**The Impact of Fourier Series Expansion on the Analysisof Asset Value Function and its Return Rates for Capital Markets.**


## Abstract

The effect of Fourier series expansion on the solution of Stochastic Differential Equation (SDE) is considered herein. The detailed measures which govern price function of return rate for capital investments are obtained periodically. Sufficient conditions of stating mathematical propositions and proving it by means of Fourier series expansion are given. These price functions were used as drift (return rate) parameter in the solution of proposed model which follow various pattern according to their propositions. This way, the desired complete solutions were obtained. Finally, the effects of the relevant parameters were demonstrated graphically for the purpose of decision making.



**Key words: SDE, Asset value, Return rates, Fourier series expansions, Stochastic Analysis, Price functions.**




## 1.Introduction

An asset in the financial market is one of the instruments that accumulates return rate to an investor or commercial traders. Therefore, assessment of asset is a procedure of defining the market value of asset prices on an investment. Asset price has turn out to be so influential for impelling economic variations, distributing economic resources to subdivisions with time and controlling the assets of the whole system that produces different returns. Return on investment for capital market is an estimate which correctly examines the profit of an investment for effective running and good management of the business.

However, Fourier series is defined as the study of processes where general periodic functions are represented in a form of sum of trigonometric functions. It utilizes the fact that any finite and time-ordered series can be adequately approximated. In Fourier series analysis comparisons are made based on amplitudes and it also possible to find significant cycles with dominating amplitudes and their periods. It has so many scientific applications; for instance, mathematics in Partial differential equations, Engineering and Physics for signal processing, imaging, and acoustic optics and also in finance for option pricing and some related stock variables, details of this can be found as follows:[1-3] etc.

On the contrary, while considering this kind of problems, analytical method which offers exact solution for suitable mathematical expectation is therefore required to enhance efficiency in terms of decision making for future developmental plans. Problems which are related to asset price and its rate of return, appropriate formulation and accurate analytical solutions are necessary requirements for the analysis of return rates in time varying investment; however, following the basic feature of the problem under-study, the analytical solution is exploited.

Though, stock market prices have been considered in different ways and results obtained in divers ways by scholars. For instance,[4] studied the stochastic analysis of stock market price model, and considered the stochastic analysis of the behavior of stock prices using a proposed log-normal distribution model. Their results showed that the proposed model is efficient for the production of stock prices.[5] studied the stochastic analysis of stock market expected returns for investors. In their results the variances of four different stocks indicated that stock1 is the best among the stocks of different companies, which is consistent with the work of [6]. Also, [7] investigated a stochastic analysis of stock market expected returns and growth-rates .

In trying to study stochastic model [8] considered the stability analysis of stochastic model for stock market prices and did analysis of the unstable nature of stock market forces applying a new differential equation model that can impact the expected returns of investors in stock exchange market with a stochastic volatility in the equation. While [9] suggested in their study, some analytical solutions of stochastic differential equations with respect to Martingale processes and discovered that the solutions of some SDEs are related to other stochastic equations with diffusion part. The second technique is to change SDE to ODE that are tried to omit diffusion part of stochastic equation by using Martingale processes.

On the other hand, [10] looked at the numerical techniques of solving stochastic differential equations like the Euler-Maruyama and Milstein methods based on the truncated Ito-Taylor expansion by solving a non-linear stochastic differential equation and approximated numerical solution using Monte Carlo simulation for each scheme. Their results showed that if the discretization value N is increasing, the Euler-Maruyama and Milstein techniques were closed to exact solution. [11] worked on the stability of both analytical and numerical solutions for non-linear stochastic delay differential equations with jumps and they observed that the compensated stochastic methods inherit stability property of the correct solution. [12] studied the solution of differential equations and stochastic differential equations of time varying investment returns and obtained precise conditions governing asset price returns rate via multiplicative and multiplicative inverse trend series. The proposed model showed an efficient and reliable multiplicative inverse trend series than the multiplicative trend in both deterministic and stochastic system. [13] studied stochastic model of the fluctuations of stock market price and obtained precise conditions for determining the equilibrium price. The model constrains the drift parameters of price process in a manner that is adequately characterized by the volatility. [14] examined the stability behaviours of stochastic differential equations(SDEs) driven by time changed Brownian motions. And a connection between the stability of the solution to the time - changed stochastic differential equations and their corresponding non-time-changed stochastic differential equations were shown using the duality theorem. [15] studied stochastic methods in practical delineation of financial models and suggested the Euler-Maruyama method as the stochastic differential equation expressions as potentially useful for delineation of asset stock price and volatility.

More so [16]studied stock price forecasting and applied the method of Fourier series analysis. The result showed the level trading does not provide a practical value in comparison to the momentum trading method.

This paper, scrutinizesthe impact of Fourier series expansion to the solutionof stochastic differential equation for capital market investments whose rate of returns is assumed to follow: linear function of price, quadratic function of price and cubic function of price over time. The SDE were solved completely by adopting Ito's theorem a close form analytical solutions were obtained.Consequently, these price functions were made in three cases where mathematical propositionsof Fourier series was obtained then proved independently.

The aim of this paper is adopting an analytical solution of stochastic differential equations and Fourier series expansions on return rates at different levelsand finding cycles with dominating amplitudes and their periods. This paper extends the work of [16] ] by combining solutions of SDEand Fourier series price functions for capital investments which was not considered by the previous efforts.

This paper is prescribed as follows: Section 2 presents the mathematical preliminaries, Results and discussion are seen in Section 3 and paper is concluded in Section 4.

**2. Mathematical Preliminaries**

Here we present few basic definitions as touching the dynamics of mathematical finance that will support the formulation of the problem , hence we have as follows:

**Definition 1. Stochastic process**: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each $t$ in the index set $T$, $X(t)$ is a random variable. Now we understand $t$ as time and call $X(t)$ the state of the procedure at time $t$. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

**Definition 2. Random Walk**: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point to the other. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k-1}^{n} X_k \tag{1}$$

where $X_k$ are independent and identically distributed random variables

**Definition 3**: A Stochastic Differential Equation(SDE) is integration of differential equation with stochastic terms. So considering the Geometric Brownian Motion (GBM) which govern price dynamics of a non-dividend paying stock as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t), \tag{2}$$

Where S denotes the asset value, $\mu$ is the stock rate of return (drift) which is also known as the average rate of the growth of asset price and $\sigma$ denotes the volatility otherwise called standard deviation of the returns. The $dz(t)$ is a Brownian motion or Wiener process which is defined on probability space $(\Omega, \mathrm{F}, \wp)$, [18]. However, stock price follows the Ito's process and the drift rate is stated as follows:

$$\mu = \left( \frac{\partial f}{\partial S_t} a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S_T^2} b_t^2 \right), \tag{3}$$

$$\sigma^2 = \frac{\partial^2 f}{\partial S_t^2} b_t^2 \tag{4}$$

**Definition 4:** A standard Brownian motion is simply a stochastic process $\{B_t\}_{t\in\tau}$ with the following properties:

i) With probability 1, $B_0 = 0$.

ii) For all $0 \le t_1 \le t_2 \le \cdots \le t_n$, the increments $B_{t2-}B_{t1}, B_{t3-}B_{t2}, B_{t2}, \cdots, B_{tn} - B_{tn-1}$ are independent.

iii) For $t \ge s \ge 0$, $B_t - B_s \sim N(0, t-s)$ .

With probability 1, the function $\rightarrow B_i$ is continuous.

**Definition 5**: Ito's process is a stochastic process $\{X_t, t \ge 0\}$ known as Ito's process which follows:

$$X_t = X_0 + \int_0^t (\mathrm{t}, \varpi) d\tau + \int_0^t b(t, \varpi) dz_t \,. \tag{5}$$

Where $a(t,\varpi)$ and $b(t,\varpi)$ are adapted random function, George and Kenneth (2019).[]

**Definition 6:(Ito's lemma). Let** $f(S,t)$ be a twice continuous differential function on $[0,\infty) \times \mathrm{A}$ and let $S_t$ denotes an Ito's process

$$dS_t = a_t dt + b_t dz(t), t \ge 0,$$

Applying Taylor series expansion of $F$ gives:

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (dS_t)^2 + \text{higer order terms } (h.\mathrm{o,t}) \ ,$$

So, ignoring h.o.t and substituting for $dS_t$ we obtain

$$dF_t = \frac{\partial F}{\partial S_t} \left( a_t dt + bdz(t) \right) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \left( a_t dt + bdz(t) \right)^2 \tag{6}$$

$$= \frac{\partial F}{\partial S_t} \left( a_t dt + bdz(t) \right) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b_t^2 dt, \tag{7}$$

$$= \left( \frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b_t^2 \right) dt + \frac{\partial F}{\partial S_t} b_t dz(t) \tag{8}$$

More so, given the variable $S(\mathrm{t})$ denotes stock price, then following GBM implies (2) and hence, the function $F\left(S,t\right)$ ,Ito's lemma gives:

$$dF = \left( \mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dz(t) \tag{9}$$

**Illustration 1:** Determinea stock price $S$ which follows a random process as

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t)$$

Let $F\left(S,t\right) = \ln S$ partial derivatives are:

$$\frac{\partial F}{\partial S} = \frac{1}{S}, \frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2} \ \ and \frac{\partial F}{\partial t} = 0.$$

Putting the above values into (9) yields the following

$$d\left(\ln S\right) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz(t),$$

Integrating both sides and talking upper and lower bounds as $0$ to $t$ gives

$$\int_0^t d\left(\ln S\right) = \int_0^t \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \int_0^t dz(t),$$

$$\ln S(t) - l\,\mathrm{n}\,S_0 = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \left( dz(\mathrm{t}) - dz(0) \right),$$

$$S(t) = S_0 \exp\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma dz(t). \tag{10}$$

where $dz$ is a standard Brownian Motion.

**2.1. Fourier Series Analysis on Stock Rate of Return for Capital Market Investments**

Let the function of an asset price be represented as $f\left(S\right)$ which is defined on the bounded interval $S \in [-1,1]$ and outside of this interval we have $f\left(S+2T\right) = f\left(S\right).$ That is to say $f\left(S\right)$ has a periodic influences on asset price which is $2T.$ Therefore, the Fourier series expansion of $f\left(S\right)$ is given as follows.

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left[ a_n Cos\left( \frac{n\pi S}{T} + b_n Sin\left( \frac{n\pi S}{T} \right) \right) \right] \tag{11}$$

where $a_n$ and $b_n$ are the Fourier coefficients given as

$$a_n = \frac{1}{T}\int_{-T}^{7} f(S) Cos \frac{n\pi S}{T} dS \quad n = 1, 2... \tag{12}$$

$$a_n = \frac{1}{T}\int_{C}^{C+27} f(S) Cos \frac{n\pi S}{T} dS \quad n = 0, 1, 2... \tag{13}$$

$$b_n = \frac{1}{T}\int_{C}^{C+27} f(S) Sin \frac{n\pi S}{T} dS \quad n = 1, 2... \tag{14}$$

Where the lower limit $C$ is equal to the interval of according to the definition of $f(S)$ and $C + 2T$ remain as the upper limit of the interval; The details of this can be seen in the following books: [19-23] etc.

Using (12) gives $a_0$ ;that is

$$a_0 = \frac{1}{T}\int_{-7}^{T} f(S) dS \tag{15}$$

In a situation where the lower limit $C$ relates to $-T$ ,we have $C = -T$ ;such that $C + 2T = -T + 2T = T$. $C + 2T = -T + 2T = T$. So we have the following coefficients to determine the stock variables or quantities.

$$a_n = \frac{1}{T}\int_{-T}^{7} f(S) Cos \frac{n\pi S}{T} dS \quad n = 0, 1, 2.. \tag{16}$$

$$b_n = \frac{1}{T}\int_{-T}^{7} f(S) Sin \frac{n\pi S}{T} dS \quad n = 1, 2.. \tag{17}$$

### 2.2 Problem Formulations

We consider trading investments periods where new dividends will not have been declared and no new assets have been purchased then the stock return follows particular processes; [13].

**Case 1:** we assume a solution where return rate of asset price is a linear function of price its self. That is $f(S_1) = S_1$ and follows Fourier series with periodic influences such as $0 < S_1 < 2\pi$ .

**Proposition 1.**

The definition of Fourier series for the capital market investment equation

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left[ a_n Cos\left( \frac{n\pi S_1}{T} + b_n Sin\left( \frac{n\pi S_1}{T} \right) \right) \right] \tag{18}$$
$$where \ f(S_1) = S_1$$

***Proof.***

We derive rate of return as a linear function of price by determining the coefficients of (18) as follows

$$a_n = \frac{1}{T}\int_C^{C+2T} f\left(S_1\right)Cos\frac{n\pi S_1}{T}dS_1 \quad = \frac{1}{\pi}\int_C^{2T} f\left(S_1\right)Cos\frac{n\pi S_1}{T}dS_1 \quad = \frac{1}{\pi}\int_C^{2T} S_1 CosnS_1 dS_1$$

Using Nedu's method of Integration by parts

$$P_n\left(S_1\right) = S_1 \quad and \quad f\left(S_1\right) = CosnS_1,$$

$$= \frac{1}{\pi}\left[\left(S_1\right)\int CosnS_1 dS_1 - (1)\int CosnS_1 dS_1 \ \right]/_0^{2\pi} = \frac{1}{\pi}\left[S_1\left(\frac{1}{n}SinnS_1\right) - 1\left(-\frac{CosnS_1}{n^2}\right)\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(S_1\frac{SinnS_1}{n}\right) - 1\left(-\frac{CosnS_1}{n^2}\right)\right]/_0^{2\pi} = \frac{1}{\pi}\left[\frac{Cos2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$if \ \ n = 0, \ a_0 = \frac{1}{\pi}\int_C^{27} f\left(S_1\right)dS_1 = \frac{1}{\pi}\int_0^{2\pi} S_1 dS = \frac{1}{\pi}\left[\frac{S_1^{\ 2}}{2}\right]/_0^{2\pi} = 2\pi$$

Similarly

$$b_n = \frac{1}{T}\int_C^{C+2T} f\left(S_1\right)Sin\frac{n\pi S_1}{T}dS_1 \quad = \frac{1}{\pi}\int_C^{2T} S_1 SinnS_1 dS_1$$

Using Nedu's method of Integration by parts

$$P_n\left(S_1\right) = S_1 \quad and \quad f\left(S_1\right) = SinnS_1,$$

$$= \frac{1}{\pi}\left[\left(S_1\right)\int SinnS_1 dS_1 - (1)\int SinnS_1 dS_1 \ \right]/_0^{2\pi} = \frac{1}{\pi}\left[S_1\left(-\frac{CosnS_1}{n}\right) - 1\left(-\frac{SinnS_1}{n^2}\right)\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi Cos2n\pi}{n}\right] = -\frac{2}{n}$$

Hence $f\left(S_1\right) = \sum_{n=1}^{\infty} -\frac{2}{n}SinS_1$

$$= \pi - 2\left[SinS_1 + \frac{1}{2}Sin2S_1 + \frac{1}{3}Sin3S_1 + \frac{1}{4}Sin4S_1 + \frac{1}{5}Sin5S_1 + \frac{1}{6}Sin6S_1 + ...\right] \qquad (19)$$

This is the net gain or loss of an investment over a specified time period. Consequently we set $f\left(S_1\right) = \mu$ of (10) which offer a complete solution of SDE with the effect of Fourier series expansion

$$S_1(t) = S_0 \exp\left(\left(\pi - 2\begin{bmatrix} SinS_1 + \frac{1}{2}Sin2S_1 + \frac{1}{3}Sin3S_1 + \frac{1}{4}Sin4S_1 + \frac{1}{5}Sin5S_1 \\ + \frac{1}{6}Sin6S_1 + ... \end{bmatrix} - \frac{\sigma^2}{2}\right)t + \sigma dz(t).\right). \tag{20}$$

**Case 2:** we derive a solution where return rate of asset price is a quadratic function . That is $f\left(S_2\right) = S_2^{\,2}$ and follows Fourier series with periodic influences such as $0 < S_2 < 2\pi$ .

**Proposition 2.**

Here let the definition of Fourier series for the capital market investment equation be given as follows:

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left[ a_n Cos\left(\frac{n\pi S_2}{T} + b_n Sin\left(\frac{n\pi S_2}{T}\right)\right)\right]. \tag{21}$$
$$where \;\; f\left(S_2\right) = S_2^{\,2}$$

***Proof.***

We want to show rate of return as a quadratic function of price by determining the coefficients of (21) as follows

$$a_n = \frac{1}{T}\int_C^{C+2T} f\left(S_2\right) Cos\frac{n\pi S_2}{T}dS_2 \quad = \frac{1}{\pi}\int_C^{2T} f\left(S_2\right) Cos\frac{n\pi S_2}{T}dS_2 \quad = \frac{1}{\pi}\int_C^{2T} S_2^{\ 2} CosnS_2 dS_2$$

Using Nedu's method of Integration by parts

$$P_n\left(S_2\right) = S_2^{\ 2} \quad and \quad f\left(S_2\right) = CosnS_2,$$

$$= \frac{1}{\pi}\left[\left(S_2^{\ 2}\right)\int CosnS_2 dS_2 - (2S_2)\int CosnS_2 dS_2 + (2)\int CosnS_2 dS_2 \ \right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[ S_2^{\ 2}\left(\frac{1}{n}SinnS_2\right) - 2S_2\left(-\frac{CosnS_2}{n^2} + 2\left(\frac{SinnS_2}{n^3}\right)\right)\right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(S_2^{\ 2}\frac{SinnS_2}{n}\right) + 2S_2\left(\frac{CosnS_2}{n^2} - \frac{2SinnS_2}{n^3}\right)\right] /_0^{2\pi} = \frac{1}{\pi}\left[\frac{2S_2Cos2nS_2}{n^2}\right] /_0^{2\pi} = \frac{1}{\pi}\left[\frac{2S_2CosnS_2}{n^2}\right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi 2\pi n}{n^2}\right] = \frac{4\pi(-1)^{2n}}{n^2\pi} = \frac{4}{n^2}, n \neq 0$$

$$if \ \ n = 0, \ a_0 == \frac{1}{\pi}\int_0^{2\pi} S_2^{\ 2} dS_2 = \frac{1}{\pi}\left[\frac{S_2^{\ 3}}{3}\right] /_0^{2\pi} = \frac{1}{n}\left(\frac{8\pi^3}{3\pi}\right) = \frac{8\pi^2}{3}$$

Similarly

$$b_n = \frac{1}{T}\int_C^{C+2T} f\left(S_2\right) Sin\frac{n\pi S_2}{T}dS_2 \quad = \frac{1}{\pi}\int_C^{2T} S_2^{\ 2} SinnS_2 dS_2$$

Using Nedu's method of Integration by parts

$$P_n\left(S_2\right) = S_2^{\ 2} \quad and \quad f\left(S_2\right) = SinnS_2,$$

$$= \frac{1}{\pi}\left[\left(S_2^{\ 2}\right)\int SinnS_2 dS_2 - (2S_2)\int SinnS_2 dS_2 \ (2)\int SinnS_2 dS_2 \ \right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[ S_2^{\ 2}\left(-\frac{CosnS_2}{n}\right) - 2S_2\left(-\frac{SinnS_2}{n^2}\right) + 2\left(\frac{CosnS_2}{n^3}\right)\right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[ -S_2^{\ 2}\frac{CosnS_2}{n} + 2S_2\frac{SinnS_2}{n^2} + 2\frac{CosnS_2}{n^3}\right] /_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2 Cos2n\pi}{n} + \frac{2Cos2n\pi}{n^3}\right] = -\left\{\frac{4\pi}{n} + \frac{2}{n^3}\right\} Cos2\pi n = -\left\{\frac{4n^2\pi + 2}{n^3}\right\}$$

Hence $f\left(S_2\right) = \frac{8\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}CosnS_2 - \left(\frac{4n^2\pi + 2}{n^3}\right)SinnS_2\right] = \frac{8\pi^2}{3} + \sum_{n=1}^{\infty} \ \frac{4}{n^2}$

$$= \frac{8\pi^2}{3} + 4CosS_2 + Cos2S_2 + \frac{4}{9}Cos3S_2 + \frac{1}{2}Cos4S_2 + \frac{4}{25}Cos5S_2 + \frac{1}{9}Cos6S_2 + ... \qquad (22)$$

Since, we are looking at rate of return that follows Fourier series with quadratic function of price. We therefore set $f\left(S_2\right)=\mu$ of (10) which results to a complete solution of SDE with the influence of Fourier series as follows.

$$S_2(t)=S_0\exp\left(\left(\begin{array}{l}\frac{8\pi^2}{3}+4CosS_2+Cos2S_2+\frac{4}{9}Cos3S_2+\frac{1}{2}Cos4S_2+\frac{4}{25}Cos5S_2\\+\frac{1}{9}Cos6S_2+...-\frac{\sigma^2}{2}\end{array}\right)t+\sigma dz(t).\right). \tag{23}$$

**Case 3:** Also we consider a solution where return rate of asset price is a cubic function . That is $f\left(S_3\right)=S_3^{\ 3}$ and follows Fourier series with periodic influences such as $0<S_3<2\pi$ .

**Proposition 3.**

Given the Fourier series for the capital market investment equation whose rate of return is cubic.

$$\frac{a_0}{2}+\sum_{n=0}^{\infty}\left[a_nCos\left(\frac{n\pi S_3}{T}+b_nSin\left(\frac{n\pi S_3}{T}\right)\right)\right]. \tag{24}$$

$$where\ \ f\left(S_3\right)=S_3^{\ 3}$$

***Proof.***

We proof rate of return as a cubic function of price by defining the coefficients of (24) as following the steps below.

$$a_n = \frac{1}{T}\int_C^{C+2T} f\left(S_3\right) Cos\frac{n\pi S_3}{T}dS_3 \quad = \frac{1}{\pi}\int_C^{2T} f\left(S_3\right) Cos\frac{n\pi S_3}{T}dS_3 \quad = \frac{1}{\pi}\int_C^{2T} S_3{}^3 CosnS_3 dS_3$$

Using Nedu's method of Integration by parts

$$P_n\left(S_3\right) = S_3{}^3 \quad and \quad f\left(S_3\right) = CosnS_3,$$

$$= \frac{1}{\pi}\left[\left(S_3{}^3\right)\int CosnS_3 dS_3 - (3S_3{}^2)\int CosnS_3 dS_3 + (6S_3)\int CosnS_3 dS_3 - \left(6\right)\int CosnS_3 dS_3\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[S_3{}^3\left(\frac{1}{n}SinnS_3\right) - 3S_3{}^2\left(-\frac{CosnS_3}{n^2}\right) + 6S_3\left(\frac{SinnS_3}{n^3}\right) - 6\left(\frac{CosnS_3}{n^4}\right)\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{3S_3{}^2 CosnS_3}{n^2} - \frac{6CosnS_3}{n^4}\right]/_0^{2\pi} = \frac{1}{\pi}\left[\frac{12S_3 Cos2n\pi}{n^2} - \frac{6}{n^4}Cos2n\pi + \frac{6}{n^4}\right]$$

$$= \frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi + \frac{6}{n^4}\left(1 - Cos2n\pi\right)\right] = \frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi + \frac{6}{n^4}\left(1 - (-1)^{2n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi\right] = \frac{12\pi^2(-1)^{2n}}{n^2\pi} = \frac{12\pi^2}{n^2}, n \neq 0$$

$$if \;\; n = 0,\; a_0 == \frac{1}{\pi}\int_0^{2\pi} S^3 dS_3 = \frac{1}{\pi}\left[\frac{S^4}{4}\right]/_0^{2\pi} = \frac{16\pi^4}{4\pi} = 4\pi^3$$

Similarly

$$b_n = \frac{1}{T}\int_C^{C+2T} f\left(S_3\right) Sin\frac{n\pi S_3}{T}dS_3 \quad = \frac{1}{\pi}\int_C^{2T} S_3{}^3 SinnS_3 dS_3$$

Using Nedu's method of Integration by parts

$$P_n\left(S_3\right) = S_3{}^3 \quad and \quad f\left(S_3\right) = SinnS_3,$$

$$= \frac{1}{\pi}\left[\left(S_3{}^3\right)\int SinnS_3 dS_3 + (3S_3{}^2)\int SinnS_3 dS_3 - (6S_3)\int SinnS_3 dS_3 + (6)\int SinnS_3 dS_3\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[S_3{}^3\left(-\frac{CosnS_3}{n}\right) + 3S_3{}^2\left(-\frac{SinnS_3}{n^2}\right) - 6S_3\left(\frac{CosnS_3}{n^3}\right) + 6\left(\frac{SinnS_3}{n^4}\right)\right]/_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-8\pi^3 Cos2n\pi}{n} - \frac{12Cos2n\pi}{n^3}\right] = -\left\{\frac{8\pi^2}{n} + \frac{12}{n^3}\right\}Cos2\pi n = -\left\{\frac{8n^2\pi^2 + 12}{n^3}\right\}$$

Hence $f\left(S_3\right) = 2\pi^3 + \sum_{n=1}^{\infty}\left[\frac{12\pi}{n^2}CosnS_3 - \left(\frac{8n^2\pi^2 + 12}{n^3}\right)SinnS_3\right] = 2\pi^3 + \sum_{n=1}^{\infty}\frac{12\pi CosnS_3}{n^2}$

$$= 2\pi^3 + 12\pi^2 Cos S_3 + 3\pi^2 Cos 2S_3 + \frac{4}{3}\pi^2 Cos 3S_3 + \frac{3}{4}\pi^2 Cos 4S_3 + \frac{12}{25}\pi^2 Cos 5S_3 + \frac{1}{3}\pi^2 Cos 6S_3 + \ldots \quad (25)$$

Therefore, this is rate of return that follows Fourier series with cubic function of price.

Hence, we set $f\left(S_3\right) = \mu$ of (10) which yields a complete solution of SDE with the impact of Fourier series

$$S_3(t) = S_0 \exp\left( \left( \begin{array}{l} 2\pi^3 + 12\pi^2 Cos S_3 + 3\pi^2 Cos 2S_3 + \frac{4}{3}\pi^2 Cos 3S_3 + \frac{3}{4}\pi^2 Cos 4S_3 \\ + \frac{12}{25}\pi \ Cos 5S_3 + \frac{1}{3}\pi^2 Cos 6S_3 + \ldots - \frac{\sigma^2}{2} \end{array} \right) t + \sigma dz(t). \right). \quad (26)$$

**2.3The Summary of PeriodicInvestment Solutions Arising in Financial Markets**

(1) Return rate which follows Fourier series as a linear function of price

$$f\left(S_1\right) = \pi - 2\left[ Sin S_1 + \frac{1}{2} Sin 2S_1 + \frac{1}{3} Sin 3S_1 + \frac{1}{4} Sin 4S_1 + \frac{1}{5} Sin 5S_1 + \frac{1}{6} + \ldots \right]$$

(2) The complete solution of SDE with effect of Fourier series as linear function of price

$$S_1(t) = S_0 \exp\left\{ \left( \pi - 2\left[ \begin{array}{l} Sin S_1 + \frac{1}{2} Sin 2S_1 + \frac{1}{3} Sin 3S_1 + \frac{1}{4} Sin 4S_1 \\ + \frac{1}{5} Sin 5S_1 + \frac{1}{6} + \ldots \end{array} \right] - \frac{\sigma^2}{2} \right) t + \sigma dz(t) \right\}$$

(3) Return rate which follows Fourier series as a quadratic function of price.

$$f\left(S_2\right)=\left[\frac{8\pi^2}{3}+4CosS_2+Cos2S_2+\frac{4}{9}Cos3S_2+\frac{1}{2}Cos4S_2+\frac{4}{25}Cos5S_2+\frac{1}{9}.\text{Cos}\,S_2+...\right]$$

(4) Complete solution of SDE with effect from of Fourier series as a quadratic function of price.

$$S_2(t)=S_0\exp\left\{\left(\left(\left[\begin{array}{l}\frac{8\pi^2}{3}+4CosS_2+Cos2S_2+\frac{4}{9}Cos3S_2+\frac{1}{2}Cos4S_2+\frac{4}{25}Cos5S_2\\+\frac{1}{9}.\text{Cos}\,S_2+...\end{array}\right]-\frac{\sigma^2}{2}\right)t+\sigma dz(t)\right)\right\}$$

(5) Return rate that follows Fourier series as a cubic function of price

$$f\left(S_3\right)=2\pi^3+12\pi^2Cos\,\text{S}_3+3\pi^2Cos2\text{S}_3+\frac{4}{3}\pi^2Cos3\text{S}_3+\frac{3}{4}\pi^2Cos4\text{S}_3+\frac{12}{25}\pi^2Cos5\text{S}_3+\frac{1}{3}Cos6\text{S}_3+...$$

(6) The complete solution of SDE with effect of Fourier series as a cubic function of price

$$\text{S}_3\left(t\right)=S_0\exp\left\{\left(\left(\left[\begin{array}{l}2\pi^3+12\pi^2Cos\,\text{S}_3+3\pi^2Cos2\text{S}_3+\frac{4}{3}\pi^2Cos3\text{S}_3+\frac{3}{4}\pi^2Cos4\text{S}_3+\frac{12}{25}\pi^2Cos5\text{S}_3\\+\frac{1}{3}\pi^2Cos6\text{S}_3+...\end{array}\right]-\frac{\sigma^2}{2}\right)t+\sigma dz(t)\right)\right\}$$

## 3. Results and Discusion

This Section presents the graphical results for the three propositions whose solutions are in (19-26). Hence the following parameter values were used in the simulation study:

$S_0=14.90,=60.24=70.15,\sigma=0.05,\ dZ=1.00,\ t=1,\ \ and\ S_1=24.25,S_2=40.12,$

$S_3=34.19\ \ and\ \pi=\frac{22}{7},$

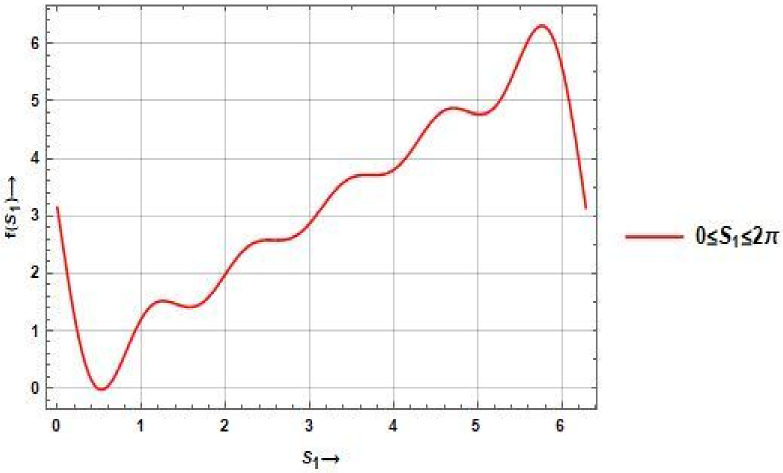


**Figure 1: The effect of Fourier series expansion when rate of return follows linear function of price.**

Figure 1 showed linear trend lines with oscillatory movement. The plot connotes the time distance between the minima and maxima in respect to the investment periods which is $2\pi$ . The significant cycles with dominating amplitudes and their periods are the maxima; which is maximum level of the investment returns. In this circumstance, any investor or trader who makes investment at this stipulated season will expect high level of returns. On the aspect of its minimum amplitude informs the investor not to make repeated investment. The various gaps in the amplitude describe odd periods which may not make any financial sense to invest to avoid financial depilation on the aspect of investors. However, the unequal amplitude ensures significant variations in terms of prices which is highly periodic in nature.

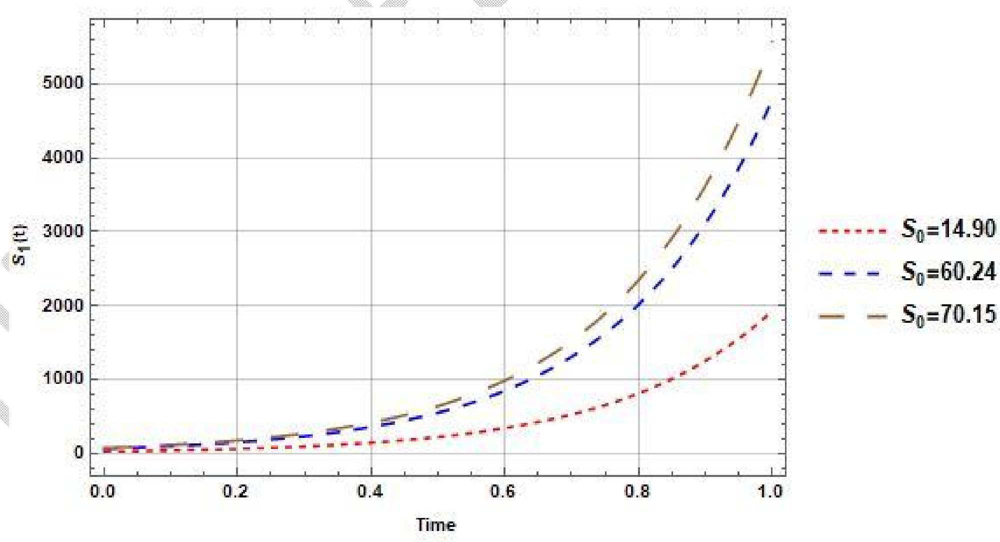


**Figure 2: The value of asset against time using variations of initial stock prices when rate of return follows Fourier series with linear function of price.**

It can be seen in Figure 2 that increase in the initial stock price; increases the value of asset price. This is very realistic because as soon as stock price continue to increase as price per unit of item over time ; that is how its value will be influenced positively and the convergence rate leads to shorter time limit. Investors are the liberty to gain more when stock prices are on constant increase.

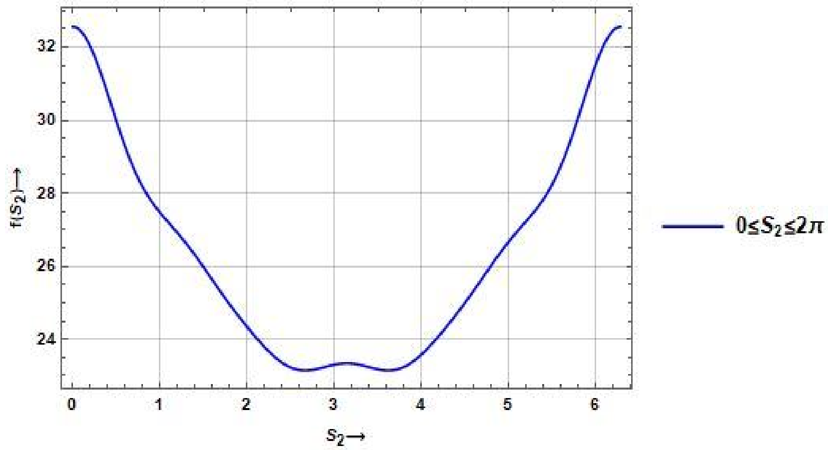


**Figure 3: The effect of Fourier series expansion when rate of return follows quadratic function of price**.

Figure 3 demonstrates a quadratic nature of periods which show-cased uncertainty, inequality in the value of assets and its return rates respectively. This is inline with results of [7],[12] and [25] etc. Also the minimum and maximum level of returns are seen in the plot which adequately informs an investors in decision making with respect to day-to-day activities of financial markets. In all, the plot is a positive investment which is high in value of asset. This remark will attract new buyers investing more in financial market hence is a business series around a secular trend.

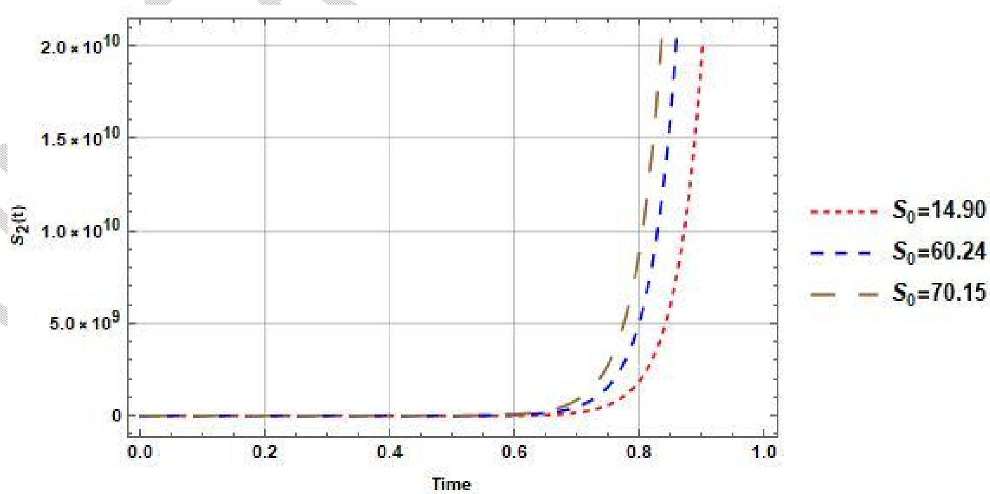


**Figure 4: The value of asset against time using variations of initial stock prices when rate of return follows Fourier series with quadratic function of price.**

The plot describes compounding returns over time which implies a constant expansion of portfolios of investments. This scenario allows investors to create large sums with small initial capital. However, all the trend movement and level of profit start at a point before growing in separate ways. That means each cost of initial stock price maximizes some level of profit in the trading activities. Clearly the plots show unbounded nature of business transactions which is indexed in millions of naira all through the trading days. See Figure 4.

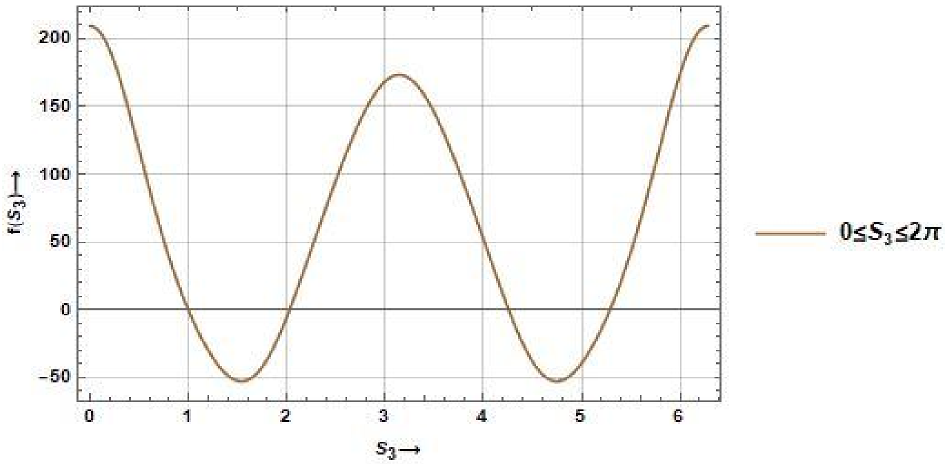


**Figure 5: The effect of Fourier series expansion when rate of return follows cubic function of price**

It can be seen in Figure 5 the oscillatory movement of cubic trend lines over the periods of buying and selling. The minimum and maximum level of each investment is noticed and the odd periods of investment which has negative financial implications to invest at that particular time and season is also seen clearly. This informs investors trading in financial market the right period to invest in other business without having any issue. More so, the equal amplitude is showing sensational variations in terms of prices which is highly periodic. The amplitude convergence rate in terms of price is obtained which shows better levels of profit during the trading activities.

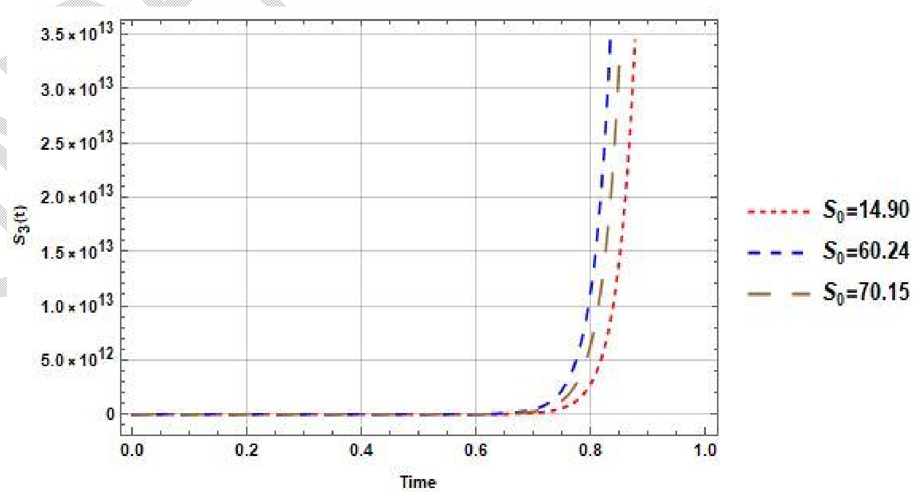

**Figure 6: The value of asset against time using variations of initial stock prices when rate of return follows Fourier series with quadratic function of price.**

Figure 6 displays exponential growth which is of great benefit to investors because there is every tendencies of making profit on every season. This result is agreement with that of [7] and [12].The trend movement was influenced by the season because the variation in the gap of each trend line is not much different compared to Figures 2 and 4 respectively. On a clear note seasons plays major roles and influences trading activities over time. The natural nature of plot is compounding which is favorable interest rate to investors as they can increase their net worth over time using small amount of cash flow.

## 4. Conclusion

The benefit of monetary assets and it return rates lies on capital investments which accrues capital such as daily, weekly, monthly, yearly and periodically etc. In this study, the effect of Fourier series expansions on the solution of Stochastic Differential Equation (SDE) was successfully exploited for the analysis of asset value function and its return rates in all seasons of buying and selling. The Fourier series expansions were expressed in terms of mathematical prepositions in three cases which was proved and used as stock return (drift parameter). The solutions of stock variables were critically observed by simulations which describe the behavior of asset value and stock markets in different variations and periods.

**Competing interests**

**There is no competing interest**

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