Review Article

Closed ideals in sequence of analytic weighted Lipschitz algebras

Abstract

We obtain a complete description of closed ideals in sequence of weighted Lipschitz algebras $(\Lambda_r)_\omega$ of sequence of analytic functions on the unit disk satisfying the following condition

$$\sum_r \frac{|f_r(z) - f_r(z+\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0).$$

where ω is a modulus of continuity satisfying some regularity conditions. In particular the closed ideals of the algebras $(\Lambda_r)_{\chi_{1-\varepsilon}}$, where $\chi_{1-\varepsilon}(2-\varepsilon) := \frac{1}{(|\log(2-\varepsilon)|+1)^{1-\varepsilon}}$, $\varepsilon > 1$, are standard and this answers Shirokov's question [N.A. Shirokov, Closed ideals of algebras of $B_{pq}^{1-\varepsilon}$.

Keywords: Closed ideals; Invariant subspaces; Banach algebra; Resolvent

1.Introduction and statement of main result

Let $\mathbb{D}$ be the unit disk of the complex plane and $\mathbb{T}$ its boundary. By $\mathcal{A}(\mathbb{D})$ we denote the usual disk algebra of all analytic functions f on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$. We define the weighted Lipschitz algebra $(\Lambda_r)_\omega(\mathbb{D}) = (\Lambda_r)_\omega$ to $(\Lambda_r)_\omega := \left\{f_r \in \mathcal{A}(\mathbb{D}) : \sup_{z\in\mathbb{D}} \sum_r \frac{|f_r(z)-f_r(z+\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0)\right\}$, where $\omega(2-\varepsilon)$ is a modulus of continuity, i.e., a nondecreasing continuous real-valued function on $[0,2]$ with $\omega(0) = 0$ and $\frac{\omega(2-\varepsilon)}{(2-\varepsilon)}$ is non increasing function such that $\lim_{(2-\varepsilon)\to 0} \frac{\omega(2-\varepsilon)}{(2-\varepsilon)} = \infty$. It is clear that $(\Lambda_r)_\omega$ is a commutative Banach algebra when equipped with the norm

$$\|f_r\|_\omega := \sum_r \|f_r\|_\infty + \sup_{z\in\overline{\mathbb{D}}} \sum_r \frac{|f_r(z) - f_r(w)|}{\omega(|\varepsilon|)} \quad ,$$

With $\|f_r\|_\infty := \sup_{z\in\mathbb{D}} |f_r(z)|$. Similarly the weighted Lipschitz algebra $(\Lambda_r)_\omega(\mathbb{T})$ is defined by

$$(\Lambda_r)_\omega := \left\{f_r \in \mathcal{A}(\mathbb{D}) : \sup_{z\in\mathbb{T}} \sum_r \frac{|f_r(z) - f_r(w)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0)\right\}$$

Shirokov showed in [7] that $(\Lambda_r)_\omega$ possesses the so-called F-property (Factorization property), i.e., for every given $f_r \in (\Lambda_r)_\omega$ and inner function U_r such that f_r / U_r belongs to the algebra $\mathcal{H}^\infty(\mathbb{D})$ of bounded analytic functions, we have $f_r / U_r \in (\Lambda_r)_\omega$ and $\left\|\sum f_r / U_r\right\|_\omega \le c \sum \|f_r\|_\omega$, for an absolute constant c (see Appendix B). Note that Tamrazov [8] proved that the algebras $(\Lambda_r)_\omega$ and $(\Lambda_r)_\omega(\mathbb{T})$ coincide for any arbitrary modulus of continuity ω (see Appendix A).

The structure of closed ideals in the disk algebra is given independently by Beurling and Rudin [2]. They proved that if $\mathfrak{I}$ is a closed ideal of $\mathcal{A}(\mathbb{D})$, then there is an inner function $U_{\mathfrak{I}}$ (the greatest common divisor of the inner parts of the nonzero functions in $\mathfrak{I}$) such that $\mathfrak{I} = \{f_r \in \mathcal{A}(\mathbb{D}) : f_{r|E_{\mathfrak{I}}^r} \equiv 0$ and $f_r / (U_r)_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})\}$, where $E_{\mathfrak{I}}^r := \{\zeta + \varepsilon \in \mathbb{T} : f_r(\zeta + \varepsilon) = 0,\ \forall f_r \in \mathfrak{I}\}$.

Using Beurling–Carleman–Domar resolvent's method combined with the F-property, we can reduce the problem of characterization of closed ideals, in some algebras of analytic functions, to a problem of approximation of outer functions (see for example [1] and references therein). Korenblum [3] has described the closed ideals of the algebra H_1^2 of analytic functions f_r such that $f_r{}'$ is in the Hardy space H^2. He proved that these ideals are standard (in the sense of Beurling–Rudin characterization of the closed ideals in the disk

algebra). Later, this result has been extended to some other Banach algebras of analytic functions. In particular by Matheson [4] and independently by Shamoyan [5] for the algebras $(\Lambda_r)_{\varphi_{1-\varepsilon}}$, where

$$\varphi_{1-\varepsilon}(2-\varepsilon) := (2-\varepsilon)^{1-\varepsilon},\ \varepsilon > 0.$$

The resolvent method is described as follows: Define $d(\zeta+\varepsilon, E^r)$ to be the distance from $(\zeta+\varepsilon) \in \mathbb{T}$ to the closed subset E^r of $\mathbb{T}$ and let $\mathfrak{T}$ be a closed ideal of the algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}$.

(i). In the first step we give an estimate to the norm of the resolvent $\sum\|((\zeta+\varepsilon)-\pi(z))^{-1}\|_{(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{T}}$ in the quotient algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{T}$, where $\pi : (\Lambda_r)_{\varphi_{1-\varepsilon}} \to (\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{T}$ is the canonical quotient map. We obtain $\|((\zeta+\varepsilon)-\pi(z))^{-1}\|_{(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{T}} \le \sum \frac{c}{d^4(\zeta+\varepsilon, E^r_{\mathfrak{T}})}$, where $1 \le |\zeta+\varepsilon| \le 2$ and c is an absolute constant. So, from Cauchy formula on the quotient algebra $(\Lambda_r)_{\varphi_{1-\varepsilon}}/\mathfrak{T}$, we deduce that all functions in $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ such that $f_r/(U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})$ and $|\sum f_r(\zeta+\varepsilon)| \le \sum d^4(\zeta+\varepsilon, E^r_{\mathfrak{T}})$, $(\zeta+\varepsilon) \in \mathbb{T}$, are in I.

(ii) The second step consists to prove that the space of all functions in $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ such that $f_r/(U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})$ and $|\sum f_r(\zeta+\varepsilon)| \le \sum d^4(\zeta+\varepsilon, E^r_{\mathfrak{T}})$, $(\zeta+\varepsilon) \in \mathbb{T}$, , is dense in the standard ideal $\{f_r \in (\Lambda_r)_{\varphi_{1-\varepsilon}} : f_{r|E^r_{\mathfrak{T}}} \equiv 0\ f_r/(U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})\}$.

A closed subset $E^r \subset \mathbb{T}$ is called a Carleson set if the Carleson condition is satisfied, to wit

$$\frac{1}{2\pi}\int_0^{2\pi} \sum \log\left(\frac{1}{d(e^{i(2-\varepsilon)}, E^r)}\right) d(2-\varepsilon) < +\infty .$$

The zeros of each given function in any Banach algebras H_1^2 , $(\Lambda_r)_{\varphi_{1-\varepsilon}}$ and other ones in which the structure of closed ideals is also studied by using the resolvent method [1,6], form a Carleson set.

For the general case the resolvent method fails to apply, as example we can consider the algebras $(\Lambda_r)_{\chi_{1-\varepsilon}}$, where $\chi_{1-\varepsilon}(2-\varepsilon) := \frac{1}{(|\log(2-\varepsilon)|+1)^{1-\varepsilon}}$, $\varepsilon > 1$ [6, p. 587]. Indeed, let $\mathfrak{T}$ be a closed ideal of $(\Lambda_r)_{\chi_{1-\varepsilon}}$ such that $E^r_{\mathfrak{T}}$ is not a Carleson set. We have

$$\left\|\sum((\zeta+\varepsilon)-\pi(z))^{-1}\right\|_{(\Lambda_r)_{\chi_{1-\varepsilon}}/\mathfrak{T}} \le \sum \frac{c}{d^4(\zeta+\varepsilon, E^r_{\mathfrak{T}})},\ \text{where } 1 \le |\zeta+\varepsilon| \le 2 \text{ and}$$

$\pi : (\Lambda_r)_{\chi_{1-\varepsilon}} \to (\Lambda_r)_{\chi_{1-\varepsilon}}/\mathfrak{T}$ is the canonical quotient map. It is clear that does not exist any power M such that $\int_{\mathbb{T}} \sum \frac{|f_r^M(e^{i(2-\varepsilon)})|}{d^4(e^{i(2-\varepsilon)}, E^r)}$ for all functions in $(\Lambda_r)_{\chi_{1-\varepsilon}}$ vanishing on $E^r_{\mathfrak{T}}$. Therefore we cannot conclude the first step of the resolvent method as described above.

From now on, ω will be a modulus of continuity such that for every $0 \le \varepsilon \le 1$ the following condition

$$\omega\left((2-\varepsilon)^{(1+\varepsilon)}\right) \ge \eta_{(1+\varepsilon)}\omega^{(1+\varepsilon)}(2-\varepsilon) \quad (\varepsilon \ge 0) \tag{1.1}$$

is satisfied, where $\eta_{(1+\varepsilon)} > 0$ is a constant depending only on $(1+\varepsilon)$.

In this work we prove that the closed ideals of the algebras $(\Lambda_r)_\omega$ are standard. For proving this we use only a special method of approximating outer functions in $(\Lambda_r)_\omega$ together with the F-property. More precisely, we obtain the following

Theorem (1.1): Let ω be a modulus of continuity satisfying (1.1). If $\mathfrak{T}$ is closed ideal of $(\Lambda_r)_\omega$, then

$$\mathfrak{T} = \left\{f_r \in (\Lambda_r)_\omega : f_{|E^r_{\mathfrak{T}}} \equiv 0 \text{ and } f_r/(U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})\right\},$$

where $E^r_{\mathfrak{T}} := \{(\zeta+\varepsilon) \in \mathbb{T} : f_r(\zeta+\varepsilon) = 0,\ \forall f_r \in \mathfrak{T}\}$ and $(U_r)_{\mathfrak{T}}$ is the greatest common divisor of the inner parts of the nonzero functions in $\mathfrak{T}$.

Consequently, we obtain the structure of closed ideals of the particular algebras $(\Lambda_r)_{\chi_{1-\varepsilon}}$.

2.Other results and proof of Theorem (1.1)

We begin by recalling that every function f_r in the disk algebra has the canonical factorization $f_r = c_{f_r} U_{f_r} O_{f_r}$, where c_{f_r} is a constant of modulus 1, $(U_r)_{f_r}$ is an inner function (that is $|U_r f_r| = 1$ a.e. on $\mathbb{T}$) and O_{f_r} the outer function given by

$$O_{f_r}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_{j=1}^n e^{i\theta_j} + z}{\sum_{j=1}^n e^{i\theta_j} - z} \log\left|f_r(\sum_{j=1}^n e^{i\theta_j})\right| d(\sum_{j=1}^n \theta_j)\right\} \quad (z \in \mathbb{D}).$$

The closed ideal of all functions in $(\Lambda_r)_\omega$ vanishing on E^r is designed by $\mathcal{J}_{E^r}$. <u>In Section 3.2 we give the proof of the following theorem.</u>

Theorem (2.1): Let ω be a modulus of continuity satisfying the following condition

$$\omega((2-\varepsilon)^2) \geq \eta_2 \omega^2(2-\varepsilon), \quad \varepsilon \geq 0 \tag{2.1}$$

where $\eta_2 > 0$ is a constant. Let $\mathfrak{T}$ be a closed ideal in $(\Lambda_r)_\omega$ such that $(U_r)_{\mathfrak{T}} \equiv 1$ and let $g_r \in \mathcal{J}_{E^r_{\mathfrak{T}}}$ be an outer function. Then g_r^2 belongs to $\mathfrak{T}$.

Remark (2.2): In the same way, as in Theorem (2.1), we can obtain that if moreover ω satisfy the following stronger condition

$$\omega((2-\varepsilon)^2) \geq \eta\, \omega\,(2-\varepsilon)\,, \qquad (\varepsilon \geq 0) \tag{2.2}$$

then g_r belongs to $\mathfrak{T}$.

In Section 3.3 we prove the following theorem.

Theorem (2.3): Let ω be a modulus of continuity satisfying the condition (46). Let $\mathfrak{T}$ be a closed ideal in $(\Lambda_r)_\omega$ and let $g_r \in (\Lambda_r)_\omega$ be a function such that $(U_r)_{g_r} O_{g_r}^2 \in \mathfrak{T}$. Then g belongs to $\mathfrak{T}$.

Proof of Theorem (1.1). We have to prove that every closed ideal of $(\Lambda_r)_\omega$ is standard. For this, let $\mathfrak{T}$ be a closed ideal of the algebra $(\Lambda_r)_\omega$. If $(U_r)_{\mathfrak{T}} \equiv 1$, let g_r be a function in $\mathcal{J}_{E^r_{\mathfrak{T}}}$. Hence, making use of the F-property of $(\Lambda_r)_\omega$, it follows $O_{g_r} \in (\Lambda_r)_\omega$, and therefore $O_{g_r} \in \mathcal{J}_{E^r_{\mathfrak{T}}}$. Thus, according to Theorem (4.3.2), we deduce $O_{g_r}^2 \in \mathfrak{T}$ and then $(U_r)_{g_r} O_{g_r}^4 \in \mathfrak{T}$. Next, by applying Theorem (4.3.4) two times, we conclude $g_r \in \mathfrak{T}$.

Now if $(U_r)_{\mathfrak{T}} \not\equiv 1$, we choose $g_r \in \mathcal{J}_{E^r_{\mathfrak{T}}}$ such that $g_r / (U_r)_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})$. Thus, the associated ideal

$$\mathcal{K}_g := \{f_r \in (\Lambda_r)_\omega : (f_r)_{g_r} \in \mathfrak{T}\}$$

is closed and by the F-property of $(\Lambda_r)_\omega$ we see easily that $(U_r)_{\mathcal{K}_{g_r}} \not\equiv 1$ and then $\mathcal{K}_g = \mathcal{J}_{(E^r)_{\mathcal{K}_{g_r}}}$. Now, since $(E^r)_{\mathcal{K}_{g_r}} \subseteq E^r_{\mathfrak{T}}$, then $O_{g_r} \in \mathcal{K}_{g_r}$. It follows that $(U_r)_{g_r} O_{g_r}^2 \in \mathfrak{T}$. So, by Theorem (4.3.4), $g_r \in \mathfrak{T}$. This completes the proof of the theorem.

3.Approximation of functions in $(\Lambda_r)_\omega$

In this section we give the proofs of Theorems (2.1) and (2.3). For simplicity we use the following Tamrazov's Theorem (see Appendix A): If f_r is a function in the disk algebra such that $f_r \in \Lambda_\omega(\mathbb{T})$, then $f_r \in (\Lambda_r)_\omega$. We need also the following simple lemma.

Lemma(3.1). Let $(f_r)_n \in (\Lambda_r)_\omega$ be a sequence of functions converging uniformly on the closed unit disk to $f_r \in (\Lambda_r)_\omega$. If

$$\sum \frac{|(f_r)_n(z) - (f_r)_n(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \qquad (as\ |\varepsilon| \to 0)\,,$$

uniformly with respect to n , then $\lim_{n\to+\infty} \sum \|(f_r)_n - f_r\|_\omega = 0$.

For $f_r \in (\Lambda_r)_\omega$, the inner function U_{f_r} is uniquely factored in the form $U_{f_r} = B_{f_r} S_{f_r}$, where B_{f_r} is the usual Blashke product associated to $Z_{f_r} \cap D, Z_{f_r} := \{z \in \mathbb{D} : f_r(z) = 0\}$ and the function

$$S_{f_r}(z) := \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \sum \frac{\sum e^{i\theta_j} + z}{\sum e^{i\theta_j} - z} d\mu_{f_r}(\sum \theta_j)\right\} \quad (z \in \mathbb{D})\,,$$

is the singular inner function associated to the singular positive measure μ_{f_r}. Note that the support $\mathrm{supp}(\mu_{f_r})$ of the singular measure μ_{f_r} is a closed subset of

$E^r_{f_r} := \{(\zeta + \varepsilon) \in \mathbb{T} : f_r(\zeta + \varepsilon) = 0\}$. For $a, a+\varepsilon \in \mathbb{T}$, we design by $(a, a+\varepsilon)$ (resp. $[a, a+\varepsilon]$) an open arc (resp. closed arc) of $\mathbb{T}$ connecting the points a and $a+\varepsilon$.

Lemma (3.2). Let ω be a modulus of continuity satisfying the condition (2.1). Let g_r be a function in $(\Lambda_r)_\omega$ and let $U_r = B_{U_r} S_{U_r} \in \mathcal{H}^\infty(\mathbb{D})$ be an inner function such that

$B_{g_r}/B_{U_r} \in \mathcal{H}^\infty(\mathbb{D})$, $\operatorname{supp}(\mu_{U_r}) \subseteq E^r_{g_r}$ and $(1/2\pi)\int_0^{2\pi} d\mu_{U_r}(\sum_{j=1}^n \theta_j) \le \widetilde{M}$, where $\widetilde{M}$ is a constant. Then $U_r O^2_{g_r}$ belongs to $(\Lambda_r)_\omega$ and we have

$$\sum \frac{\left|U_r(\zeta+\varepsilon)O^2_{g_r}(\zeta+\varepsilon) - U_r(\zeta)O^2_{g_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.1}$$

uniformly with respect to U_r. If moreover ω satisfy the condition (2.2), then $U_r O_{g_r}$ belongs to $(\Lambda_r)_\omega$ and we have

$$\sum \frac{\left|U_r(\zeta+\varepsilon)O_{g_r}(\zeta+\varepsilon) - U_r(\zeta)O_{g_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.2}$$

uniformly with respect to U_r.

Proof. Let $\zeta+\varepsilon, \zeta \in \mathbb{T}$ be two distinct points such that $d(\zeta+\varepsilon, E^r_{g_r}) \le d(\zeta, E^r_{g_r})$. It is clear that

$$\sum \frac{\left|U_r(\zeta+\varepsilon)O_{g_r}(\zeta+\varepsilon) - U_r(\zeta)O_{g_r}(\zeta)\right|}{\omega(\varepsilon)}$$
$$\le \sum \frac{\left|B_{U_r}(\zeta+\varepsilon)O^2_{g_r}(\zeta+\varepsilon) - B_{U_r}(\zeta)O^2_{g_r}(\zeta)\right|}{\omega(\varepsilon)} + \sum \left|O^2_g(\zeta)\right| \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)}.$$

By the F-property of $(\Lambda_r)_\omega$ we have $O_{g_r} \in (\Lambda_r)_\omega$ and $B_{U_r}O^2_{g_r} \in (\Lambda_r)_\omega$. Then to prove (3.1), it is sufficient to prove that

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.3}$$

First we suppose that $\varepsilon \ge \left(\frac{d(\zeta, E^r_{g_r})}{2}\right)^2$. Then

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} \le 8\eta_2^{-1} \sum \left(\frac{\left|O_{g_r}(\zeta)\right|}{\omega(d(\zeta, E^r_{g_r}))}\right)^2 = o(1) \quad (\text{as} \quad \varepsilon \to 0) \tag{3.4}$$

Now, let suppose that $\varepsilon \ge \left(\frac{d(\zeta, E^r_{g_r})}{2}\right)^2$. Then $[\zeta+\varepsilon, \zeta] \subset \mathbb{T}\backslash E_{g_r}$ and therefore $d(z, E^r_{g_r}) \ge d(\zeta, E^r_{g_r})$, for every $z \in [\zeta+\varepsilon, \zeta]$. There is $z \in [\zeta+\varepsilon, \zeta]$ such that

$$\sum \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\varepsilon} = \sum \left|S'_{U_r}(z)\right| \le \frac{1}{\pi}\int_0^{2\pi} \sum \frac{1}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\mu_{U_r}(\sum_{j=1}^n \theta_j) \le \sum \frac{2\widetilde{M}}{d^2(z, E^r_{g_r})}.$$

It follows

$$\sum \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\varepsilon} \le \sum \frac{2\widetilde{M}}{d^2(\zeta, E^r_{g_r})}.$$

We obtain

$$\sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} = \sum \left|O^2_{g_r}(\zeta)\right| \frac{\left|S_{U_r}(\zeta+\varepsilon) - S_{U_r}(\zeta)\right|}{\omega(\varepsilon)} \frac{\varepsilon}{\omega(\varepsilon)}$$
$$\le 2M\eta_2^{-1} \sum \left(\frac{\left|O_{g_r}(\zeta)\right|}{\omega(d(\zeta, E^r_{g_r}))}\right)^2 \frac{\omega(d^2(\zeta, E^r_{g_r}))}{d^2(\zeta, E^r_{g_r})} \frac{\varepsilon}{\omega(\varepsilon)} = o(1) \quad (\text{as} \quad \varepsilon \to 0). \tag{3.5}$$

So (3.3) follows from (3.4) and (3.5).Consequently $U_r O^2_{g_r}$ belongs to $(\Lambda_r)_\omega$. If moreover ω satisfy the condition (2.2), we can argue similarly to prove (3.2). This finishes the proof of the lemma.

Lemma (3.3). Let f_r be a function in Λ_ω. Let $\delta > 0$, $N \in \mathbb{N}$ and $\{\text{an}: 0 \le n \le N\}$ be a finite number of points in $E^r_{f_r}$. Then

$$\lim_{\delta \to 0} \sum \left\|\psi_{\delta,N} f_r - f_r\right\|_\omega = 0,$$

where $\psi_{\delta,N}(z) := \prod_{n=0}^{n=N} \frac{z\overline{a_n}-1}{z\overline{a_n}-1-\delta}$, $z\in\mathbb{D}$

Proof. Without loss of the generality we can suppose that $N = 0$ and $a_0 = 1$. Set $\psi_\delta(z) := \frac{z-1}{z-1-\delta}$, $z \in \mathbb{D}$, and suppose that $\{1\} \in E_{f_r}^r$. We have to show that $\lim_{\delta\to 0}\sum\|\psi_\delta f_r - f_r\|_\omega = 0$. Let $\zeta + \varepsilon, \zeta \in \mathbb{T}$ be two distinct points such that $\varepsilon \geq 0|$. We have

$$\sum \frac{|\psi_\delta(\zeta+\varepsilon)f_r(\zeta+\varepsilon) - \psi_\delta(\zeta)f_r(\zeta)|}{\omega(\varepsilon)}$$

$$\leq |\psi_\delta(\zeta+\varepsilon)|\sum\frac{|f_r(\zeta+\varepsilon)-f_r(\zeta)|}{\omega(\varepsilon)}$$

$$+\sum|f_r(\zeta)|\frac{|\psi_\delta(\zeta+\varepsilon)-\psi_\delta(\zeta)|}{\omega(\varepsilon)}. \tag{3.6}$$

Suppose first that $\varepsilon \geq 0$. Then

$$\sum|f_r(\zeta)|\frac{|\psi_\delta(\zeta+\varepsilon)-\psi_\delta(\zeta)|}{\omega(\varepsilon)} \leq 2\sum\frac{|f_r(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon\to 0). \tag{3.7}$$

Now if $\varepsilon \geq 0$, then $|z-1| \geq |\zeta-1|$ for every $z \in [\zeta+\varepsilon, \zeta]$. We obtain

$$\sum|f_r(\zeta)|\frac{|\psi_\delta(\zeta+\varepsilon)-\psi_\delta(\zeta)|}{\omega(\varepsilon)} = \sum|f_r(\zeta)|\frac{|\psi_\delta(\zeta+\varepsilon)-\psi_\delta(\zeta)|}{(\varepsilon)}\frac{|f_r(\zeta)|}{\omega(\varepsilon)}$$

$$= \sum|f_r(\zeta)||\psi'_\delta(Z)|\frac{\varepsilon}{\omega(\varepsilon)} \quad (Z\in[\varepsilon]) \leq \sum\frac{|f_r(\zeta)|}{\omega(|\zeta-1|)}\frac{\omega(|\zeta-1|)}{|\zeta-1|}\frac{\varepsilon}{\omega(\varepsilon)}$$

$$= o(1) \quad (\text{as } \varepsilon\to 0) \tag{3.8}$$

From (3.6), (3.7) and (3.8) we deduce

$$\sum\frac{|\psi_\delta(\zeta+\varepsilon)f_r(\zeta+\varepsilon)-\psi_\delta(\zeta)f_r(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon\to 0),$$

uniformly with respect to $\delta > 0$. So the result follows by applying Lemma (3.1) to the family of functions $\psi_\delta f_r, \delta > 0$. This completes the proof of the lemma.

We denote by K^C the complement in $\mathbb{T}$ of the subset K of $\mathbb{T}$. For a closed subset E^r of $\mathbb{T}$, we have $(E^r)^C = \cup_{n\in\mathbb{N}}(a_n, a_n+\varepsilon_n)$, where $(a_n, a_n+\varepsilon_n) \subset (E^r)^C$ and $a_n, a_n+\varepsilon_n \in E^r$. We define Ω_{E^r} to be the family of all the unions of arcs $(a_n, a_n+\varepsilon_n)$, where $(a_n, a_n+\varepsilon_n) \subset (E^r)^C$ and a_n, $a_n+\varepsilon_n \in E^r$. For a given function f_r in the disk algebra and $\Gamma \in \Omega_E$ (E^r is a closed subset of $\mathbb{T}$), let us define the outer function $(f_r)_\Gamma \in \mathcal{H}^\infty(\mathbb{D})$ associated to the outer factor of f_r by

$$(f_r)_\Gamma(z) := \exp\left\{\frac{1}{2\pi}\int_\Gamma\sum\frac{\sum_{j=1}^n e^{i\theta_j}+z}{\sum_{j=1}^n e^{i\theta_j}-z}\,log\left|f_r\left(\sum_{j=1}^n e^{i\theta_j}\right)\right|d\left(\sum_{j=1}^n\theta_j\right)\right\} \quad (z\in\mathbb{D}).$$

Then, we assert

Lemma (3.4). Let $\mathfrak{I}$ be a closed ideal of $(\Lambda_r)_\omega$, $f_r \in (\Lambda_r)_\omega$ an outer function and let $h \in \mathcal{J}_{E_{\mathfrak{I}}^r}$ a function such that $h_{f_r} \in \mathfrak{I}$. Let $\Gamma \in \Omega_{E_{\mathfrak{I}}^r}$ be such that $\mathbb{T}\backslash\overline{\Gamma}$ is union of a finite number of arcs $(a, a+\varepsilon) \subseteq \mathbb{T}\backslash E_{\mathfrak{I}}^r$ $(a, a+\varepsilon \in E_{\mathfrak{I}}^r)$. If $h(f_r)_\Gamma \in (\Lambda_r)_\omega$, then $h(f_r)_\Gamma \in \mathfrak{I}$.

Proof. For simplicity we suppose that $\mathbb{T}\backslash\overline{\Gamma} = (a, a+\varepsilon) := \gamma$, where $a, a+\varepsilon \in E_{\mathfrak{I}}^r$ and $(a, a+\varepsilon) \subseteq \mathbb{T}\backslash E_{\mathfrak{I}}^r$. Let $\varepsilon > 0$ be such that $\gamma_\varepsilon := (ae^{i\varepsilon}, (a+\varepsilon)e^{-i\varepsilon}) \subset \gamma$. For $\delta > 0$, we set

$$\phi_{\delta,\varepsilon}(z) := \left(\frac{z\bar{a}e^{-i\varepsilon}-1}{z\bar{a}e^{-i\varepsilon}-1-\delta}\right)\left(\frac{z\overline{a+\varepsilon}e^{i\varepsilon}-1}{z\overline{a+\varepsilon}e^{i\varepsilon}-1-\delta}\right) \quad (z\in\mathbb{D}).$$

It is clear that $\phi_{\delta,\varepsilon} \in (\Lambda_r)_\omega$ and that $\phi_{\delta,\varepsilon}(ae^{i\varepsilon}) = \phi_{\delta,\varepsilon}((a+\varepsilon)e^{-i\varepsilon}) = 0$. Then, according to <u>Proposition 3.6.1</u> (below), we see that the function $(f_r)_{\gamma_\varepsilon}$ multiplied by the square of $\phi_{\delta,\varepsilon}$ belongs to $(\Lambda_r)_\omega$, i.e., $\phi_{\delta,\varepsilon}^2(f_r)_{\gamma_\varepsilon} \in (\Lambda_r)_\omega$. Similarly, we get $\phi_{\delta,\varepsilon}^2(f_r)_{\gamma_\varepsilon^c} \in (\Lambda_r)_\omega$. Now, for

$\pi : (\Lambda_r)_\omega \to (\Lambda_r)_\omega / \mathfrak{T}$ being the canonical quotient map, it follows

$$0 = \sum \pi\left(\phi^4_{\delta,\varepsilon} h_{f_r}\right) = \sum \pi\left(h\phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon}\right)\pi\left(\phi^2_{\delta,\varepsilon}(f_r)_{\gamma_\varepsilon}\right).$$

Since, the function $\phi^2_{\delta,\varepsilon}(f_r)_{\gamma_\varepsilon}$ is invertible in the quotient algebra $(\Lambda_r)_\omega / \mathfrak{T}$, then

$$h\phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon} \in \mathfrak{T} \qquad (\delta, \varepsilon > 0).$$

Using the fact

$$\lim_{\varepsilon \to 0} \sum \left\| \phi^2_{\delta,\varepsilon}(f_r)_{\gamma^c_\varepsilon} - \phi^2_{\delta,0}(f_r)_\Gamma \right\|_\omega = 0,$$

we can check

$$\phi^2_{\delta,0} h f_\Gamma \in \mathfrak{T} \qquad (\delta > 0).$$

Now, since $h(f_r)_\Gamma \in (\Lambda_r)_\omega$ and $h(f_r)_\Gamma(a) = h(f_r)_\Gamma(a + \varepsilon) = 0$, we can deduce from Lemma (4.3.6) that

$$\lim_{\delta \to 0} \left\| \phi^2_{\delta,0} h(f_r)_\Gamma - h(f_r)_\Gamma \right\|_\omega = 0.$$

So $h(f_r)_\Gamma \in \mathfrak{T}$. This completes the proof of the lemma.

Let $\mathfrak{T}$ be a closed ideal of the algebra $(\Lambda_r)_\omega$. We have $U_\mathfrak{T} = B_\mathfrak{T} S_\mathfrak{T}$. The inner function $B_\mathfrak{T}$ is the usual Blashke product associated to the zero set $Z_\mathfrak{T} \cap \mathbb{D}$, where $Z_\mathfrak{T} := \{z \in \overline{\mathbb{D}} : f_r(z) = 0 \text{ for all } f_r \in \mathfrak{T}\}$. The positive singular measure $\mu_\mathfrak{T}$ associated to the singular inner function $S_\mathfrak{T}$ is the greatest common divisor of all μ_{f_r}, $f_r \in \mathfrak{T}$. Note that $\operatorname{supp}(\mu_\mathfrak{T})$ is included in $E^r_\mathfrak{T}$. For a subset K of T, we set

$$(S_{f_r})_K(z) := \exp\left\{ -\frac{1}{2\pi} \int_K \sum \frac{\sum_{j=1}^n e^{i\theta_j} + z}{\sum_{j=1}^n e^{i\theta_j} - z} d\mu_{f_r}\left(\sum_{j=1}^n \theta_j\right) \right\} \qquad (f_r \in \mathcal{A}(\mathbb{D}))$$

Lemma (3.5) Let f_r be a function in a closed ideal $\mathfrak{T}$ of the algebra $(\Lambda_r)_\omega$. Then $B_\mathfrak{T}(S_{f_r})_{E_\mathfrak{T}} O_{f_r}$ belongs to $\mathfrak{T}$.

Proof. Let $f_r \in \mathfrak{T}$. Define $B_{f_r,n}$ and $B_{\mathfrak{T},n}$ to be respectively the Blashke product with zeros $Z_{f_r} \cap \mathbb{D}_n$ and $Z_\mathfrak{T} \cap \mathbb{D}_n$, where $\mathbb{D}_n := \{z \in \mathbb{D} : |z| < \frac{n-1}{n},\ n \in \mathbb{N}\}$. Fix $n \in \mathbb{N}$. The function $B_{f_r,n} / B_{\mathfrak{T},n}$ is invertible in the quotient algebra $(\Lambda_r)_\omega / \mathcal{J}_n$, where $\mathcal{J}_n := \{g \in (\Lambda_r)_\omega : g B_{\mathfrak{T},n} \in \mathfrak{T}\}$. Then $f_r / B_{\mathfrak{T},n} \in \mathcal{J}_n$. It follows that $B_{\mathfrak{T},n}(f_r / B_{\mathfrak{T},n}) \in \mathfrak{T}$. It is clear that

$$\lim_{n \to +\infty} \sum \left\| B_{\mathfrak{T},n}\left(f_r / B_{f_r,n}\right) - B_\mathfrak{T} S_\mathfrak{T} O_f \right\|_\infty = 0.$$

By using Corollary B.2 in Appendix B (F-property of Λ_ω), we obtain

$$\lim_{n \to +\infty} \sum \left\| B_{\mathfrak{T},n}\left(f_r / B_{f_r,n}\right) - B_\mathfrak{T} S_\mathfrak{T} O_{f_r} \right\|_\omega = 0.$$

So, $B_\mathfrak{T} S_\mathfrak{T} O_{f_r} \in \mathfrak{T}$. Let $\varepsilon > 0$ be such that $\gamma_\varepsilon := (ae^{i\varepsilon}, (a + \varepsilon)e^{-i\varepsilon}) \subset \gamma := (a, a + \varepsilon)$, where $a, a + \varepsilon \in E^r_\mathfrak{T}$ and $(a, a + \varepsilon) \subseteq \mathbb{T} \setminus E^r_\mathfrak{T}$. By using the F-property of $(\Lambda_r)_\omega$, the function $B_\mathfrak{T}(S_{f_r})\gamma^c_\varepsilon O_{f_r}$ belongs to $(\Lambda_r)_\omega$. We set

$$\phi_\varepsilon(z) := \left(z\bar{a}e^{-i\varepsilon} - 1\right)\left(z\overline{a + \varepsilon}e^{-i\varepsilon} - 1\right) \quad (z \in \mathbb{D}).$$

From <u>Proposition 3.6.1</u> below, the function $(g_r)_\varepsilon := \phi^2_\varepsilon (f_r)_{\gamma_\varepsilon}$ belongs to $(\Lambda_r)_\omega$. Since $(\mu_{(S_{f_r})_{\gamma_\varepsilon}}) \subseteq E^r_{(g_r)_\varepsilon}$, then we deduce from Lemma (4.3.5) that $(S_{f_r})_{\gamma_\varepsilon}(g_r)^2_\varepsilon \in (\Lambda_r)_\omega$. We have

$$0 = \sum \pi\left((g_r)^2_\varepsilon B_\mathfrak{T} S_\mathfrak{T} O_{f_r}\right) = \sum \pi\left(\left(S_{f_r}\right)_{\gamma_\varepsilon}(g_r)^2_\varepsilon\right) \times \pi\left(B_\mathfrak{T}(S_{f_r})_{\gamma^c_\varepsilon} O_{f_r}\right)$$

where $\pi : (\Lambda_r)_\omega \to (\Lambda_r)_\omega / \mathfrak{T}$, is the canonical quotient map. The function $(S_{f_r})_{\gamma_\varepsilon} g^2_\varepsilon$ is invertible in the quotient algebra $(\Lambda_r)_\omega / \mathfrak{T}$, then $B_\mathfrak{T}(S_{f_r})_{\gamma^c_\varepsilon} O_{f_r} \in \mathfrak{T}$. It is clear that

$$\lim_{\varepsilon \to 0} \sum \left\| B_\mathfrak{T}(S_{f_r})_{\gamma^c_\varepsilon} O_{f_r} - B_\mathfrak{T}(S_{f_r})_{\gamma^c} O_{f_r} \right\|_\infty = 0.$$

Then, using Corollary B.2 in Appendix B, we obtain

$$\lim_{\varepsilon\to 0}\left\|B_{\mathfrak{T}}(S_f)_{\gamma_\varepsilon^c}\,O_f - B_{\mathfrak{T}}(S_f)_{\gamma^c}\,O_f\right\|_\omega = 0\,.$$

So $B_{\mathfrak{T}}(S_f)_{\gamma^c}\,O_f \in \mathfrak{T}$. Similarly we can prove that $B_{\mathfrak{T}}(S_f)_{\Gamma_N^c}\,O_f \in \mathfrak{T}$, where $\Gamma_N := \bigcup_{n\le N}(a_n, a_n+\varepsilon_n) \in \Omega_{E_{\mathfrak{T}}}$. We have

$$\lim_{N\to+\infty}\left\|B_{\mathfrak{T}}(S_f)_{\Gamma_N^c}\,O_f - B_{\mathfrak{T}}(S_f)_{E_{\mathfrak{T}}}\,O_f\right\|_\infty = 0\,.$$

Using again Corollary B.2 we deduce

$$\lim_{N\to+\infty}\left\|B_{\mathfrak{T}}(S_f)_{\Gamma_N^c}\,O_f - B_{\mathfrak{T}}(S_f)_{E_{\mathfrak{T}}}\,O_f\right\|_\omega = 0\,.$$

Then $B_{\mathfrak{T}}(S_f)_{E_{\mathfrak{T}}}\,O_f \in \mathfrak{T}$. This proves the lemma.

3.2. Proof of Theorem (2.1)

For the proof of Theorem (2.1), we need the following proposition about approximation of functions in Λ_ω.

Proposition (3.6) Let ω be a modulus of continuity satisfying the condition (2.1). Let $f \in \Lambda_\omega$ be a function such that $\|f\|_\omega \le 1$ and E a closed subset of $\mathbb{T}$. Let $g \in \mathcal{J}_E$ be an outer function and S singular inner function such that $\operatorname{supp}(\mu_S) \subseteq E_g$. Then

(i) the functions Sg^2 and $Sg^2 f_{\Gamma_N^c}$ belong to Λ_ω, for every $N \in \mathbb{N}$,

(ii) we have $\lim_{N\to+\infty}\left\|Sg^2 f_{\Gamma_N^c} - Sg^2\right\|_\omega = 0$, where $\Gamma_N := \bigcup_{n\le N}(a_n, a_n+\varepsilon_n) \in \Omega_E$. If moreover ω satisfy the stronger condition (2.2), then

(i)′. the functions Sg and $Sgf_{\Gamma_N^c}$ belong to Λ_ω, for every $N \in \mathbb{N}$,

(ii)′. we have $\lim_{N\to+\infty}\left\|Sgf_{\Gamma_N^c} - Sg\right\|_\omega = 0$.

Proof. From Lemma (3.2) we have $Sg^2 \in \Lambda_\omega$. It is clear that $\{Sgf_{\Gamma_N^c}\}_{N\in\mathbb{N}}$ is a sequence of functions in the disk algebra. Let us note that if

$$\frac{|S(z)g^2(z)f_\Gamma(z) - S(z-\varepsilon)g^2(z-\varepsilon)f_\Gamma(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (as\ |\varepsilon| \to 0) \tag{3.9}$$

uniformly with respect to $\Gamma \in \Omega_E$, then assertion (i) follows immediately. Furthermore, assertion (ii) can be deduced by applying Lemma (3.1).

Thus, it suffices to show only (3.9). For this, we fix $\Gamma \in \Omega_E$ and we let $\zeta+\varepsilon$ and ζ be two distinct points in $\mathbb{T}$ such that $d(\zeta+\varepsilon, E) \ge d(\zeta, E)$. It is clear that

$$\frac{|S(\zeta+\varepsilon)g^2(\zeta+\varepsilon)f_\Gamma(\zeta+\varepsilon) - S(\zeta)g^2(\zeta)f_\Gamma(\zeta)|}{\omega(\varepsilon)}$$

$$\le |f_\Gamma(\zeta+\varepsilon)|\frac{|S(\zeta+\varepsilon)g^2(\zeta+\varepsilon) - S(\zeta)g^2(\zeta)|}{\omega(\varepsilon)} + |g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon) - f_\Gamma(\zeta)|}{\omega(\varepsilon)}.$$

Since $Sg^2 \in \Lambda_\omega$ (by Lemma 3.2), then the proof of (3.9) reduces to

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon) - f_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (as\ |\varepsilon| \to 0) \tag{3.10}$$

Case (i). For $\varepsilon \ge (\frac{d(\zeta,E)}{2})^2$, we have

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon) - f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le 2\frac{|g^2(\zeta)|}{\omega(\varepsilon)} \le 8\eta_2^{-1}\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^2$$

$$= o(1) \qquad (as\ |\varepsilon| \to 0)\,. \tag{3.11}$$

Case (ii). For $\varepsilon \le (\frac{d(\zeta,E)}{2})^2$ with $\zeta \not\in \Gamma$. It follows $[\zeta+\varepsilon, \zeta] \subset (\Gamma \cup E)^c$. Then $z \not\equiv \Gamma$ and $d(z,E) \ge d(\zeta,E)$ for every $z \in [\zeta+\varepsilon, \zeta]$. There is $z \in [\zeta+\varepsilon, \zeta]$ such that

$$\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} = |f_\Gamma'(z)| \le \frac{1}{\pi}\int_\Gamma \frac{\left|log\left|f\left(\sum_{j=1}^n e^{i\theta_j}\right)\right|\right|}{\left|\sum_{j=1}^n e^{i\theta_j}-z\right|} d\left(\sum_{j=1}^n \theta_j\right) \le \frac{c_f}{d^2(z,E)}.$$

It follows

$$\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le \frac{c_f}{d^2(\zeta,E)}.$$

Therefore, we have

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} = \frac{|g^2(\zeta)|\varepsilon}{\omega(\varepsilon)}\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\varepsilon}$$
$$\le c_f\frac{|g^2(\zeta)|\varepsilon}{\omega(\varepsilon)d^2(\zeta,E)} \le c_f\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^2\frac{\omega(d^2(\zeta,E))\varepsilon}{\omega(\varepsilon)d^2(\zeta,E)} = o(1)$$
$$(\text{as } \varepsilon\to 0) \qquad (3.12)$$

Case 3. In this case let us assume that $\varepsilon \le (\frac{d(\zeta,E)}{2})^2$ and $\zeta \in \Gamma$, and therefore $\left|f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon)\right| = \left|f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta)\right| = 1$. From

$$f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta) = f_\Gamma(\zeta+\varepsilon)f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon) - f_\Gamma(\zeta)f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta)$$
$$= f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon)\big(f(\zeta+\varepsilon)-f(\zeta)\big) + f(\zeta+\varepsilon)\left(f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon)-f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta)\right)$$
$$= f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon)\big(f(\zeta+\varepsilon)-f(\zeta)\big)$$
$$- f(\zeta)f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta+\varepsilon)f_{\mathbb{T}\setminus\Gamma}^{-1}(\zeta)\left(f_{\mathbb{T}\setminus\Gamma}(\zeta+\varepsilon)\right.$$
$$\left.- f_{\mathbb{T}\setminus\Gamma}(\zeta)\right) \qquad (3.13)$$

combined with (3.12), we deduce

$$|g^2(\zeta)|\frac{|f_\Gamma(\zeta+\varepsilon)-f_\Gamma(\zeta)|}{\omega(\varepsilon)} \le |g^2(\zeta)|\frac{|f(\zeta+\varepsilon)-f(\zeta)|}{\omega(\varepsilon)}$$
$$+|f(\zeta)||g^2(\zeta)|\frac{\left|f_{\mathbb{T}\setminus\Gamma}(\zeta+\varepsilon)-f_{\mathbb{T}\setminus\Gamma}(\zeta)\right|}{\omega(\varepsilon)} = o(1) \qquad (\text{as } \varepsilon\to 0) \qquad (3.14)$$

From inequalities (3.11), (3.12) and (3.14), we see that (3.10) holds. If moreover ω satisfy the condition (2.2), we can deduce similarly the assertions (i)′ and (ii)′. This completes the proof of the proposition.

Proof of Theorem (2.1). Let $\mathfrak{T}$ be a closed ideal of the algebra Λ_ω such that $U_{\mathfrak{T}} \equiv 1$. Let g be an outer function in $\mathcal{J}_{E_{\mathfrak{T}}}$ and consider $f \in \mathfrak{T}$. From Lemma (3.5), we deduce that $(S_f)_{E_{\mathfrak{T}}}O_f \in \mathfrak{T}$. Then, $(S_f)_{E_{\mathfrak{T}}}g^2O_f \in \mathfrak{T}$. From Lemma (3.4) and Proposition 3.6.1, we deduce that $(S_f)_{E_{\mathfrak{T}}}g^2f_{\Gamma_N^c} \in \mathfrak{T}$, for every $N \in \mathbb{N}$, where $\Gamma_N := \bigcup_{n\le N}(a_n, a_n+\varepsilon_n) \in \Omega_{E_{\mathfrak{T}}}$. Thus, by Proposition 3.6.2, $(S_f)_{E_{\mathfrak{T}}}g^2 \in \mathfrak{T}$. Now choose a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{T}$ such that the greatest common divisor of the inner parts of f_n is equal to 1 and such that $(1/2\pi)\int_0^{2\pi} d\mu_{f_n}(\theta) \le 1$. This infers that $k_n := (S_{f_n})_{E_{\mathfrak{T}}}g^2 \in \mathfrak{T}$, for every $n \in \mathbb{N}$. By Lemma (3.2) we have

$$\frac{|k_n(\zeta+\varepsilon)-k_n(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon\to 0),$$

uniformly with respect to n. Using the fact

$$\lim_{n\to+\infty}\|k_n-g^2\|_\infty = 0$$

and Lemma (3.1), we deduce

$$\lim_{n\to+\infty}\|k_n-g^2\|_\omega = 0.$$

So $g^2 \in \mathfrak{T}$. This finishes the proof of the theorem.

3.3. Proof of Theorem (2.3)

We begin by proving the following proposition.

Proposition (3.7). Let ω be a modulus of continuity satisfying the condition (1.1). Fix $\varepsilon > 0$. Let $g \in \Lambda_\omega$ be a function such that $\|g\|_\omega \leq 1$ and let E be a closed subset of E_g. Then

(i) the functions $U_g O_g^{(1+\varepsilon)}$ and $U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c}$ belong to Λ_ω, for every $N \in \mathbb{N}$,

(ii) we have $\lim_{N\to+\infty} \left\| U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} - U_g O_g^{(1+\varepsilon)} \right\|_\omega = 0$, where $\Gamma_N := \bigcup_{n\leq N}(a_n, a_n + \varepsilon_n) \in \Omega_E$.

Proof. By using the F-property of Λ_ω and the following

$$\begin{aligned} U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta) \\ = O_g^{\varepsilon}(\zeta+\varepsilon)\Big(U_g(\zeta+\varepsilon)O_g(\zeta+\varepsilon) - U_g(\zeta)O_g(\zeta)\Big) - U_g(\zeta)O_g^{\varepsilon}(\zeta+\varepsilon)\Big(O_g(\zeta+\varepsilon) - O_g(\zeta)\Big) \\ + U_g(\zeta)\Big(O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - O_g^{(1+\varepsilon)}(\zeta)\Big) \qquad (\zeta+\varepsilon, \zeta \in \mathbb{T}), \end{aligned}$$

the function $U_g O_g^{(1+\varepsilon)}$ belongs to Λ_ω. It is clear that $\left\{U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c}\right\}_{N\in\mathbb{N}}$ is sequence of functions in the disk algebra and

$$\lim_{N\to+\infty} \left\| U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} - U_g O_g^{(1+\varepsilon)} \right\|_\infty = 0 .$$

Note that if

$$\frac{\left| U_g O_g^{(1+\varepsilon)} g_\Gamma(\zeta+\varepsilon) - U_g O_g^{(1+\varepsilon)} g_\Gamma(\zeta) \right|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0), \tag{3.15}$$

uniformly with respect to $\Gamma \in \Omega_E$, then assertion (i) holds as well as assertion (ii), which follows from (3.15) combined with Lemma (3.1).Below, we have to prove (3.15). Let $\zeta+\varepsilon, \zeta \in \mathbb{T}$ be two different points such that $d(\zeta+\varepsilon, E) \geq d(\zeta, E)$. It is clear that

$$\begin{aligned} &\frac{\left| U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon)g_\Gamma(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta)g_\Gamma(\zeta) \right|}{\omega(\varepsilon)} \\ &\leq |g_\Gamma(\zeta+\varepsilon)| \frac{\left| U_g(\zeta+\varepsilon)O_g^{(1+\varepsilon)}(\zeta+\varepsilon) - U_g(\zeta)O_g^{(1+\varepsilon)}(\zeta) \right|}{\omega(\varepsilon)} + |g(\zeta)|^{(1+\varepsilon)} \frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)} . \end{aligned}$$

Then, to prove (3.15) it is sufficient to prove that

$$|g(\zeta)|^{(1+\varepsilon)} \frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.16}$$

1. We suppose that $\varepsilon \geq \left(\frac{d(\zeta,E)}{2}\right)^{(1+\varepsilon)}$, we obtain

$$|g(\zeta)|^{(1+\varepsilon)} \frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\omega(\varepsilon)} \leq 2\frac{|g(\zeta)|^{(1+\varepsilon)}}{\omega(\varepsilon)} \leq 2^{(2+\varepsilon)}\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^{(1+\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.17}$$

2. In this case we suppose that $\varepsilon \leq \left(\frac{d(\zeta,E)}{2}\right)^{(1+\varepsilon)}$ and that $\zeta \notin \Gamma$. It follows $[\zeta+\varepsilon, \zeta] \subset \mathbb{T}\backslash E$.Then $z \notin \Gamma$ and $|z - \sum_{j=1}^n e^{i\theta_j}| \geq \frac{1}{2}|\zeta - \sum_{j=1}^n e^{i\theta_j}|$ for every $z \in [\zeta+\varepsilon, \zeta]$ and for every $\sum_{j=1}^n e^{i\theta_j} \in \Gamma$. There is $z \in [\zeta+\varepsilon, \zeta]$ such that

$$\frac{|g_\Gamma(\zeta+\varepsilon) - g_\Gamma(\zeta)|}{\varepsilon} = |g_\Gamma'(z)| \leq a_\Gamma(z),$$

where $a_\Gamma(z) := \frac{1}{\pi}\int_\Gamma \frac{\left|log\left|g(\sum_{j=1}^n e^{i\theta_j})\right|\right|}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\theta_j$. Since $a_\Gamma(z) \leq 4a_\Gamma(\zeta)$, then

$$\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{|\xi-\zeta|}\le 4a_\Gamma(\zeta)\,.$$

2.1. First we suppose that $a_\Gamma(\zeta)\le\frac{1}{d^{(1+\varepsilon)}(\zeta,E)}$. It is clear that

$$\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{d^{(1+\varepsilon)}(\zeta,E)\omega(\varepsilon)}=\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^{(1+\varepsilon)}\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}$$
$$\le\inf\left\{\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega\big(d(\zeta,E)\big)}\right)^{(1+\varepsilon)},\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}=o(1)\quad(\text{as }\varepsilon\to 0).$$

Therefore we obtain,

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)}\le\frac{|g(\zeta)|^{(1+\varepsilon)}\varepsilon}{\omega(\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\varepsilon}\le 4\frac{|g(\zeta)|^{(1+\varepsilon)}\varepsilon}{d^4(\zeta,E)\omega(\varepsilon)}$$
$$=o(1)\quad(\text{as }\varepsilon\to 0).\tag{3.18}$$

2.2. Next we suppose that $a_\Gamma(\zeta)\ge\frac{1}{d^{\varepsilon}(\zeta,E)\varepsilon^{1/(1+\varepsilon)}}$. Set $\lambda_\zeta:=1-\varepsilon^{1/(1+\varepsilon)}$. Then

$$|g(\lambda_\zeta\zeta)|=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\lambda_\zeta^2}{\left|\sum_{j=1}^n e^{i\theta_j}-\lambda_\zeta\zeta\right|^2}\log\left|g\left(\sum_{j=1}^n e^{i\theta_j}\right)d\Big(\sum_{j=1}^n\theta_j\Big)\right|\right\}$$
$$\le\exp\left\{\frac{1}{2\pi}\int_\Gamma\frac{\varepsilon^{1/(1+\varepsilon)}}{\left|\sum_{j=1}^n e^{i\theta_j}-\lambda_\zeta\zeta\right|^2}\log\left|g\left(\sum_{j=1}^n e^{i\theta_j}\right)d\Big(\sum_{j=1}^n\theta_j\Big)\right|\right\}\le\exp\left\{-\frac{1}{4}\varepsilon^{1/(1+\varepsilon)}a_\Gamma(\zeta)\right\}$$
$$\le\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}.$$

It is clear that

$|g(\zeta)|^{(1+\varepsilon)}\le 2^{\varepsilon}|g(\zeta)-g(\lambda_\zeta\zeta)|^{(1+\varepsilon)}+2^{\varepsilon}\big|g(\lambda_\zeta\zeta)\big|^{(1+\varepsilon)}$. We obtain

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)}$$
$$\le 2^{\varepsilon}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|^{(1+\varepsilon)}}{\omega(\varepsilon)}|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|$$
$$+2^{\varepsilon}\big|g(\lambda_\zeta\zeta)\big|^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\varepsilon}$$
$$\le 2^{(1+\varepsilon)}\eta_{(1+\varepsilon)}^{-1}o(1)+2^{\varepsilon}\frac{\varepsilon}{\omega(\varepsilon)}\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}\frac{\int_{\mathbb{T}}\left|log\big|g\big(\sum_{j=1}^n e^{i\theta_j}\big)\big|\right|d\big(\sum_{j=1}^n\theta_j\big)}{d^2(\zeta,E)}$$
$$=o(1)\quad(\text{as }\varepsilon\to 0).\tag{3.19}$$

2.3. Now we suppose that

$\frac{1}{d^{(1+\varepsilon)}(\zeta,E)}\le a_\Gamma(\zeta)\le\frac{1}{d^{\varepsilon}(\zeta,E)|\varepsilon^{1/(1+\varepsilon)}}$. set

$\mu_\zeta:=1-\frac{1}{a_\Gamma(\zeta)d^{\varepsilon}(\zeta,E)}$. Then $\varepsilon^{1/(1+\varepsilon)}\le 1-\mu_\zeta\le d(\zeta,E)$. It follows that

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{\omega\big(d(\zeta,E)\big)}{d(\zeta,E)}\right)^{\varepsilon}\le\left(\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\right)^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}\le\eta_{(1+\varepsilon)}^{-1}\,,$$

and

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)} \le \eta_{(1+\varepsilon)}^{-(1/1+\varepsilon)}\left(\frac{\varepsilon}{\omega(\varepsilon)}\right)^{\frac{\varepsilon}{(1+\varepsilon)}}.$$

Then

$$\frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon} \le \inf\left\{\eta_{(1+\varepsilon)}^{-1}\left(\frac{|g(\zeta)|}{\omega(d(\zeta,E))}\right)^{\varepsilon}, \eta_{(1+\varepsilon)}^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{\varepsilon}{\omega(\varepsilon)}\right)^{\frac{\varepsilon}{(1+\varepsilon)}}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0).$$

Also we have $|g(\mu_\zeta\zeta)| \le \exp\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\}$. We obtain

$$|g(\zeta)|^{(1+\varepsilon)}\frac{|g_\Gamma(\zeta+\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} \le |g(\zeta)|^{(1+\varepsilon)}\frac{\varepsilon}{\omega(\varepsilon)}a_\Gamma(z)$$
$$(z \in [\zeta+\varepsilon,\zeta])$$
$$\le |g(\zeta)|^{\varepsilon}\frac{|g(\zeta)-g(\mu_\zeta\zeta)|\varepsilon}{(1-\mu_\zeta)\omega(\varepsilon)}(1-\mu_\zeta)a_\Gamma(z) \qquad \le |g(\zeta)|^{\varepsilon}|g(\mu_\zeta\zeta)|\frac{\varepsilon}{\omega(\varepsilon)}a_\Gamma(z)$$
$$\le \frac{\omega(1-\mu_\zeta)}{1-\mu_\zeta}\frac{\varepsilon}{\omega(\varepsilon)}\left(\frac{|g(\zeta)|}{d(\zeta,E)}\right)^{\varepsilon} + \frac{\varepsilon}{\omega(\varepsilon)}\exp\left\{-\frac{1}{4d^{\varepsilon}(\zeta,E)}\right\}\frac{\int_{\mathbb{T}}\left|log|g(\sum_{j=1}^{n}e^{i\theta_j})|\right|d(\sum_{j=1}^{n}\theta_j)}{d^2(\zeta,E)}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.20}$$

Consequently, from (3.18), (3.19) and (3.20) we obtain that if $\varepsilon \le (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$ and $\zeta \notin \Gamma$, then

$$\left|g^{(1+\varepsilon)}(\zeta)\right|^{(1+\varepsilon)}\frac{|g_\Gamma(z-\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.21}$$

3. In this case we suppose that $\varepsilon \le (\frac{d(\zeta,E)}{2})^{(1+\varepsilon)}$ and that $\zeta \in \Gamma$. We use the equality (3.13) to transfer this case into the case $\zeta \notin \Gamma$. Hence we use (3.21) to obtain also in this case that

$$\left|g^{(1+\varepsilon)}(\zeta)\right|^{(1+\varepsilon)}\frac{|g_\Gamma(z-\varepsilon)-g_\Gamma(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0) \tag{3.22}$$

Now (3.16) follows from inequalities (3.17), (3.21) and (3.22). This completes the proof of the proposition.

Proof of Theorem (2.3). Let $\mathfrak{T}$ be a closed ideal in Λ_ω and g a function in Λ_ω. Then $O_g \in \Lambda_\omega$ and $O_g^{(1+\varepsilon)} \in \Lambda_\omega$, for every $\varepsilon > 0$. Suppose that $U_g O_g^{(1+\varepsilon)} \in \mathfrak{T}$. Then $U_g O_g^{(\varepsilon+3)} \in \mathfrak{T}$, for every $\varepsilon > 0$.

From Proposition 3.7.1 we have $U_g O_g^{(1+\varepsilon)} g_{\Gamma_N^c} \in \Lambda_\omega$, and hence, $U_g O_g^{(\varepsilon+2)} g_{\Gamma_N^c} \in \Lambda_\omega$. It follows, by using Lemma (3.4), that $U_g O_g^{(\varepsilon+2)} g_{\Gamma_N^c} \in \mathfrak{T}$, for every $\varepsilon > 0$ and for every $N \in \mathbb{N}$. Hence, from Proposition 3.7.(ii), $U_g O_g^{(\varepsilon+2)} \in \mathfrak{T}$ for every $\varepsilon > 0$. Again we use Lemma (3.4) and Proposition (3.7) to deduce that $U_g O_g^{(\varepsilon+2)} \in \mathfrak{T}$ for every $\varepsilon > 0$. This infers that $g \in \mathfrak{T}$.

Appendix A. An equivalent norm in Λ_ω

In this section we give a simple proof of the following Tamrazov's Theorem [8].

Theorem A.1. (See [8].) Let ω be any arbitrary modulus of continuity and let f be a function in $\Lambda_\omega(\mathbb{T}) \cap \mathcal{A}(\mathbb{D})$. Then f belongs to Λ_ω. We need the following key lemma.

Lemma A.2. Let ω be any arbitrary modulus of continuity and let $f \in \Lambda_\omega(\mathbb{T}) \cap \mathcal{A}(\mathbb{D})$ be a function such that $\|f\|_{\Lambda_\omega(\mathbb{T})} \le 1$. For every $\delta \ge 0$, we have

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)\right\}$$
$$\le o(\omega(1-|z|)) + A\delta \qquad (\text{as } |z| \to 1),$$

where A is an absolute constant.

Proof. Let $0 < \varepsilon < 1$ and $c_\varepsilon > 0$ such that for any $\zeta + \varepsilon, \zeta \in T$ satisfying $\varepsilon \le c_\varepsilon$, we have $|f(\zeta + \varepsilon) - f(\zeta)| \le \varepsilon\omega(\varepsilon)$. Divide T into the following three parts

$$\Gamma_1 := \{\zeta + \varepsilon \in \mathbb{T} : |(\zeta + \varepsilon) - z/|z|| \le 1 - |z| \le c_\varepsilon\},$$
$$\Gamma_2 := \{\zeta + \varepsilon \in \mathbb{T} : 1 - |z| \le |(\zeta + \varepsilon) - z/|z|| \le c_\varepsilon\},$$
$$\Gamma_3 := \{\zeta + \varepsilon \in \mathbb{T} : c_\varepsilon \le |(\zeta + \varepsilon) - z/|z||\}.$$

We have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\left|f\left(\sum_{j=1}^n e^{i\theta_j}\right) - f(z/|z|)\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\le \frac{1}{2\pi}\int_{\Gamma_1} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\varepsilon\omega\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\quad + \frac{1}{2\pi}\int_{\Gamma_2} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\varepsilon\omega\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\quad + \frac{1}{2\pi}\int_{\Gamma_3} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\varepsilon\omega\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\quad := I_1 + I_2 + I_3 . \qquad (A.1)
\end{aligned}$$

It is clear that

$$I_1 \le \left(\frac{1}{2\pi}\int_{\Gamma_1} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\left(\sum_{j=1}^n \theta_j\right)\right) \log(\varepsilon\omega(1-|z|) + \delta). \qquad (A.2)$$

Next we have

$$\begin{aligned}
I_2 &= \frac{1}{2\pi}\int_{\Gamma_2} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\varepsilon\omega\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\le \frac{1}{2\pi}\int_{\Gamma_2} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\varepsilon\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| \frac{\omega(1-|z|)}{1-|z|} + \delta\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\le \frac{1}{2\pi}\int_{\Gamma_2} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\frac{\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right|}{1-|z|}(\varepsilon\omega(1-|z|) + \delta)\right) d\left(\sum_{j=1}^n \theta_j\right) \\
&\le \left(\frac{1}{2\pi}\int_{\Gamma_1} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} d\left(\sum_{j=1}^n \theta_j\right)\right) \log(\varepsilon\omega(1-|z|) + \delta) \\
&\quad + c\int_{\varepsilon \ge 0} \frac{\log(2-\varepsilon)}{(2-\varepsilon)^2} d(2-\varepsilon), \qquad (A.3)
\end{aligned}$$

where c is an absolute constant. Let c'_ε be a positive number such that $c'_\varepsilon \le c_\varepsilon$ and for every $z \in \mathbb{D}$ satisfying $1 - |z| \le c'_\varepsilon$, we have $\frac{1}{2\pi}\int_{\Gamma_3} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j - z}\right|^2} d\left(\sum_{j=1}^n \theta_j\right) \le \varepsilon$. Hence, as in (73), we obtain

$$I_3 = \frac{1}{2\pi}\int_{\Gamma_3} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2} \log\left(\omega\left|\sum_{j=1}^n e^{i\theta_j} - z/|z|\right| + \delta\right) d\left(\sum_{j=1}^n \theta_j\right)$$

$$\leq \left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\omega(1-|z|)+\delta) \quad + c\int_{\varepsilon\geq 0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$= \left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta) \quad + c\int_{\varepsilon\geq 0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$+\left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)log\left(\frac{\omega(1-|z|)+\delta}{\varepsilon\omega(1-|z|)+\delta}\right)$$

$$\leq \left(\frac{1}{2\pi}\int_{\Gamma_3}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}d(\sum_{j=1}^{n}\theta_j)\right)\log(\varepsilon\omega(1-|z|)+\delta) + c\int_{\varepsilon\geq 0}\frac{\log(2-\varepsilon)}{(2-\varepsilon)^2}d(2-\varepsilon)$$

$$-\varepsilon\log(\varepsilon), \qquad (A.4)$$

for every $z \in \mathbb{D}$ satisfying $1-|z| \leq c'_\varepsilon$. From (A.2), (A.3) and (A.4) we obtain

$$\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|+\delta\right)d(\sum_{j=1}^{n}\theta_j)\right\}$$
$$\leq A(\varepsilon\omega(1-|z|)+\delta) \quad (z\in\mathbb{D} \text{ and } 1-|z|\leq c'_\varepsilon) \qquad (A.5)$$

where A is an absolute constant. This completes the proof of the lemma.

We use Lemma A.2 to prove the following one

Lemma A.3. Let ω be any arbitrary modulus of continuity and let $f \in \Lambda_\omega(\mathbb{T})$. Then

$$|f(z)-f(\zeta)| = o\big(\omega(|z-\zeta|)\big) \qquad (\text{as } |z-\zeta|\to 0, z\in\mathbb{D} \text{ and } \zeta\in\mathbb{T}). \qquad (A.6)$$

Proof. We can suppose that $\|f\|_{\Lambda_\omega(\mathbb{T})} \leq 1$. Let $0<\varepsilon<1$ and $c_\varepsilon>0$ be a number such that for any $\zeta+\varepsilon, \zeta\in\mathbb{T}$ satisfying $\varepsilon\leq c_\varepsilon$, we have $|f(\zeta+\varepsilon)-f(\zeta)|\leq\varepsilon\omega(\varepsilon)$. Fix $\zeta\in\mathbb{T}$ and fix $z\in\mathbb{D}$ such that $|z-\zeta|\leq c_\varepsilon/2$ and $|z|\geq 1/4$. We have $|z-\zeta|^2=(1-|z|)^2+|z||z/|z|-\zeta|^2\geq\frac{1}{4}|z/|z|-\zeta|^2$. Hence $|z/|z|-\zeta|\leq 2|z-\zeta|\leq c_\varepsilon$. We obtain

$$|f(z)-f(\zeta)|\leq|f(z)-f(z/|z|)|+|f(z/|z|)-f(\zeta)|\leq|f(z)-f(z/|z|)|+\varepsilon\omega(|z/|z|-\zeta|)$$
$$\leq\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^{n}e^{i\theta_j}-z\right|^2}\log\left(\left|f\left(\sum_{j=1}^{n}e^{i\theta_j}\right)-f(z/|z|)\right|\right)d(\sum_{j=1}^{n}\theta_j)\right\}+2\varepsilon\omega(|z-\zeta|).$$

Now, we use Lemma A.2 to deduce the result of the lemma.

Proof of Theorem A.1. Let $0<\varepsilon<1$. From Lemma A.3 there is $0<c_\varepsilon<1/2$ such that for every $z\in\mathbb{D}$ and for every $\zeta\in\mathbb{T}$ satisfying $|z-\zeta|\leq c_\varepsilon$ we have
$|f(z)-f(\zeta)|\leq\varepsilon\omega(|z-\zeta|)$. Let $z\in\mathbb{D}$ satisfying $|\varepsilon|\leq c_\varepsilon/2$ and
$\inf\{|z|,|z-\varepsilon|\}=|z-\varepsilon|\geq 1-c_\varepsilon$. It follows that
$|\frac{z}{|z|}-(z-\varepsilon)/|z-\varepsilon||\leq|z(z-\varepsilon)|^{-1/2}|\varepsilon|\leq 2|\varepsilon|\leq c_\varepsilon$.

(i). First we assume that $|\varepsilon|\geq 1-|z-\varepsilon|$. We obtain

$$|f(z)-f(z-\varepsilon)|$$
$$\leq|f(z)-f(z/|z|)|+|f(z/|z|)-f(z-\varepsilon/|z-\varepsilon|)| \quad +|f(z-\varepsilon)-f(z-\varepsilon/|z-\varepsilon|)|$$
$$\leq 4\varepsilon\omega(|\varepsilon|). \qquad (A.7)$$

(ii). Now we suppose that $|\varepsilon|\leq 1-|z-\varepsilon|$. We apply the maximum principle theorem in $\mathbb{D}$ to the analytic function
$z\mapsto\frac{f(z)-f(z-\varepsilon)}{\varepsilon}$ we get

$$\left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| \le \sup_{(\zeta+\varepsilon)\in\mathbb{T}}\left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right| = \frac{\left|f\left((\zeta+\varepsilon)_{(z-\varepsilon)}\right)-f(z-\varepsilon)\right|}{\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|}$$
$$\left((\zeta+\varepsilon)_{(z-\varepsilon)}\in\mathbb{T}\right).$$

2.1. If $\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right| \le c_\varepsilon$, then

$$\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} = \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right|\frac{|\varepsilon|}{\omega(|\varepsilon|)} \le 2\frac{\|f\|_\infty}{c_\varepsilon} = o(1) \quad (\text{as } |\varepsilon|\to 0). \qquad (A.8)$$

2.2. If $|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)| \le c_\varepsilon$, then
$|f((\zeta+\varepsilon)_{(z-\varepsilon)})-f(z-\varepsilon)| \le \varepsilon\omega(|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)|)$. It follows that

$$\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} = \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right|\frac{|\varepsilon|}{\omega(|\varepsilon|)} \le \frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)}$$
$$= \left|\frac{f\left((\zeta+\varepsilon)_{(z-\varepsilon)}\right)-f(z-\varepsilon)}{(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)}\right|\frac{1-|z-\varepsilon|}{\omega(1-|z-\varepsilon|)} \quad \left((\zeta+\varepsilon)_{(z-\varepsilon)}\in\mathbb{T}\right)$$
$$\le \varepsilon\frac{\omega\left(\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|\right)}{\left|(\zeta+\varepsilon)_{(z-\varepsilon)}-(z-\varepsilon)\right|}\frac{1-|(z-\varepsilon)|}{\omega(1-|(z-\varepsilon)|)} \le \varepsilon. \qquad (A.9)$$

From inequalities (A.7), (A.8) and (A.9), there is $c'_\varepsilon > 0$ such that if $|\varepsilon| \le c'_\varepsilon$ and $\inf\{|z|, |(z-\varepsilon)|\} \ge 1-c'_\varepsilon$, then

$$\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} \le \varepsilon. \qquad (A.10)$$

3. If $z\in\mathbb{D}$ are such that $\sup\{|z|, |z-\varepsilon|\} \le 1-c'_\varepsilon$ we have

$$\frac{|f(z)-f(z-\varepsilon)|}{\omega(|\varepsilon|)} = \left|\frac{f(z)-f(z-\varepsilon)}{\varepsilon}\right|\frac{|\varepsilon|}{\omega(|\varepsilon|)} \le \sup_{|\zeta|\le 1-c'_\varepsilon}|f'(\zeta)|\frac{|\varepsilon|}{\omega(|\varepsilon|)}$$
$$= o(1) \quad (\text{as } |\varepsilon|\to 0). \qquad (A.11)$$

From (A.10) and (A.11) we deduce the result. The proof of the theorem is completed.

Appendix B. Factorization property in Λ_ω

The F-property of Λ_ω, for any arbitrary modulus of continuity ω, is given by Shirokov [7]. For completeness we give here the proof.

Theorem B.1. *(*See [7].) Let ω be any arbitrary modulus of continuity. Let f be a function in Λ_ω and let U be inner function such that $f/U \in \mathcal{H}^\infty(\mathbb{D})$. Then $f/U \in \Lambda_\omega$ and

$$\frac{|f/U(z)-f/U(z-\varepsilon)|}{\omega(|\varepsilon|)} = o(1) \quad (\text{as } |\varepsilon|\to 0),$$

uniformly with respect to U. Also $\|f/U\|_\omega \le c\|f\|_\omega$, where c is an absolute constant.

Corollary B.2. Let ω be any arbitrary modulus of continuity. Let f and g be functions in Λ_ω and let $\{U_n\}_{n\in\mathbb{N}}$ be sequence of inner functions such that $f/U_n \in \mathcal{H}^\infty(\mathbb{D})$ for every $n\in\mathbb{N}$. If $\lim_{n\to+\infty}\|f/U_n - g\|_\infty = 0$, then $\lim_{n\to+\infty}\|f/U_n - g\|_\omega = 0$.

Proof. The proof immediately follows from Theorem B.1 and Lemma (3.1).

We begin the proof of Theorem B.1 by establishing several lemmas.

Lemma B.3. Let ω be any arbitrary modulus of continuity. Let f be a function in Λ_ω. Then

$$|O_f(z)| \le o\left(\omega(1-|z|)\right) + A|f(z/|z|)| \quad (\text{as } |z|\to 1),$$

where $A > 0$ is an absolute constant.

Proof. For $z\in\mathbb{D}$, we have

$$\log|O_f(z)| = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j}-z\right|^2}\log\left|f\left(\sum_{j=1}^n e^{i\theta_j}\right)\right| d\left(\sum_{j=1}^n \theta_j\right)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{\left|\sum_{j=1}^n e^{i\theta_j}-z\right|^2} \log\left(\left|f\left(\sum_{j=1}^n e^{i\theta_j}\right)-f(z/|z|)\right|+|f(z/|z|)|\right) d(\sum_{j=1}^n \theta_j).$$

Now, we use Lemma A.2 to complete the proof of the lemma.

For a function $f \in \mathcal{H}^\infty(\mathbb{D})$ we set

$$a_f(\zeta+\varepsilon) := \sum_{n\geq 0} \frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2} + \frac{1}{\pi}\int_{\mathbb{T}} \frac{1}{\left|\sum_{j=1}^n e^{i\theta_j}-(\zeta+\varepsilon)\right|^2} d\mu_f\left((\sum_{j=1}^n \theta_j)\right),$$

where $\{a_n : n \in \mathbb{N}\} = Z_f \cap \mathbb{D}$ (for all n, a_n is repeated according to it is multiplicity) and μ_f is the positive singular measure associated to the singular factor S_f of f.

Lemma B.4. Let f be a function in the disk algebra with inner factor $U_f \not\equiv 1$.
Let $\zeta+\varepsilon \in \mathbb{T}\backslash E_f$ and let $\varepsilon \leq 0$ be such that $-\varepsilon \leq d(\zeta+\varepsilon, Z_f)$. Then

$$\left|U_f((1+\varepsilon)(\zeta+\varepsilon))\right| \leq \exp\left\{\frac{\varepsilon}{8} a_f(\zeta+\varepsilon)\right\}.$$

Proof. We have

$$\begin{aligned}
\log\left|S_f\big((1+\varepsilon)(\zeta+\varepsilon)\big)\right| &= -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|(1+\varepsilon)(\zeta+\varepsilon)|^2}{\left|\sum_{j=1}^n e^{i\theta_j}-(\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^n \theta_j) \\
&\leq -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{4}\frac{-\varepsilon}{\left|\sum_{j=1}^n e^{i\theta_j}-(\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^n \theta_j) \\
&= -\frac{\varepsilon}{8\pi}\int_{\mathbb{T}} \frac{1}{4}\frac{1}{\left|\sum_{j=1}^n e^{i\theta_j}-(\zeta+\varepsilon)\right|^2} d\mu_f(\sum_{j=1}^n \theta_j). \qquad \text{(B.1)}
\end{aligned}$$

It is clear that

$$|\varepsilon|^2 = \big||z|-|z-\varepsilon|\big|^2 + |z||z-\varepsilon||z/|z|-(z-\varepsilon)/|z-\varepsilon||^2 \quad (z \in \mathbb{D}).$$

Now we estimate $|B_f((1+\varepsilon)(\zeta+\varepsilon))|$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\frac{(1+\varepsilon)(\zeta+\varepsilon)-a_n}{(\zeta+\varepsilon)-(1+\varepsilon)a_n}\right|^2 &= \frac{\big((1+\varepsilon)-|a_n|\big)^2+(1+\varepsilon)|a_n||(\zeta+\varepsilon)-a_n/|a_n||^2}{(\varepsilon|a_n|)^2+(1+\varepsilon)|a_n||(\zeta+\varepsilon)-a_n/|a_n||^2} \\
&= 1-(2\varepsilon+\varepsilon^2)\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2} \leq 1+\frac{\varepsilon}{4}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}.
\end{aligned}$$

Hence

$$log\left|\frac{(1+\varepsilon)(\zeta+\varepsilon)-a_n}{(\zeta+\varepsilon)-(1+\varepsilon)a_n}\right| \leq \frac{\varepsilon}{8}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}$$

Therefore

$$\left|B_f\big((1+\varepsilon)(\zeta+\varepsilon)\big)\right| \leq \exp\left\{\frac{\varepsilon}{8}\sum_{n\geq 0}\frac{1-|a_n|^2}{|(\zeta+\varepsilon)-a_n|^2}\right\}. \qquad \text{(B.2)}$$

From (B.1) and (B.2) we obtain

$$\left|U_f((1+\varepsilon)(\zeta+\varepsilon))\right| \leq \exp\left\{\frac{\varepsilon}{8} a_f((\zeta+\varepsilon))\right\}.$$

This proves the lemma.

Lemma B.5. Let f be a function in Λ_ω and let U be an inner function such that $f/U \in \mathcal{H}^\infty(\mathbb{D})$. Then

$$|f(\zeta)| = o\left(\omega\left(\frac{1}{a_U(\zeta)}\right)\right) \quad \big(as\ d(\zeta, Z_f) \to 0, \zeta \in \mathbb{T}\backslash E_f\big).$$

Proof. Let $\varepsilon > 0$. There is $c_\varepsilon > 0$ such that if $|\varepsilon| \le c_\varepsilon$, $z \in \mathbb{T}$, then we have $|f(z) - f(z-\varepsilon)| \le \varepsilon\omega(|\varepsilon|)$. From Lemma B.3, there is $0 < c'_\varepsilon < c_\varepsilon$ such that if $\varepsilon \le -c_\varepsilon$, then $\left|O_f\big((1+\varepsilon)\zeta\big)\right| \le (1+\varepsilon)(\varepsilon\omega(-\varepsilon) + |f(\zeta)|)$, where $\varepsilon > 0$ is an absolute constant. Let $\zeta \in \mathbb{T}\backslash E_f$ such that $d(\zeta, Z_f) \le c'_\varepsilon$.

1. We assume that $a_U(\zeta) \le \frac{8(1+\varepsilon)}{d(\zeta,Z_f)}$. Then we obtain

$$|f(\zeta)| \le \varepsilon\omega(d(\zeta,Z_f)) \le \varepsilon\omega(\tfrac{8(1+\varepsilon)}{a_U(\zeta)}) \le 8(1+\varepsilon)\varepsilon\omega(\tfrac{1}{a_U(\zeta)}).$$

2. Now assume that $a_U(\zeta) \ge \frac{8(1+\varepsilon)}{d(\zeta,Z_f)}$. then $1-(1+\varepsilon)_\zeta \le d\big(\zeta, Z_f\big)$,

where $(1+\varepsilon)_\zeta := 1 - \frac{(1+\varepsilon)^2}{a_U(\zeta)}$. We have $U_f/U \in \mathcal{H}^\infty(\mathbb{D})$, then $a_U(\zeta) \le a_f(\zeta)$ and by using Lemma B.4 we obtain

$$\left|U_f((1+\varepsilon)_\zeta\zeta)\right| \le \exp\left\{-\frac{1-(1+\varepsilon)_\zeta}{8}a_f(\zeta)\right\} \le \exp\left\{\frac{\varepsilon}{8}a_U(\zeta)\right\} = \exp\{-(1+\varepsilon)\}.$$

Since $d(\zeta, Z_f) \le c'_\varepsilon$, then $1-(1+\varepsilon)_\zeta \le c'_\varepsilon$ and we have

$$\begin{aligned}\left|f((1+\varepsilon)_\zeta\zeta)\right| &= \left|U_f((1+\varepsilon)_\zeta\zeta)\right|\left|O_f((1+\varepsilon)_\zeta\zeta)\right| \\ &\le (1+\varepsilon)\exp\{-(1+\varepsilon)\}\big(\varepsilon\omega\big(1-(1+\varepsilon)_\zeta\big) + |f(\zeta)|\big).\end{aligned}$$

Hence

$$\begin{aligned}|f(\zeta)| &\le \left|f(\zeta) - f((1+\varepsilon)_\zeta\zeta)\right| + \left|f((1+\varepsilon)_\zeta\zeta)\right| \\ &\le \varepsilon\omega\big(1-(1+\varepsilon)_\zeta\big) + (1+\varepsilon)\exp\{-(1+\varepsilon)\}\big(\varepsilon\omega\big(1-(1+\varepsilon)_\zeta\big) + |f(\zeta)|\big).\end{aligned}$$

It follows that $|f(\zeta)| \le 3\varepsilon\omega(1-(1+\varepsilon)_\zeta) \le 24(1+\varepsilon)\varepsilon\omega(\frac{1}{a_U(\zeta)})$. This completes the proof of the lemma.

Proof of Theorem B.1. Now, we can deduce the proof of Theorem B.1 by using Lemma B.5. Indeed, from Theorem A.1 it is sufficient to prove that $f/U \in \Lambda_\omega(\mathbb{T})$, that is

$$\frac{|f(\zeta+\varepsilon)/U(\zeta+\varepsilon) - f(\zeta)/U(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (\text{as } \varepsilon \to 0 \text{ and } \zeta+\varepsilon, \zeta \in \mathbb{T}).$$

Let $\zeta+\varepsilon, \zeta \in \mathbb{T}$ be two distinct points such that $d(\zeta+\varepsilon, Z_f) \ge d(\zeta, Z_f)$. We have

$$\frac{|f(\zeta+\varepsilon)/U(\zeta+\varepsilon) - f(\zeta)/U(\zeta)|}{\omega(\varepsilon)} \le \frac{|f(\zeta+\varepsilon) - f(\zeta)|}{\omega(\varepsilon)} + |f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)}.$$

Then it suffices to prove

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)} = o(1) \quad (as\ \varepsilon \to 0). \qquad (B.3)$$

1. First we suppose that $\varepsilon \ge \frac{1}{2}d(\zeta, Z_f)$. Then

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon) - U(\zeta)|}{\omega(\varepsilon)} \le 2\frac{|f(\zeta)|}{\omega(\frac{1}{2}d(\zeta,Z_f))} \le 4\frac{|f(\zeta)|}{\omega(d(\zeta,Z_f))}$$

$$= o(1) \quad (\text{as } \varepsilon \to 0). \qquad (B.4)$$

2. Next we suppose that $\varepsilon \le \frac{1}{2}d(\zeta, Z_f)$. Then $[\zeta+\varepsilon, \zeta] \subset T\backslash E_f$. There is $z \in [\zeta+\varepsilon, \zeta]$ such that $\frac{|U(\zeta+\varepsilon)-U(\zeta)|}{\omega(\varepsilon)} = |U'(z)| \le \sum_{n\ge0}\frac{1-|a_n|^2}{|z-a_n|^2} + \frac{1}{\pi}\int_{\mathbb{T}}\frac{1}{\left|\sum_{j=1}^n e^{i\theta_j} - z\right|^2}d\mu(\sum_{j=1}^n \theta_j) := a_U(z) \le 4a_U(\zeta) \le \frac{c_f}{d^2(\zeta,Z_f)}$.

2.1. If $a_U(\zeta) \le \frac{1}{\varepsilon}$. Then, by using Lemma B.5, we deduce that

$$|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)} = o(1) \quad \big(\text{as } d(\zeta, Z_f) \to 0\big).$$

therefore

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon)-U(\zeta)|}{\omega(\varepsilon)} \le 4|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)} \le \inf\left\{4|f(\zeta)|a_U(\zeta)\frac{\varepsilon}{\omega(\varepsilon)}\,,\frac{c_f\|f\|_\infty}{d^2(\zeta,Z_f)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0)\,. \tag{B.5}$$

2.2. Now assume that $a_U(\zeta) \ge \frac{1}{\varepsilon}$. Then, by Lemma B.5, we obtain $|f(\zeta)| = o(\omega(\varepsilon))$, as $d(\zeta, Z_f) \to 0$. Therefore

$$|f(\zeta)|\frac{|U(\zeta+\varepsilon)-U(\zeta)|}{\omega(\varepsilon)} \le \inf\left\{2\frac{|f(\zeta)|}{\omega(\varepsilon)}\,,\frac{c_f\|f\|_\infty}{d^2(\zeta,Z_f)}\frac{\varepsilon}{\omega(\varepsilon)}\right\}$$
$$= o(1) \quad (\text{as } \varepsilon \to 0)\,. \tag{B.6}$$

Consequently (B.3) follows from inequalities (B.4), (B.5) and (B.6). The proof of the theorem is completed.

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