ON GENERATING MEASURES OF FUZZY CROSS-ENTROPY BY EMPLOYING MEASURES OF FUZZY WEIGHT ENTROPY

Abstract

By utilizing the ideas of fuzzy-weighted entropy and fuzzy-weighted directed divergence and appropriately selecting the weights, several new measures of fuzzy cross-entropy have been generated. Concurrently, some new measures of fuzzy weighted entropy have been suggested based on the new measures of fuzzy cross-entropy generated.

Keywords : Fuzzy entropy, Fuzzy weighted entropy, Fuzzy cross-entropy etc.

1. INTRODUCTION

Consider the measure of weighted entropy,

$$H(P:W) = -\sum_{i=1}^{n} w_i \emptyset(p_i) \tag{1.1}$$

and the corresponding measure of fuzzy weighted entropy,

$$H(A:W) = -\sum_{i=1}^{n} w_i \emptyset\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right). \tag{1.2}$$

Where, $\emptyset(\,.\,)$ is a continuous convex twice-differential function.Its maximum subject to natural constraint,

$$\sum_{i=1}^{n}\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right) = 1 \tag{1.3}$$

occurs when,

$$w_1\emptyset'(p_1) = w_2\emptyset'(p_2) = \cdots = w_n\emptyset'(p_n) \tag{1.4}$$

$i.e.$ $$w_1\,\Phi\left(\mu_A(x_1)\right) = w_2\,\Phi\left(\mu_A(x_2)\right) = \cdots = w_n\,\Phi\left(\mu_A(x_n)\right) \tag{1.5}$$

where, $$\Phi\left(\mu_A(x_i)\right) = \emptyset'(p_i)\,;\; i = 1, \dots, n.$$

If the maximum occurs when $p_i = q_i$, $i = 1, \dots, n$, then we get

$$\frac{w_1}{\left[\Phi\left(\mu_B(x_1)\right)\right]^{-1}} = \frac{w_2}{\left[\Phi\left(\mu_B(x_2)\right)\right]^{-1}} = \cdots = \frac{w_n}{\left[\Phi\left(\mu_B(x_n)\right)\right]^{-1}} \tag{1.6}$$

So that (1.1) becomes,

$$-\sum_{i=1}^{n}\frac{\emptyset\left(\mu_A(x_i)+(1-\mu_A(x_i))\right)}{\Phi\left(\mu_B(x_i)+(1-\mu_B(x_i))\right)} \text{ or } \sum_{i=1}^{n}\frac{\emptyset\left(\mu_A(x_i)+(1-\mu_A(x_i))\right)}{\Phi\left(\mu_B(x_i)+(1-\mu_B(x_i))\right)}. \quad (1.7)$$

According as $\Phi\left(\mu_B(x_i)+(1-\mu_B(x_i))\right)=\emptyset'(q_i)$ is positive or negative.

In either case, the measure is a continuous twice-differentiable convex function of $p_1,\dots,p_n$. Which has its minimum value at $q_1,\dots,q_n$. As such, this can be used as a measure of directed-divergence or cross-entropy of P from Q, where

$$P = p_1,\dots,p_n\ ;\ Q = q_1,\dots,q_n.$$

If $q_1,\dots,q_n$ is such that either $\emptyset'(q_i)>0$ or <0 for all i's is true, then the condition on $\emptyset'(q_i)$ is satisfied for all i. For this purpose, it is sufficient but not necessary that $\emptyset\,(x)$ be monotonically rising or decreasing, and all of the aforementioned conditions hold for the corresponding fuzzy situations.

We can obtain a measure of fuzzy cross-entropy that corresponds to each measure of fuzzy weighted entropy of type (1.2). Unfortunately, this metric is only relevant to a particular class of a priori distributions and not to all probability distributions.

2. OUR WORK

1. Guiasu's [2] measure of fuzzy weighted entropy

$$G(A{:}W) = -\sum_{i=1}^{n} w_i\left[\mu_A(x_i)\ln\mu_A(x_i) + \left((1-\mu_A(x_i)\right)\ln(1-\mu_A(x_i))\right]. \quad (2.1)$$

The corresponding measure of fuzzy cross-entropy

$$D_1(A{:}B) = \frac{\sum_{i=n}^{n}[\mu_B(x_i)\ln\mu_B(x_i)+(1-\mu_B(x_i))\ln(1-\mu_B(x_i))]-[\mu_A(x_i)\ln\mu_A(x_i)+(1-\mu_A(x_i))\ln(1-\mu_A(x_i))]}{\sum_{i=1}^{n}1+\ln\mu_B(x_i)+(1-\mu_B(x_i))\ln(1-\mu_B(x_i))} \quad (2.2)$$

2. Havrda-Charvat [3] measure of entropy

$$H(P) = \frac{1}{\alpha(1-\alpha)}\sum_{i=1}^{n} p_i^{\alpha} - p_i \ ,\alpha \neq 1\,,\alpha > 0. \quad (2.3)$$

The corresponding measure of fuzzy entropy

$$H(A) = \frac{1}{\alpha(1-\alpha)}\sum_{i=1}^{n}\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - \mu_A(x_i) - (1-\mu_A(x_i))$$

$$= \frac{1}{\alpha(1-\alpha)}\sum_{i=1}^{n}\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - 1 \quad (2.4)$$

The corresponding measure of fuzzy weighted entropy

$$H(A{:}W) = \frac{1}{\alpha(1-\alpha)}\sum_{i=1}^{n} w_i[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - 1] \quad (2.5)$$

The corresponding measure of fuzzy cross-entropy

$$D_2(A:B) = \frac{1}{\alpha(1-\alpha)} \frac{\sum_{i=1}^{n} \mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha} - \mu_B^{\alpha}(x_i)-(1-\mu_B(x_i))^{\alpha}}{\sum_{i=1}^{n} \alpha\mu_B^{\alpha-1}(x_i)+ \alpha(1-\mu_B(x_i))^{\alpha-1}} \quad (2.6)$$

Provided $\alpha\mu_B^{\alpha-1}(x_i) + \alpha(1-\mu_B(x_i))^{\alpha-1} < 0$

3. Burg's [1] measure of entropy

$$B(P) = \sum_{i=1}^{n} \ln p_i \quad (2.7)$$

The corresponding measure of fuzzy entropy

$$B(A) = \sum_{i=1}^{n} \ln \mu_A(x_i) + \ln(1-\mu_A(x_i)) \quad (2.8)$$

and the corresponding measure of fuzzy weighted entropy

$$B(A:W) = \sum_{i=1}^{n} w_i[\ln \mu_A(x_i) + \ln(1-\mu_i(x_i))] \quad (2.9)$$

The corresponding measure of fuzzy cross-entropy

$$D_3(A:B) = \frac{\sum_{i=1}^{n} \ln \mu_A(x_i)+\ln(1-\mu_A(x_i)-\ln \mu_B(x_i)-\ln(1-\mu_B(x_i)))}{1/\sum_{i=1}^{n} \ln \mu_B(x_i)+\ln(1-\mu_B(x_i))} \quad (2.10)$$

$$= \sum_{i=1}^{n}\left(\ln \mu_B(x_i) + \ln\left(1-\mu_B(x_i)\right)\right)$$

$$\left(\ln \mu_A(x_i) + \ln\left(1-\mu_A(x_i)\right) - \ln \mu_B(x_i) - \ln\left(1-\mu_B(x_i)\right)\right)$$

$$D_3(A:B) = \sum_{i=1}^{n}\left(\ln \mu_B(x_i) \ln \frac{\mu_A(x_i)(1-\mu_A(x_i))}{2(1-\mu_B(x_i))^2} + \ln(1-\mu_B(x_i)) \ln \frac{\mu_A(x_i)(1-\mu_A(x_i))}{2}\right) \quad (2.11)$$

4. Vajda's [8] measure of entropy

$$V(P) = \sum_{i=1}^{n}(p_i - p_i^2) \quad (2.12)$$

and the corresponding measure of fuzzy entropy

$$V(A) = \sum_{i=1}^{n} \mu_A(x_i) + (1-\mu_A(x_i)) - \mu_A^2(x_i) - \left(1-\mu_A(x_i)\right)^2 \quad (2.13)$$

The corresponding measure of fuzzy weighted entropy

$$V(A:W) = \sum_{i=1}^{n} w_i \left[1-\mu_A^2(x_i) - \left(1-\mu_A(x_i)\right)^2\right] \quad (2.14)$$

The corresponding measure of fuzzy cross-entropy

$$D_4(A:B) = \frac{\sum_{i=1}^{n}(\mu_A(x_i)+(1-\mu_A(x_i))-\mu_A^2(x_i)-(1-\mu_A(x_i))^2-\mu_B(x_i)-(1-\mu_B(x_i))+\mu_B^2(x_i)+(1-\mu_B(x_i))^2)}{\sum_{i=1}^{n} 1-2\mu_B(x_i)-2(1-\mu_B(x_i))}$$

$$= 2\sum_{i=1}^{n} \mu_B(x_i) - \mu_B^2(x_i) - \mu_A(x_i) + \mu_A^2(x_i) \quad (2.15)$$

5. Kapur's [4] measure of entropy

$$K(P) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}\sum_{i=1}^{n}(1 + ap_i)\ln(1 + ap_i) - \frac{1}{a}\sum_{i=1}^{n}(1 + a)\ln(1 + a)p_i \quad (2.16)$$

and the corresponding measure of fuzzy entropy

$$\begin{aligned} K(A) = &-\sum_{i=1}^{n} \mu_A(x_i)\ln\mu_A(x_i) + (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)) \\ &+ \frac{1}{a}\sum_{i=1}^{n}(1 + a\mu_A(x_i))\ln(1 + a\mu_A(x_i)) + (1 + a - a\mu_A(x_i))\ln(1 + a - a\mu_A(x_i)) \\ &- \frac{1}{a}\sum_{i=1}^{n}(1 + a)\ln(1 + a) + \left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right) \end{aligned} \quad (2.17)$$

so the corresponding measure of fuzzy weighted entropy is

$$\begin{aligned} K(A:W) = &-\sum_{i=1}^{n} w_i[\mu_A(x_i)\ln\mu_A(x_i) + (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)) \\ &+ \frac{1}{a}(1 + a\mu_A(x_i))\ln(1 + a\mu_A(x_i)) + (1 + a - a\mu_A(x_i))\ln(1 + a - a\mu_A(x_i)) \\ &- \frac{1}{a}(1 + a)\ln(1 + a) + \left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\Big] \end{aligned} \quad (2.18)$$

The corresponding measure of fuzzy cross-entropy

$$D_5(A:B) = \frac{\sum_{i=1}^{n}[\mu_B(x_i)\ln\mu_B(x_i) + \left(1-\mu_B(x_i)\right)\ln\left(1-\mu_B(x_i)\right) - \mu_A(x_i)\ln\mu_A(x_i) - \left(1-\mu_A(x_i)\right)\ln\left(1-\mu_A(x_i)\right) - \frac{1}{a}\left(\left(1+a\mu_B(x_i)\right)\ln\left(1-a\mu_B(x_i)\right) + \left(1+a-a\mu_B(x_i)\right)\ln\left(1+a-a\mu_B(x_i)\right)\right) + \frac{1}{a}\left(\left(1+a\mu_A(x_i)\right)\ln\left(1+a\mu_A(x_i)\right) + (1+a-a\mu_A(x_i)\ln\left(1+a-a\mu_A(x_i)\right)\right))]}{\sum_{i=1}^{n}[\ln\mu_B(x_i) + \ln(1-\mu_B(x_i)) - \ln(1+a\mu_B(x_i)) - \ln(1+a-a\mu_B(x_i))}$$

$$= \sum_{i=1}^{n} \frac{\mu_B(x_i)\ln\frac{\mu_B(x_i)}{1-\mu_B(x_i)} - \mu_A(x_i)\ln\frac{\mu_A(x_i)}{1-\mu_A(x_i)} + \ln\frac{1-\mu_B(x_i)}{1-\mu_A(x_i)} + \frac{1}{a}\ln\left(\frac{1+a+a^2\mu_A(x_i)-a^2\mu_A^2(x_i)}{1+a+a^2\mu_B(x_i)-a^2\mu_B^2(x_i)}\right) + \mu_A(x_i)\ln\left(\frac{1+a\mu_A(x_i)}{1+a-a\mu_A(x_i)}\right) - \mu_B(x_i)\ln\left(\frac{1+a\mu_B(x_i)}{1+a-a\mu_B(x_i)}\right) + \ln\left(\frac{1+a-a\mu_A(x_i)}{1+a-a\mu_B(x_i)}\right)}{\ln\left(\mu_B(x_i)\left(1-\mu_B(x_i)\right)\right) - \ln(1+a)}$$

$$D_5(A:B) = \sum_{i=1}^{n} \frac{\mu_B(x_i)\ln\frac{\mu_B(x_i)}{1-\mu_B(x_i)}.\frac{1+a-a\mu_B(x_i)}{1+a\mu_B(x_i)} + \mu_A(x_i)\ln\frac{1+a\mu_A(x_i)}{1+a-a\mu_A(x_i)}.\frac{1-\mu\mu_A(x_i)}{\mu_A(x_i)} + \ln\frac{(1-\mu_B(x_i))}{(1-\mu_A(x_i))}.\frac{(1+a-a\mu_A(x_i))}{(1+a-a\mu_B(x_i))} + \frac{1}{a}\left(\frac{1+a+a^2\mu_A(x_i)-a^2\mu_A^2(x_i)}{1+a+a^2\mu_B(x_i)-a^2\mu_B^2(x_i)}\right)}{\ln(\mu_B(x_i))(1-\mu_B(x_i)) - \ln(1+a)} \quad (2.19)$$

6. Bose-Einstein [5] measure of entropy

$$E(P) = \sum_{i=1}^{n} -p_i\ln p_i + (1 + p_i)\ln(1 + p_i) - 2\ln 2p_i \quad (2.20)$$

and the corresponding measure of fuzzy entropy

$$E(A) = \sum_{i=1}^{n} -\left(\mu_A(x_i)\ln\mu_A(x_i) + (1-\mu_A(x_i))\ln\left(1-\mu_A(x_i)\right)\right) + \left(\left(1+\mu_A(x_i)\right)\ln\left(1+\mu_A(x_i)\right) + \left(2-\mu_A(x_i)\right)\ln\left(2-\mu_A(x_i)\right) - 2\ln 2\right) \tag{2.21}$$

so the corresponding measure of fuzzy weighted entropy

$$E(A:W) = -\sum_{i=1}^{n} w_i\Big[-(\mu_A(x_i)\ln\mu_A(x_i) + (1-\mu_A(x_i))\ln(1-\mu_A(x_i))) + \left(\left(1+\mu_A(x_i)\right)\ln\left(1+\mu_A(x_i)\right) + \left(2-\mu_A(x_i)\right)\ln\left(2-\mu_A(x_i)\right) - 2\ln 2\right)\Big]. \tag{2.22}$$

The corresponding measure of fuzzy cross-entropy

$$D_6(A:B) = \sum_{i=1}^{n} \frac{\begin{array}{c}\mu_B(x_i)\ln\mu_B(x_i) + (1-\mu_B(x_i))\ln(1-\mu_B(x_i)) - \mu_A(x_i)\ln\mu_A(x_i) - (1-\mu_A(x_i))\ln(1-\mu_A(x_i)) \\ -(1+\mu_B(x_i))\ln(1+\mu_B(x_i)) - (2-\mu_B(x_i))\ln(2-\mu_B(x_i)) + (1+\mu_A(x_i))\ln(1+\mu_A(x_i)) + \\ (2-\mu_A(x_i))\ln(2-\mu_A(x_i))\end{array}}{\ln\mu_B(x_i) + \ln(1-\mu_B(x_i)) - \ln(1+\mu_B(x_i)) - \ln(2-\mu_B(x_i))} \tag{2.23}$$

so that,

$$D_6(A:B) = \sum_{i=1}^{n} \frac{\mu_B(x_i)\ln\frac{\mu_B(x_i)(2-\mu_B(x_i))}{(1+\mu_B(x_i))(1-\mu_B(x_i))} + \mu_A(x_i)\ln\frac{(1+\mu_A(x_i))(1-\mu_A(x_i))}{\mu_A(x_i)(2-\mu_A(x_i))} + \ln\frac{(1+\mu_A(x_i))(1-\mu_B(x_i))(2-\mu_A(x_i))^2}{(1+\mu_B(x_i))(1+\mu_A(x_i))(2-\mu_B(x_i))^2}}{\ln\frac{\mu_B(x_i)(1-\mu_B(x_i))}{(1+\mu_B(x_i))(2-\mu_B(x_i))}} \tag{2.24}$$

7. Fermi-Dirac [5] measure of entropy

$$F(P) = \sum_{i=1}^{n} -p_i\ln p_i - (1-p_i)\ln(1-p_i) \tag{2.25}$$

so the corresponding measure of fuzzy entropy

$$F(A) = \sum_{i=1}^{n} -2\mu_A(x_i)\ln\mu_A(x_i) - 2\left(1-\mu_A(x_i)\right)\ln\left(1-\mu_A(x_i)\right) \tag{2.26}$$

and the corresponding measure of fuzzy weighted entropy

$$F(A:W) = \sum_{i=1}^{n} w_i\left[-2\mu_A(x_i)\ln\mu_A(x_i) - 2\left(1-\mu_A(x_i)\right)\ln\left(1-\mu_A(x_i)\right)\right] \tag{2.27}$$

The corresponding measure of fuzzy cross-entropy

$$D_7(A:B) = \sum_{i=1}^{n}\Big[\mu_B(x_i)\ln\mu_B(x_i) + \left(1-\mu_B(x_i)\right)\ln\left(1-\mu_B(x_i)\right) - \mu_A(x_i)\ln\mu_A(x_i) - \left(1-\mu_A(x_i)\right)\ln\left(1-\mu_A(x_i)\right)\Big] \tag{2.28}$$

8. Renyi's [6] measure of entropy

$$R(P) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i \,, \quad \alpha \neq 1, \quad \alpha > 0 \tag{2.29}$$

so the corresponding measure of fuzzy entropy

$$R(A) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} (\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}) \,, \alpha \neq 1 \,, \alpha > 0 \tag{2.30}$$

It's corresponding measure of fuzzy weighted entropy

$$R(A:W) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} w_i (\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}) \,, \alpha \neq 1 \,, \alpha > 0 \tag{2.31}$$

The corresponding measure of fuzzy cross-entropy

$$D_8(A:B) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} \frac{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - \mu_B^{\alpha}(x_i) - (1-\mu_B(x_i))^{\alpha}}{\alpha \mu_B^{\alpha-1}(x_i) + \alpha \mu_B^{\alpha-1}(x_i)} \,, \alpha \neq 1 \,, \alpha > 0 \tag{2.32}$$

9. Sharma and Taneja's [7] measure of entropy

$$T(P) = \frac{1}{\beta-\alpha} \left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta} \right) \,, \alpha \neq \beta \tag{2.33}$$

so the corresponding measure of fuzzy entropy

$$T(A) = \frac{1}{\beta-\alpha} \sum_{i=1}^{n} \left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - \mu_A^{\beta}(x_i) - (1-\mu_A(x_i))^{\beta} \right] \,, \alpha \neq \beta \tag{2.34}$$

and the corresponding measure of fuzzy weighted entropy

$$T(A:W) = \frac{1}{\beta-\alpha} \sum_{i=1}^{n} w_i \left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - \mu_A^{\beta}(x_i) - (1-\mu_A(x_i))^{\beta} \right] \,, \alpha \neq \beta \tag{2.35}$$

The corresponding measure of fuzzy cross-entropy

$$D_9(A:B) = \frac{1}{\beta-\alpha} \sum_{i=1}^{n} \frac{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - \mu_B^{\alpha}(x_i) - (1-\mu_B(x_i))^{\alpha} - \mu_A^{\beta}(x_i) - (1-\mu_A(x_i))^{\beta} + \mu_B^{\beta}(x_i) + (1-\mu_B(x_i))^{\beta}}{\beta \mu_B^{\beta-1}(x_i) + \beta(1-\mu_B(x_i))^{\beta-1} + \alpha \mu_B^{\alpha-1}(x_i) + \alpha(1-\mu_B(x_i))^{\alpha-1}} \,, \alpha \neq \beta \tag{2.36}$$

3. CONCLUDING REMARKS

There are numerous probabilistic and non-probabilistic ways to calculate entropy. We may create a fuzzy entropy measure and a weighted fuzzy entropy measure for each measure of probabilistic entropy, and vice versa. The literature on fuzzy measurements can be improved by using the understanding of probabilistic measures, which have many similarities to both types of measures. Both types of entropies can be used to gain a deeper understanding of the concept of uncertainty so that it can be controlled for the benefit of mankind, while there are some parallels and differences between the two types of entropy measures.

3. REFERENCES

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