Case (ii). For $\varepsilon \le (\frac{d(\zeta,E)}{2})^2$ with $\zeta /\in \Gamma$. Following is $[\zeta+\varepsilon,\zeta] \subset (\Gamma \cup E)^c$. Then $z \not\equiv \Gamma$ and $d(z,E) \ge d(\zeta,E)$ for each $z \in [\zeta+\varepsilon,\zeta\,]$. That is $z \in [\zeta+\varepsilon,\zeta\,]$ like that

$$\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} = |f_\Gamma'(z)| \le \frac{1}{\pi}\int_\Gamma \frac{\left|log\left|f\left(\sum_{j=1}^n e^{i\theta_j}\right)\right|\right|}{\left|\sum_{j=1}^n e^{i\theta_j}-z\right|} d(\sum_{j=1}^n \theta_j) \le \frac{c_f}{d^2(z,E)}.$$

Following is

$$\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le \frac{c_f}{d^2(\zeta,E)}.$$

Consequently, we have

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} = \frac{|g^2(\zeta)|\varepsilon}{\omega(\varepsilon)}\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\varepsilon}$$
$$\le c_f\,\frac{|g^2(\zeta)|\varepsilon}{\omega(\varepsilon)d^2(\zeta,E)} \le c_f\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^2\frac{\omega\big(d^2(\zeta,E)\big)\varepsilon}{\omega(\varepsilon)d^2(\zeta,E)} = o(1)$$
$$(\text{as } \ \varepsilon \to 0) \qquad (3.12)$$

Case 3. In this instance, let’s assume that $\varepsilon \le (\frac{d(\zeta,E)}{2})^2$ and $\zeta \in \Gamma$, that follows $\left|f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)\right| = \left|f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta)\right| = 1$. From

$$f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta) = f_\Gamma(\zeta+\varepsilon)f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)-f_\Gamma(\zeta)f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta)$$
$$= f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)\big(f(\zeta+\varepsilon)-f(\zeta)\big)+f(\zeta+\varepsilon)\left(f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)-f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta)\right)$$
$$= f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)\big(f(\zeta+\varepsilon)-f(\zeta)\big)$$
$$-f(\zeta)f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta+\varepsilon)f_{\mathbb{T}\backslash\Gamma}^{-1}(\zeta)\Big(f_{\mathbb{T}\backslash\Gamma}(\zeta+\varepsilon)$$
$$-f_{\mathbb{T}\backslash\Gamma}(\zeta)\Big) \qquad (3.13)$$

in light (3.12), and deduction

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le |g^2(\zeta)|\frac{|f(\zeta+\varepsilon)-f(\zeta)|}{\omega(\varepsilon)}$$
$$+|f(\zeta)||g^2(\zeta)|\frac{\left|f_{\mathbb{T}\backslash\Gamma}(\zeta+\varepsilon)-f_{\mathbb{T}\backslash\Gamma}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \qquad (\text{as } \ \varepsilon \to 0) \qquad (3.14)$$

We can see that (3.10) holds from the inequalities (3.11), (3.12) and (3.14). Additionally ω, if the condition (2.2) is met, we can infer the claim (i)′ and (ii)′ in a similar manner. The argument’s proof is now complete.

Theorem (2.1) Proof. Let $\mathfrak{T}$ there exits a closed algebraic ideal Λ_ω such that $U_{\mathfrak{T}} \equiv 1$. Consider $f \in \mathfrak{T}$ and let g be an outer function in $\mathcal{J}_{E_{\mathfrak{T}}}$. Lemma (3.5) leads us to conclude that $(S_f)_{E_{\mathfrak{T}}}O_f \in \mathfrak{T}$ .Then, $(S_f)_{E_{\mathfrak{T}}}g^2O_f \in \mathfrak{T}$. We conclude that Lemma (3.4) and Proposition 3.6.1, allow us to conclude that $(S_f)_{E_{\mathfrak{T}}}g^2f_{\Gamma_N^c} \in \mathfrak{T}$, for each $N \in \mathbb{N}$, where $\Gamma_N := \bigcup_{n\le N}(a_n,a_n+\varepsilon_n) \in \Omega_{E_{\mathfrak{T}}}$. So, according to Proposition 3.6.2, $(S_f)_{E_{\mathfrak{T}}}g^2 \in \mathfrak{T}$. Now pick a set of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{T}$ such that the inner portions of f_n have a greatest common factor of 1 and such that

$(1/2\pi)\int_0^{2\pi} d\mu_{f_n}(\theta) \le 1$. This suggests that

$k_n := (S_{f_n})_{E_{\mathfrak{T}}}g^2 \in \mathfrak{T}$, for each $n \in \mathbb{N}$. Lemma (3.2) gives us

$$\frac{|k_n(\zeta+\varepsilon)-k_n(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \ \varepsilon \to 0),$$

consistently with regard to n. Using the reality

$$\lim_{n\to+\infty}\|k_n - g^2\|_\infty = 0$$

Lemma (3.1), and we get at

$$\lim_{n\to+\infty}\|k_n - g^2\|_\omega = 0.$$

So $g^2 \in \mathfrak{T}$. The theorem's proof is now complete.

3.3. Proof of Theorem (2.3)

We start by establishing the following claim.

Proposition (3.7). Let ω be a continuity modulus that satisfies condition (1.1). Fix $\varepsilon > 0$. Let E be a closed subset of E_g and let $g \in \Lambda_\omega$ be a function such that $\|g\|_\omega \leq 1$. Then

(i) the roles $U_g O_g^{(1+\varepsilon)}$ and $U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c}$ belong to Λ_ω, for each $N \in \mathbb{N}$,

(ii) we know $\lim_{N\to+\infty}\left\| U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} - U_g O_g^{(1+\varepsilon)}\right\|_\omega = 0$, where $\Gamma_N := \bigcup_{n\leq N}(a_n, a_n + \varepsilon_n) \in \Omega_E$.

Proof. By utilizing Λ_ω and the F-property , the following

$$\begin{aligned} &U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta)\\ &\quad= O_g^{\varepsilon}(\zeta+\varepsilon)\left(U_g(\zeta+\varepsilon)O_g(\zeta+\varepsilon) - U_g(\zeta)O_g(\zeta)\right) - U_g(\zeta)O_g^{\varepsilon}(\zeta+\varepsilon)\left(O_g(\zeta+\varepsilon) - O_g(\zeta)\right)\\ &\quad+ U_g(\zeta)\left(O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - O_g^{(1+\varepsilon)}(\zeta)\right) \qquad (\zeta+\varepsilon, \zeta \in \mathbb{T}),\end{aligned}$$

belonging to Λ_ω is the function $U_g O_g^{(1+\varepsilon)}$. It is evident that $\left\{U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c}\right\}_{N\in\mathbb{N}}$ there are a series of functions in disk algebra and

$$\lim_{N\to+\infty}\left\| U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} - U_g O_g^{(1+\varepsilon)}\right\|_\infty = 0 .$$

Recall that if

$$\frac{\left| U_g O_g^{(1+\varepsilon)} g_\Gamma(\zeta+\varepsilon) - U_g O_g^{(1+\varepsilon)} g_\Gamma(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0), \tag{3.15}$$

consistently with respect to $\Gamma \in \Omega_E$, then assertions (i) and (ii), which are inferred from (3.15) and Lemma (3.1) are true. Below, we must substantiate (3.15). Let there $\zeta + \varepsilon$, $\zeta \in \mathbb{T}$ be two distint points such that $d(\zeta+\varepsilon, E) \geq d(\zeta, E)$. It is evident that

$$\begin{aligned}&\frac{\left| U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon)g_\Gamma(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta)g_\Gamma(\zeta)\right|}{\omega(\varepsilon)}\\ &\leq |g_\Gamma(\zeta+\varepsilon)|\frac{\left| U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta)\right|}{\omega(\varepsilon)} + |g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)}.\end{aligned}$$

Then, to demonstrate (3.15) it is suffices to demonstrate that

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.16}$$

1. We believe that $\varepsilon \geq (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$, we are successful

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)} \leq 2\frac{|g(\zeta)|^{(1+\varepsilon)}}{\omega(\varepsilon)} \leq 2^{(2+\varepsilon)}\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^{(1+\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.17}$$

2. In this instance, we posit that $\varepsilon \leq (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$ and that $\zeta \notin \Gamma$. It comes $[\zeta+\varepsilon, \zeta] \subset \mathbb{T}\backslash E$ after.Afterward, $z \notin \Gamma$ and $|z - \sum_{j=1}^{n} e^{i\theta_j}| \geq \frac{1}{2}|\zeta - \sum_{j=1}^{n} e^{i\theta_j}|$ for each $z \in [\zeta+\varepsilon, \zeta]$ and for every $\sum_{j=1}^{n} e^{i\theta_j} \in \Gamma$. There are $z \in [\zeta+\varepsilon, \zeta]$ such things

$$\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\varepsilon}=|g'_\Gamma(z)|\le a_\Gamma(z),$$

where $a_\Gamma(z):=\frac{1}{\pi}\int_\Gamma \frac{\left|log\left|g(\sum_{j=1}^n e^{i\theta_j})\right|\right|}{\left|\sum_{j=1}^n e^{i\theta_j}-z\right|^2}d\theta_j$. Since $a_\Gamma(z)\le 4a_\Gamma(\zeta)$, then

$$\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{|\xi-\zeta|}\le 4a_\Gamma(\zeta)\,.$$

2.1. First, let's assume that $a_\Gamma(\zeta)\le\frac{1}{d^{(1+\varepsilon)}(\zeta,E)}$. It is obvious that

$$\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{d^{(1+\varepsilon)}(\zeta,E)\omega(\varepsilon)}=\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^{(1+\varepsilon)}\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}$$
$$\le\inf\left\{\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^{(1+\varepsilon)},\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}=o(1)\quad(\text{as }\ \varepsilon\to0).$$

Thus, we are able to:,

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)}\le\frac{|g(\zeta)|^{(1+\varepsilon)}\varepsilon}{\omega(\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\varepsilon}\le4\frac{|g(\zeta)|^{(1+\varepsilon)}\varepsilon}{d^4(\zeta,E)\omega(\varepsilon)}$$
$$=o(1)\quad(\text{as }\ \varepsilon\to0).\qquad(3.18)$$

2.2. After that, let's assume that $a_\Gamma(\zeta)\ge\frac{1}{d^{\varepsilon}(\zeta,E)\varepsilon^{1/(1+\varepsilon)}}$. Set $\lambda_\zeta:=1-\varepsilon^{1/(1+\varepsilon)}$, after that

$$\left|g(\lambda_\zeta\zeta)\right|=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\lambda_\zeta^2}{\left|\sum_{j=1}^n e^{i\theta_j}-\lambda_\zeta\zeta\right|^2}\log\left|g\left(\sum_{j=1}^n e^{i\theta_j}\right)d\Big(\sum_{j=1}^n\theta_j\Big)\right|\right\}$$
$$\le\exp\left\{\frac{1}{2\pi}\int_\Gamma\frac{\varepsilon^{1/(1+\varepsilon)}}{\left|\sum_{j=1}^n e^{i\theta_j}-\lambda_\zeta\zeta\right|^2}\log\left|g\left(\sum_{j=1}^n e^{i\theta_j}\right)d\Big(\sum_{j=1}^n\theta_j\Big)\right|\right\}\le\exp\left\{-\frac{1}{4}\varepsilon^{1/(1+\varepsilon)}a_\Gamma(\zeta)\right\}$$
$$\le\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}.$$

It is evident that

$|g(\zeta)|^{(1+\varepsilon)}\le2^{\varepsilon}|g(\zeta)-g(\lambda_\zeta\zeta)|^{(1+\varepsilon)}+2^{\varepsilon}\left|g(\lambda_\zeta\zeta)\right|^{(1+\varepsilon)}$. We discover

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)}$$
$$\le2^{\varepsilon}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|^{(1+\varepsilon)}}{\omega(\varepsilon)}|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|$$
$$+2^{\varepsilon}\left|g(\lambda_\zeta\zeta)\right|^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\varepsilon}$$
$$\le2^{(1+\varepsilon)}\eta_{(1+\varepsilon)}^{-1}o(1)+2^{\varepsilon}\frac{\varepsilon}{\omega(\varepsilon)}\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}\frac{\int_{\mathbb{T}}\left|log\left|g\big(\sum_{j=1}^n e^{i\theta_j}\big)\right|\right|d\big(\sum_{j=1}^n\theta_j\big)}{d^2(\zeta,E)}$$
$$=o(1)\quad(\text{as }\ \varepsilon\to0).\qquad(3.19)$$

2.3. Now, let's assume that

$\frac{1}{d^{(1+\varepsilon)}(\zeta,E)}\le a_\Gamma(\zeta)\le\frac{1}{d^{\varepsilon}(\zeta,E)|\varepsilon^{1/(1+\varepsilon)}}$ situation was set

$\mu_\zeta:=1-\frac{1}{a_\Gamma(\zeta)d^{\varepsilon}(\zeta,E)}$. Thus $\varepsilon^{1/(1+\varepsilon)}\le1-\mu_\zeta\le d(\zeta,E)$, it follows

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{\varepsilon} \le \left(\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)} \le \eta_{(1+\varepsilon)}^{-1},$$

and

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)} \le \eta_{(1+\varepsilon)}^{-(1/1+\varepsilon)}\left(\frac{\varepsilon}{\omega(\varepsilon)}\right)^{\frac{\varepsilon}{(1+\varepsilon)}}.$$

Then

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon} \le \inf\left\{\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^{\varepsilon}, \eta_{(1+\varepsilon)}^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{\varepsilon}{\omega(\varepsilon)}\right)^{\frac{\varepsilon}{(1+\varepsilon)}}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0).$$

We also have $|g(\mu_\zeta\zeta)| \le \exp\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\}$. We discover

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} \le |g(\zeta)|^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}a_\Gamma(z)$$
$$(z \in [\zeta+\varepsilon,\zeta])$$
$$\le |g(\zeta)|^{\varepsilon}\frac{|g(\zeta)-g(\mu_\zeta\zeta)|\varepsilon}{(1-\mu_\zeta)\omega(\varepsilon)}(1-\mu_\zeta)a_\Gamma(z) \quad \le |g(\zeta)|^{\varepsilon}|g(\mu_\zeta\zeta)|\frac{\varepsilon}{\omega(\varepsilon)}a_\Gamma(z)$$
$$\le \frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon} + \frac{\varepsilon}{\omega(\varepsilon)}\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}\frac{\int_{\mathbb{T}}\left|log\left|g\left(\sum_{j=1}^{n}e^{i\theta_j}\right)\right|\right|d\left(\sum_{j=1}^{n}\theta_j\right)}{d^2(\zeta,E)}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.20}$$

As a result, we may infer from (3.18), (3.19) and (3.20) that if $\varepsilon \le (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$ and $\zeta \notin \Gamma$, then

$$\left|g^{(1+\varepsilon)}(\zeta)\right|^{(1+\varepsilon)}\frac{|g_\Gamma(z-\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.21}$$

3. In this instance, we assume that $\varepsilon \le (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$ and that $\zeta \in \Gamma$. We integrate this instance into the case using the equality (3.13). There for, we utilize (3.21) to also obtain in this situation that

$$\left|g^{(1+\varepsilon)}(\zeta)\right|^{(1+\varepsilon)}\frac{|g_\Gamma(z-\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.22}$$

Inequalities (3.17), (3.21) and (3.22) now lead to (3.16). The argument's proof is now complete.

Theorem (2.3) Proof . Suppose there $\mathfrak{I}$ is a closed ideal in Λ_ω and g a function in Λ_ω. Then $O_g \in \Lambda_\omega$ and $O_g^{(1+\varepsilon)} \in \Lambda_\omega$, for each $\varepsilon > 0$. Imagine that
$U_g O_g^{(1+\varepsilon)} \in \mathfrak{I}$. After that $U_g O_g^{(\varepsilon+3)} \in \mathfrak{I}$, for each $\varepsilon > 0$.
As a result of proposition 3.7.1 we have $U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} \in \Lambda_\omega$, and $U_g O_g^{(\varepsilon+2)} g_{\Gamma_N^c} \in \Lambda_\omega$. Using Lemma (3.4), it is evident that $U_g O_g^{(\varepsilon+2)} g_{\Gamma_N^c} \in \mathfrak{I}$, for every $\varepsilon > 0$ and for every $N \in \mathbb{N}$. As a result of proposition 3.7.(ii), $U_g O_g^{(\varepsilon+2)} \in \mathfrak{I}$ for every $\varepsilon > 0$. Again, we arrive to the conclusion that $U_g O_g^{(\varepsilon+2)} \in \mathfrak{I}$ for every $\varepsilon > 0$ using Lemma (3.4) and Proposition (3.7). This suggest that $g \in \mathfrak{I}$.

Appendix A. A comparable standard in Λ_ω

The following Tamrazov's Theorem [8] is simply proved in this section.

Theorem A.1. (Refer to [8].) Let f be a function in $\Lambda_\omega(\mathbb{T}) \cap \mathcal{A}(\mathbb{D})$ and let ω be any arbitrary modulus of continuity. Then f member of Λ_ω. The next key lemma is required.

Lemma A.2. Let $f \in \Lambda_\omega(\mathbb{T}) \cap \mathcal{A}(\mathbb{D})$ be a function such that $\|f\|_{\Lambda_\omega(\mathbb{T})} \le 1$ and let ω be any arbitrary modulus of continuity. To each $\delta \ge 0$, we have

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z\,/\,|z|)\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)\right\}$$
$$\leq o\big(\omega(1-|z|)\big)+A\delta \qquad (\text{as } |z|\to 1),$$

where A is a fixed constant.

Proof. Allow $0<\varepsilon<1$ and $c_\varepsilon>0$ such that $\varepsilon\leq c_\varepsilon$ we have $|f(\zeta+\varepsilon)-f(\zeta)|\leq\varepsilon\omega(\varepsilon)$ for any $\zeta+\varepsilon,\zeta\in T$ fulfilling, Separate T into the next three sections.

$$\Gamma_1:=\{\zeta+\varepsilon\in\mathbb{T}:|(\zeta+\varepsilon)-z/|z||\leq 1-|z|\leq c_\varepsilon\},$$
$$\Gamma_2:=\{\zeta+\varepsilon\in\mathbb{T}:1-|z|\leq|(\zeta+\varepsilon)-z/|z||\leq c_\varepsilon\},$$
$$\Gamma_3:=\{\zeta+\varepsilon\in\mathbb{T}:c_\varepsilon\leq|(\zeta+\varepsilon)-z/|z||\}.$$

Ours has

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$\leq\frac{1}{2\pi}\int_{\Gamma_1}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\varepsilon\omega\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$+\frac{1}{2\pi}\int_{\Gamma_2}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\varepsilon\omega\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$+\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\varepsilon\omega\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$:=I_1+I_2+I_3\,. \qquad (A.1)$$

It is obvious that

$$I_1\leq\left(\frac{1}{2\pi}\int_{\Gamma_1}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta). \qquad (A.2)$$

We then have

$$I_2=\frac{1}{2\pi}\int_{\Gamma_2}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\varepsilon\omega\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$\leq\frac{1}{2\pi}\int_{\Gamma_2}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\varepsilon\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|\frac{\omega(1-|z|)}{1-|z|}+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$
$$\leq\frac{1}{2\pi}\int_{\Gamma_2}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\frac{\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|}{1-|z|}(\varepsilon\omega(1-|z|)+\delta)\right)d(\sum_{j=1}^{n}\theta_j)$$
$$\leq\left(\frac{1}{2\pi}\int_{\Gamma_1}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta)$$
$$+c\int_{\varepsilon\geq 0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)\,, \qquad (A.3)$$

where c is a fixed constant. Let c'_ε be a positive number such that we have $\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\leq\varepsilon$ and $c'_\varepsilon\leq c_\varepsilon$ for every $z\in\mathbb{D}$ satisfying $1-|z|\leq c'_\varepsilon$. As a result, as in (73), we get

$$I_3=\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\omega\left|\sum_{j=1}^{n}e^{i\theta_j}-z/|z|\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)$$

$$\leq\left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\omega(1-|z|)+\delta)\quad+c\int_{\varepsilon\geq0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$=\left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta)\quad+c\int_{\varepsilon\geq0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$+\left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)log\left(\frac{\omega(1-|z|)+\delta}{\varepsilon\omega(1-|z|)+\delta}\right)$$

$$\leq\left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta)+c\int_{\varepsilon\geq0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$-\varepsilon\log(\varepsilon),\qquad (A.4)$$

for each $z\in\mathbb{D}$ fulfilling $1-|z|\leq c'_\varepsilon$. We obtain from (A.2), (A.3) and (A.4)

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)\right\}$$

$$\leq A(\varepsilon\omega(1-|z|)+\delta)\quad(z\in\mathbb{D}\text{ and }1-|z|\leq c'_\varepsilon)\qquad (A.5)$$

where A is the unchanging constant. The lemma's proof is now finished.

Lemma A.2 is used to support the following claim.

Lemma A.3. Let ω the continuity modulus be any arbitrary number, and let $f\in\Lambda_\omega(\mathbb{T})$. Then

$$|f(z)-f(\zeta)|=o(\omega(|z-\zeta|))\qquad(\text{as }|z-\zeta|\to0, z\in\mathbb{D}\text{ and }\zeta\in\mathbb{T}).\qquad (A.6)$$

Proof. We may assume that $\|f\|_{\Lambda_\omega(\mathbb{T})}\leq1$. Let $0<\varepsilon<1$ and $c_\varepsilon>0$ be a number such that we have $|f(\zeta+\varepsilon)-f(\zeta)|\leq\varepsilon\omega(\varepsilon)$ for any $\zeta+\varepsilon,\zeta\in\mathbb{T}$ satisfying $\varepsilon\leq c_\varepsilon$. Fix $\zeta\in\mathbb{T}$ and fix $z\in\mathbb{D}$ it a way that $|z-\zeta|\leq c_\varepsilon/2$ and $|z|\geq1/4$. To date $|z-\zeta|^2=(1-|z|)^2+|z||z/|z|-\zeta|^2\geq\frac{1}{4}|z/|z|-\zeta|^2$. Hence $|z/|z|-\zeta|\leq2|z-\zeta|\leq c_\varepsilon$. We discover

$$|f(z)-f(\zeta)|\leq|f(z)-f(z/|z|)|+|f(z/|z|)-f(\zeta)|\leq|f(z)-f(z/|z|)|+\varepsilon\omega(|z/|z|-\zeta|)$$

$$\leq\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|\right)d(\sum_{j=1}^{n}\theta_j)\right\}+2\varepsilon\omega(|z-\zeta|).$$

Now, we apply Lemma A.2 to infer the lemma's conclusion.

Theorem A.1 Proof. Let $0<\varepsilon<1$. From Lemma A.3 there is $0<c_\varepsilon<1/2$ such that for each $z\in\mathbb{D}$ and every $\zeta\in\mathbb{T}$ satisfying $|z-\zeta|\leq c_\varepsilon$ we have

$|f(z)-f(\zeta)|\leq\varepsilon\omega(|z-\zeta|)$. Let $z\in\mathbb{D}$ satisfying $|\varepsilon|\leq c_\varepsilon/2$ and

$\inf\{|z|,|z-\varepsilon|\}=|z-\varepsilon|\geq1-c_\varepsilon$. Thus, it follows

$|\frac{z}{|z|}-(z-\varepsilon)/|z-\varepsilon||\leq|z(z-\varepsilon)|^{-1/2}|\varepsilon|\leq2|\varepsilon|\leq c_\varepsilon$.

(i). First, let's assume $|\varepsilon|\geq1-|z-\varepsilon|$. We discover

$$
\begin{aligned}
&|f(z)-f(z-\varepsilon)| \\
&\qquad \le |f(z)-f(z/|z|)| + |f(z/|z|)-f(z-\varepsilon/|z-\varepsilon|)| \;+ |f(z-\varepsilon)-f(z-\varepsilon/|z-\varepsilon|)| \\
&\qquad \le 4\varepsilon\omega(|\varepsilon|). \qquad\qquad\qquad\qquad (A.7)
\end{aligned}
$$

(ii). Now we assume that $|\varepsilon| \le 1-|z-\varepsilon|$. We incorporate the maximum principle theorem into $\mathbb{D}$ the analytic function

$z \mapsto \frac{f(z)-f(z-\varepsilon)}{\varepsilon}$ we get

$$
\left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| \le \sup_{(\zeta+\varepsilon)\in\mathbb{T}} \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| = \frac{\left|f\left((\zeta+\varepsilon)_{(z-\varepsilon)}\right)-f(z-\varepsilon)\right|}{\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|} \quad \left((\zeta+\varepsilon)_{(z-\varepsilon)} \in \mathbb{T}\right).
$$

2.1. In case $\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right| \le c_\varepsilon$, then

$$
\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} = \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| \frac{|\varepsilon|}{\omega(|\varepsilon|)} \le 2\frac{\|f\|_\infty}{c_\varepsilon} = o(1) \quad (\text{as } |\varepsilon| \to 0). \qquad (A.8)
$$

2.2. If $|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)| \le c_\varepsilon$, then

$|f((\zeta+\varepsilon)_{(z-\varepsilon)})-f(z-\varepsilon)| \le \varepsilon\omega(|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)|)$. Thus, it follows

$$
\begin{aligned}
\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} &= \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| \frac{|\varepsilon|}{\omega(|\varepsilon|)} \le \frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} \\
&= \left|\frac{f\left((\zeta+\varepsilon)_{(z-\varepsilon)}\right)-f(z-\varepsilon)}{(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)}\right| \frac{1-|z-\varepsilon|}{\omega(1-|z-\varepsilon|)} \quad \left((\zeta+\varepsilon)_{(z-\varepsilon)} \in \mathbb{T}\right) \\
&\le \varepsilon \frac{\omega\left(\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|\right)}{\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|} \frac{1-|(z-\varepsilon)|}{\omega(1-|(z-\varepsilon)|)} \le \varepsilon\,. \qquad (A.9)
\end{aligned}
$$

From the inequalities (A.7), (A.8) and (A.9), it follows that if $|\varepsilon| \le c'_\varepsilon$ and $\inf\{|z|, |(z-\varepsilon)|\} \ge 1-c'_\varepsilon$, then

$$
\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} \le \varepsilon\,. \qquad (A.10)
$$

3. Should be $z \in \mathbb{D}$ such that $\sup\{|z|, |z-\varepsilon|\} \le 1-c'_\varepsilon$ we have

$$
\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} = \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| \frac{|\varepsilon|}{\omega(|\varepsilon|)} \le \sup_{|\zeta|\le 1-c'_\varepsilon} |f'(\zeta)| \frac{|\varepsilon|}{\omega(|\varepsilon|)}
$$

$$
= o(1) \quad (\text{as } |\varepsilon| \to 0)\,. \qquad (A.11)
$$

We determine the outcome from (A.10) and (A.11). The theorem's proof is finished.

Appendix B. Factorization property in Λ_ω

Shirokov [7] provides the F-property of Λ_ω, for any arbitrary modulus of continuity ω. We provide the evidence here for completeness.

Theorem B.1. *(*Refer to [7].) Let the continuity modulus ω be any arbitrary number. Let there f be a function in Λ_ω and let there U be an inner function such that $f/U \in \mathcal{H}^\infty(\mathbb{D})$. Next $f/U \in \Lambda_\omega$ comes

$$
\frac{|f/U(z)-f/U(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (\text{as } |\varepsilon| \to 0)\,,
$$

consistently with regard to U. Additionally, $\|f/U\|_\omega \le c\|f\|_\omega$, where c is an unchaging constant.

Corollary B.2. Assume that the continuity modulus ω is any arbitrary value. Allow f and g be functions in Λ_ω and allow $\{U_n\}_{n\in\mathbb{N}}$ for sequence of inner functions to be created so that $f/U_n \in \mathcal{H}^\infty(\mathbb{D})$ for each $n \in \mathbb{N}$. If $\lim_{n\to+\infty} \|f/U_n - g\|_\infty = 0$, then $\lim_{n\to+\infty} \|f/U_n - g\|_\omega = 0$.

Proof. Following directly from Theorem B.1 and Lemma (3.1), the proof is given.

We develop a number of lemmas before starting the proof of Theorem B.1.

Lemma B.3. Let ω be any arbitrary continuity modulus, according to lemma B.3. Allow f a function in Λ_ω to be. Then

$$
\left|O_f(z)\right| \le o\left(\omega(1-|z|)\right) + A|f(z/|z|)| \qquad (\text{as } |z| \to 1),
$$

where $A > 0$ is the unchanging constant.

Proof. Because $z \in \mathbb{D}$, we have

$$\log\left|O_f(z)\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{\left|\sum_{j=1}^{n} e^{i\theta_j} - z\right|^2} \log\left|f\left(\sum_{j=1}^{n} e^{i\theta_j}\right)\right| d(\sum_{j=1}^{n}\theta_j)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{\left|\sum_{j=1}^{n} e^{i\theta_j} - z\right|^2} \log\left(\left|f\left(\sum_{j=1}^{n} e^{i\theta_j}\right) - f(z/|z|)\right| + |f(z/|z|)|\right) d(\sum_{j=1}^{n}\theta_j).$$

We now utilize Lemma A.2 to finish the lemma's proof.

For a function $f \in \mathcal{H}^\infty(\mathbb{D})$ we specify

$$a_f(\zeta+\varepsilon) := \sum_{n\geq 0} \frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2} + \frac{1}{\pi}\int_{\mathbb{T}} \frac{1}{\left|\sum_{j=1}^{n} e^{i\theta_j} - (\zeta+\varepsilon)\right|^2} d\mu_f\left((\sum_{j=1}^{n}\theta_j)\right),$$

where $\{a_n : n \in \mathbb{N}\} = Z_f \cap \mathbb{D}$ (because n, a_n of its multiplicity, everything is repeated) and μ_f is the unique positive singular measure connected to the singular factor S_f of f.

Lemma B.4. Assume that f is a disk algebraic with an inner factor $U_f \not\equiv 1$.
Let there $\zeta+\varepsilon \in \mathbb{T}\backslash E_f$, $\varepsilon \leq 0$ be such thing $-\varepsilon \leq d(\zeta+\varepsilon, Z_f)$. Then

$$\left|U_f((1+\varepsilon)(\zeta+\varepsilon))\right| \leq \exp\left\{\frac{\varepsilon}{8} a_f(\zeta+\varepsilon)\right\}.$$

Proof . To date

$$\log\left|S_f\left((1+\varepsilon)(\zeta+\varepsilon)\right)\right| = -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|(1+\varepsilon)(\zeta+\varepsilon)|^2}{\left|\sum_{j=1}^{n} e^{i\theta_j} - (\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^{n}\theta_j)$$

$$\leq -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{4}\frac{-\varepsilon}{\left|\sum_{j=1}^{n} e^{i\theta_j} - (\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^{n}\theta_j)$$

$$= -\frac{\varepsilon}{8\pi}\int_{\mathbb{T}} \frac{1}{4}\frac{1}{\left|\sum_{j=1}^{n} e^{i\theta_j} - (\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^{n}\theta_j). \qquad \text{(B.1)}$$

It is obvious that

$$|\varepsilon|^2 = \left||z| - |z-\varepsilon|\right|^2 + |z||z-\varepsilon||z/|z| - (z-\varepsilon)/|z-\varepsilon||^2 \quad (z \in \mathbb{D}).$$

We now estimate $|B_f((1+\varepsilon)(\zeta+\varepsilon))|$. For everyone $n \in \mathbb{N}$, we have

$$\left|\frac{(1+\varepsilon)(\zeta+\varepsilon)-a_n}{(\zeta+\varepsilon)-(1+\varepsilon)a_n}\right|^2 = \frac{\left((1+\varepsilon)-|a_n|\right)^2 + (1+\varepsilon)|a_n||(\zeta+\varepsilon)-a_n/|a_n||^2}{(\varepsilon|a_n|)^2 + (1+\varepsilon)|a_n||(\zeta+\varepsilon)-a_n/|a_n||^2}$$

$$= 1-(2\varepsilon+\varepsilon^2)\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2} \leq 1 + \frac{\varepsilon}{4}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}.$$

Hence

$$log\left|\frac{(1+\varepsilon)(\zeta+\varepsilon)-a_n}{(\zeta+\varepsilon)-(1+\varepsilon)a_n}\right| \leq \frac{\varepsilon}{8}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}$$

Therefore

$$\left|B_f\left((1+\varepsilon)(\zeta+\varepsilon)\right)\right| \leq \exp\left\{\frac{\varepsilon}{8}\sum_{n\geq 0}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}\right\}. \qquad \text{(B.2)}$$

From (B.1) and (B.2) we achieve

$$\left|U_f((1+\varepsilon)(\zeta+\varepsilon))\right| \leq \exp\left\{\frac{\varepsilon}{8} a_f((\zeta+\varepsilon))\right\}.$$

This demonstrates the lemma.

Lemma B.5. Assume that f and U that an inner function is Λ_ω such that $f/U \in \mathcal{H}^\infty(\mathbb{D})$. Then

$$|f(\zeta)| = o\left(\omega\left(\frac{1}{a_U(\zeta)}\right)\right) \quad \left(as\ d(\zeta, Z_f) \to 0, \zeta \in \mathbb{T}\backslash E_f\right).$$

Proof. Let $\varepsilon > 0$. The situation is $c_\varepsilon > 0$ such that if $|\varepsilon| \le c_\varepsilon$, $z \in \mathbb{T}$, then we have $|f(z) - f(z-\varepsilon)| \le \varepsilon\omega(|\varepsilon|)$. From Lemma B.3, it follows that if $\varepsilon \le -c_\varepsilon$, then $\left|O_f\left((1+\varepsilon)\zeta\right)\right| \le (1+\varepsilon)(\varepsilon\omega(-\varepsilon) + |f(\zeta)|)$ and $0 < c'_\varepsilon < c_\varepsilon$, where $\varepsilon > 0$ is an absolute constant. Let $\zeta \in \mathbb{T}\backslash E_f$ it be as such $d(\zeta, Z_f) \le c'_\varepsilon$.

1. We believe that $a_U(\zeta) \le \frac{8(1+\varepsilon)}{d(\zeta, Z_f)}$. Then we achieve

$$|f(\zeta)| \le \varepsilon\omega(d(\zeta, Z_f)) \le \varepsilon\omega(\tfrac{8(1+\varepsilon)}{a_U(\zeta)}) \le 8(1+\varepsilon)\varepsilon\omega(\tfrac{1}{a_U(\zeta)}).$$

2. Assume that $a_U(\zeta) \ge \frac{8(1+\varepsilon)}{d(\zeta, Z_f)}$ to begin with $1-(1+\varepsilon)_\zeta \le d(\zeta, Z_f)$, where $(1+\varepsilon)_\zeta := 1 - \frac{(1+\varepsilon)^2}{a_U(\zeta)}$. Lemma B.4 is applied after we have $U_f/U \in \mathcal{H}^\infty(\mathbb{D})$, and the result is

$$\left|U_f((1+\varepsilon)_\zeta\zeta)\right| \le \exp\left\{-\frac{1-(1+\varepsilon)_\zeta}{8} a_f(\zeta)\right\} \le \exp\left\{\frac{\varepsilon}{8} a_U(\zeta)\right\} = \exp\{-(1+\varepsilon)\}.$$

Because $d(\zeta, Z_f) \le c'_\varepsilon$, then $1-(1+\varepsilon)_\zeta \le c'_\varepsilon$ and we have

$$\begin{aligned}\left|f((1+\varepsilon)_\zeta\zeta)\right| &= \left|U_f((1+\varepsilon)_\zeta\zeta)\right|\left|O_f((1+\varepsilon)_\zeta\zeta)\right| \\ &\le (1+\varepsilon)\exp\{-(1+\varepsilon)\}\left(\varepsilon\omega\left(1-(1+\varepsilon)_\zeta\right) + |f(\zeta)|\right).\end{aligned}$$

Hence

$$\begin{aligned}|f(\zeta)| &\le \left|f(\zeta) - f((1+\varepsilon)_\zeta\zeta)\right| + \left|f((1+\varepsilon)_\zeta\zeta)\right| \\ &\le \varepsilon\omega\left(1-(1+\varepsilon)_\zeta\right) + (1+\varepsilon)\exp\{-(1+\varepsilon)\}\left(\varepsilon\omega\left(1-(1+\varepsilon)_\zeta\right) + |f(\zeta)|\right).\end{aligned}$$

Thus, it follows $|f(\zeta)| \le 3\varepsilon\omega(1-(1+\varepsilon)_\zeta) \le 24(1+\varepsilon)\varepsilon\omega(\frac{1}{a_U(\zeta)})$. The lemma's proof is now complete.

Proof of Theorem B.1. Using Lemma B.5, Now, we can now derive the proof of Theorem B.1 Indeed, Theorem A.1 provides enough evidence to demonstrate that $f/U \in \Lambda_\omega(\mathbb{T})$, that is

$$\frac{|f(\zeta+\varepsilon)/U(\zeta+\varepsilon) - f(\zeta)/U(\zeta)|}{\omega(\varepsilon)} = o(1) \qquad (\text{as } \varepsilon \to 0 \text{ and } \zeta+\varepsilon, \zeta \in \mathbb{T}).$$

Let $\zeta+\varepsilon, \zeta \in \mathbb{T}$ there be two separate points so that $d(\zeta+\varepsilon, Z_f) \ge d(\zeta, Z_f)$. To date

$$\frac{|f(\zeta+\varepsilon)/U(\zeta+\varepsilon) - f(\zeta)/U(\zeta)|}{\omega(\varepsilon)} \le \frac{|f(\zeta+\varepsilon) - f(\zeta)|}{\omega(\varepsilon)} + |f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)}.$$

When proof is sufficient ,

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (as\ \varepsilon \to 0). \qquad \text{(B.3)}$$

1. First, let's assume that $\varepsilon \ge \frac{1}{2} d(\zeta, Z_f)$. Then

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)} \le 2\frac{|f(\zeta)|}{\omega(\frac{1}{2}d(\zeta, Z_f))} \le 4\frac{|f(\zeta)|}{\omega(d(\zeta, Z_f))}$$

$$= o(1) \quad (\text{as } \varepsilon \to 0). \qquad \text{(B.4)}$$

2. After that, we assume $\varepsilon \le \frac{1}{2} d(\zeta, Z_f)$. Then $[\zeta+\varepsilon, \zeta] \subset T\backslash E_f$. There is $z \in [\zeta+\varepsilon, \zeta]$ such a

$$\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)} = |U'(z)| \le \sum_{n\ge 0}\frac{1-|a_n|^2}{|z-a_n|^2} + \frac{1}{\pi}\int_{\mathbb{T}}\frac{1}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\mu(\textstyle\sum_{j=1}^n \theta_j) := a_U(z) \le 4a_U(\zeta) \le \frac{c_f}{d^2(\zeta, Z_f)}.$$

2.1. If $a_U(\zeta) \le \frac{1}{\varepsilon}$. Using Lemma B.5, we can therefore conclude that

$$|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)} = o(1) \quad (\text{as } d(\zeta, Z_f) \to 0).$$

therefore

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon)-U(\zeta)|}{\omega(\varepsilon)} \leq 4|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)} \leq \inf\left\{4|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)}, \frac{c_f\|f\|_\infty}{d^2(\zeta, Z_f)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0). \tag{B.5}$$

2.2. Now let's suppose that $a_U(\zeta) \geq \frac{1}{\varepsilon}$. Then, using Lemma B.5, we get $|f(\zeta)| = o(\omega(\varepsilon))$, as follows $d(\zeta, Z_f) \to 0$. Therefore

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon)-U(\zeta)|}{\omega(\varepsilon)} \leq \inf\left\{2\frac{|f(\zeta)|}{\omega(\varepsilon)}, \frac{c_f\|f\|_\infty}{d^2(\zeta, Z_f)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0). \tag{B.6}$$

Inequalities (B.4), (B.5) and (B.6) lead to (B.3) as a result. The theorem's proof is finished.

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