Original Research Article

Three Parameter Transmuted Exponential Distribution

ABSTRACT

Many distribution functions,can be explored in numerous dimensions with the extended form of the distribution function.In this paper an extended form of exponential distribution is studied by the quadratic rank transmutation map, called three parameter transmuted exponential distribution, where the base distribution is exponential distribution with two parameters $\theta > 0$ *and* $\beta > 0$, shape and location parameters respectively. Some statistical properties have been studied for the said distribution including moments, quantile function, moment generating function, reliability analysis, mills ratio, reverse hazard rate function, order statistics, Bonferroni and Lorenz curves and indices,mean deviation about mean and median, estimation of parameter. The applicability have been explored by comparing value of -2ℓ, AIC and BIC using real data set.

Keywords: Quadratic rank transmutation map, Moments, Quantile, Maximum Likelihood Estimation, Reliability.

1. INTRODUCTION

In this paper, we study an extended form of the distribution function for a particular distribution. Consider the exponential distribution with two parameters λ and β, where the probability density function is given by

$$f(x) = \frac{1}{\lambda}\exp\left[-\frac{x-\theta}{\lambda}\right]; \qquad x > \theta,\ \lambda > 0 \quad (1)$$

With the cumulative distribution function (cdf)

$$F(x) = \left[1-\exp\left(-\frac{x-\theta}{\lambda}\right)\right] \quad (2)$$

Now, the exponential distribution with one parameter is defined as

$$f(x) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right) \quad \text{Or} \quad f(x) = \lambda\exp(-\lambda x);\ x > 0,\ \lambda > 0 \quad (3)$$

With

$$F(x) = \left[1-\exp\left(-\frac{x}{\lambda}\right)\right] \quad (4)$$

here, λ *or* θ is not duration of time, it is a rate as the parameter λ *or* θ of a poisson process. For example, the number of customers arriving at a particular shop, number of earthquake per year at a particular area, number of misprinted pages in a book, etc.

When $\theta = 0$ and $\lambda = 1$, then

$$f(x) = \exp(-x); \qquad x > 0 \quad (5)$$

which is the probability density function of standard exponential distribution. This forms of the exponential distribution is the particular case of Gamma distribution.

There are various extended form of exponential distribution namely generalized, weighted, mixture, exponentiated, truncated etc. There is another form of exponential distribution known as transmuted exponential distribution. The transmuted exponential distribution is introduced by adding an additional parameter to the existing distribution, with the goal of addressing certain issues that arise in financial mathematics.This is introduced by Shaw and Buckley in 2007 and named the family as quadratic transmuted family of distributions. The cdf of the family is

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, x \in R, \quad (6)$$

Where λ is a transmuted parameter, $\lambda \in [-1,1]$ and $G(x)$ is the cdf of the baseline distribution [16].

Recently, using rank transmutation map many transmuted distributions have been proposed by various researchers, for example Aryal and Tsokos [9] have generated a flexible family of probability distributions taking extreme value distribution as the base value distribution by introducing a new parameter by the technique of quadratic rank transmutation map.Merovci [14] has developed a transmuted exponentiated exponential distribution. Merovci and Puka [11] have proposed a transmuted Pareto distribution. Merovci [12] have proposed the transmuted Rayleigh distribution. Merovci [13] has proposed transmuted generalized Rayleigh distribution.Owoloko et al. [16] have developed a transmuted exponential distribution where the base was one parameter exponential distribution. Pobocikovai et al. [17] have developed a transmuted Weibull distribution. Hussian [7] have proposed the transmuted exponentiated Gamma distribution.Khan and King [9] have gaeneralized the three parameter modified Weibull distribution and application in real set of data with its statistical properties. Merovci and Elbatal [15] have proposed transmuted Lindley - Geometricdistribution and its application on real set of data. Elbatal and Aryal [6] have developed a transmuted Dagum distribution and discussed its statistical properties with its application on real set of data.Khan et al. [8] have developed the transmuted Kumaraswamy distribution.Khan [10] has developed a three parameter transmuted Rayleigh distribution with its some structural properties. Abd El-Monsef and Sohsah [2] have developed a new discrete compound distribution namely Poisson transmuted Lindley distribution. Ullah and Shahzad [20] have propose a new distribution using the technique of transmutation map.Azzwidden and Al-Zou'bi [1] have introduced the transmuted Gamma Gompertze distribution.Rahman et al.[18] have discussed a review about the transmuted families of distributions.Tripathi and Mishra [19] have developed a new distribution namely transmuted inverse XGamma distribution and its statistical properties.

2. METHODOLOGY

2.1 Transmuted Family of Distributions

Shaw and Buckley (2007) developed a novel approach for managing problems associated with financial mathematics by introducing a new parameter to an existing distribution. They called this family of distributions the quadratic transmuted family. The family's CDF has the simple quadratic form given below

$$G(x) = (1+\lambda)F(x) - \lambda F(x)^2, x \in R, \quad (7)$$

Where λ is a transmuted parameter; $\lambda \in [-1,1]$ and $F(x)$ is the cdf of the baseline distribution [18].

2.2 Statistical Properties

In case of statistical properties, moments, moment generating function, order statistics, quantile function, reliability analysis, mean deviation about mean and median has been done for the developed distribution and these can be calculated by following formulas

The r^{th} moment can be defined as

$$E[X^r] = \int_0^\infty x^r g(x) dx \quad (8)$$

The quantile function, x_q can be defined as

$$G(x_q) = q \quad (9)$$

If X is a random variable with probability function $f(x)$ then moment generating function can be defined as

$$M_x(t) = E[e^{tx}]$$

$$= \int_0^\infty e^{tx} f(x) dx \quad (10)$$

If $X_1, X_2 \ldots X_n$ be n independent and identically distributed variables from a continuous population with cumulative distribution function (cdf) $G(x)$ and probability density function (pdf) $g(x)$. If these variables are arranged in ascending order of magnitude $X_{(1)} \leq X_{(2)} \leq \ldots . X_{(n)}$, then the pdf of r-th order statistics $X_{(r)}$ can be written as

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} g_X(x)[G_X(x)]^{r-1}[1-G_X(x)]^{n-r} \quad (11)$$

Mathematically, the survival and hazard function are given by

$$Survival \quad function, \; S(x) = 1 - G(x) \quad (12)$$

$$Hazard \quad function, \; H(x) = \frac{g(x)}{1-G(x)} \quad (13)$$

$$Mills \quad Ratio = \frac{1}{H(x)} \quad (14)$$

$$And \qquad Reverse \quad Hazard \quad Rate \quad Function = \frac{f(x)}{F(x)}$$

The Bonferroni Index (BI) and Bonferroni Curve (BC) can be obtained as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx \quad \text{and} \quad L(p) = \frac{1}{\mu} \int_0^q x f(x) dx \quad (15)$$

If X has a three parameter transmuted exponential distribution with mean $E(X)=\mu$ and $Median(X)=M$, then we can derive the mean deviation about the mean = μ and about the *median =M* by following equations

$$\delta_1(x)=\int_0^\infty |x-mean|\,f(x)dx \quad and \quad \delta_2(x)=\int_0^\infty |x-Median|\,f(x)dx \quad (16)$$

3. TRANSMUTED EXPONENTIAL DISTRIBUTION (TED)

If X is a random variable having exponential distribution with two parameters $\theta > 0$ (location parameter) and $\beta > 0$ (scale parameter), then

$$F(x)=\left[1-\exp\left\{-\frac{x-\theta}{\beta}\right\}\right] \quad ; x>\theta, \beta>0 \quad (17)$$

$$f(x)=\frac{1}{\beta}\exp\left[-\frac{x-\theta}{\beta}\right] \quad ; x>\theta, \beta>0 \quad (18)$$

Then, the transmuted distribution can be defined by Shaw and Buckley (2007) as

$$G(x)=(1+\lambda)F(x)-\lambda F(x)^2, x\in R, \quad (19)$$

Where λ is a transmuted parameter; $\lambda\in[-1,1]$ and $F(x)$ is the cdf of the baseline distribution.

Now, by putting (17) into (19), the cumulative distribution function (cdf) of three parameter transmuted exponential distribution (TED) is obtained, where the base line distribution iss two parameter exponential distribution.

$$G(x)=(1+\lambda)F(x)-\lambda F(x)^2, x\in R,$$

$$G(x)=(1+\lambda)\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}-\lambda\left[1-\exp\left\{-\frac{x-\theta}{\beta}\right\}\right]^2$$

$$=1-\exp\left(-\frac{x-\theta}{\beta}\right)+\lambda\exp\left(-\frac{x-\theta}{\lambda}\right)-\lambda\exp\left\{-\left(\frac{x-\theta}{\beta}\right)\right\}^2$$

$$=\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x-\theta}{\lambda}\right)\right\} \quad (20)$$

Differentiating (20) with respect to *x* we get the probability density function (pdf)

$$g(x)=\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1+\lambda-2\lambda\left\{1-\exp\left(\frac{x-\theta}{\beta}\right)\right\}\right]$$

$$=\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left\{-\frac{x-\theta}{\beta}\right\}\right] \quad (21)$$

When, $\eta=1$, then the cdf and pdf of this proposed distribution are

$$G(x)=\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\} \quad (22)$$

$$g(x)=\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left\{-\frac{x-\theta}{\beta}\right\}\right] \quad (23)$$

Special cases

Substituting $\theta = 0$, in equation (23) reduces to pdf of transmuted exponential distribution in which the base line distribution was one parameter exponential distribution.
Substituting $\theta = 0, \ \beta = 1$, in equation (23) reduces to pdf of transmuted standard exponential distribution.
Substituting $\theta = 0, \ \beta = 1 \ and \ \lambda = 1$, in equation (23) reduces to pdf of standard exponential distribution

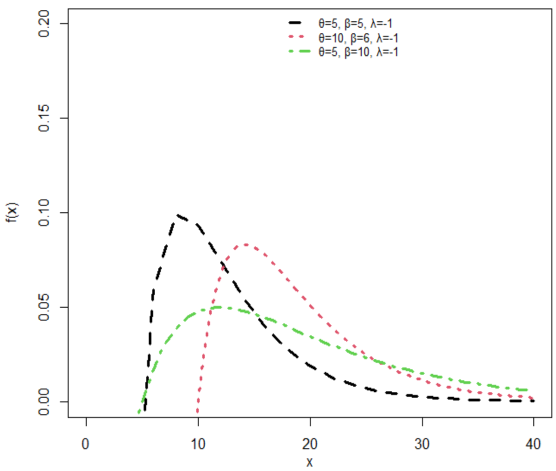

Fig. 1. plot of pdf when λ= -1

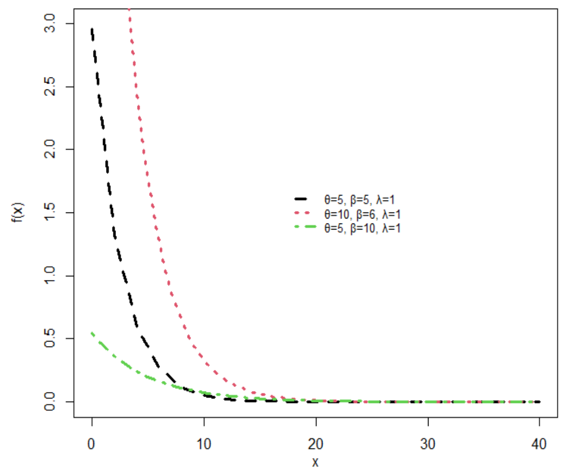

Fig. 2. plot of pdf when λ=1

From the above Fig. 1. and Fig. 2., we observe that the shape of transmuted exponential distribution is increasing then decreasing when the transmuted parameter, $\lambda = -1$; otherwise it is gradually decreasing when, $\lambda = 1$ respectively.

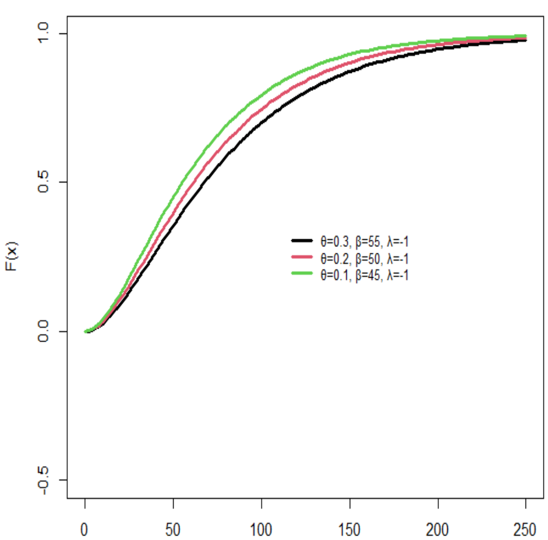

Fig. 3. Plot of cdf when λ= -1

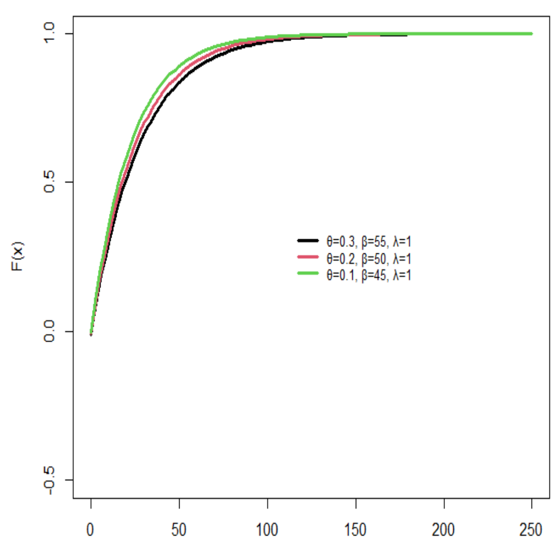

Fig. 4. Plot of cdf when λ=1

From the above Fig. 3. and Fig. 4., we observe that cdf of the proposed distribution tends to 1, when λ takes both positive and negative values.

4.STATISTICAL PROPERTIES

Some statistical properties of three parameter TED are discussed in this section

4.1 Moments

The r^{th} moment can be defined as

$$E\left[X^r\right] = \int_{\theta}^{\infty} x^r g(x) dx$$

$$= \theta^r \left[1 - {}^rC_1 \frac{\beta}{\theta}\left(\frac{\lambda}{2}-1\right) + 2.\,{}^rC_2 \left(\frac{\beta}{\theta}\right)^2\left(\frac{\lambda}{2}-1\right) + \dots\dots\dots + (-1)^r . r.\left(\frac{\beta}{\theta}\right)^r\left(\frac{\lambda}{2}-1\right)\right] \tag{24}$$

$When \quad r = 1$

$$E(X) = \theta\left(1 - \frac{\beta}{\theta}\right)\left(\frac{\lambda}{2}-1\right)$$

$$Mean \ = \theta - \frac{\beta\lambda}{2} + \beta$$

$$Variance, \ V(X) = E\left(X^2\right) - \{E(X)\}^2$$

$$V(X) = \theta^2 - \theta\beta\lambda - 2\beta\theta + \beta^2\lambda - 2\beta^2 - \theta^2\beta^2 + \lambda\beta^2 - \beta^2\frac{\lambda^2}{4} - 2\theta\beta$$

$$= 2\lambda\beta^2 - 3\beta^2 - \theta\beta\lambda - \beta^2\frac{\lambda^2}{4}$$

By putting *r=3, 4* skewness and kurtosis are obtained as

$$E\left(X^3\right) = \int_{\theta}^{\infty} x^3 f(x) dx$$

$$= \theta^3 - 3\beta^3\frac{\lambda}{2} + 3\beta^3 - 3\frac{\theta^2\lambda}{2} + 3\theta^2 + 3\theta\lambda - 6\theta$$

$$E\left(X^4\right) = \int_{\theta}^{\infty} x^4 f(x) dx$$

$$= \theta^4 - 2\theta^3\beta\lambda + 4\theta^3\beta + 6\theta^2\beta^2\lambda - 12\theta^2\beta^2 - 6\theta\beta^3\lambda + 12\theta\beta^3 + 2\beta^4\lambda - 4\beta^4$$

$$Skewness \ = \ \frac{E[X^3] - 3E[X^2]\mu + 2\mu^3}{\sigma^3}$$

$$= \frac{1}{\sigma^3}\left[\begin{array}{l}\theta^3 - 3\beta^3\frac{\lambda}{2} + 3\beta^3 - 3\theta^2\frac{\lambda}{2} + 3\theta^2 + 3\theta\lambda - 6\theta \\ -3\left(\theta^2 - \theta\beta\lambda + 2\theta\beta + \beta^2\lambda - 2\beta^2\right)\mu + 2\mu^3\end{array}\right] \tag{25}$$

$$kurtosis = \frac{E[X^4] - 4E[X^3]\mu + 6E[X^2]\mu^2 - 3\mu^4}{\sigma^4}$$

$$= \frac{1}{\sigma^4}\begin{bmatrix} \left(\theta^4 - 2\theta^3\beta\lambda + 4\theta^3\beta + 6\theta^2\beta^2\lambda - 12\theta^2\beta^2 - 6\theta\beta^3\lambda + 12\theta\beta^3 + 2\beta^4\lambda - 4\beta^4\right) \\ -4\left(\theta^3 - 3\beta^3\frac{\lambda}{2} + 3\beta^3 - 3\theta^2\frac{\lambda}{2} + 3\theta^2 + 3\theta\lambda - 6\theta\right)\mu \\ +6\left(\theta^2 - \theta\beta\lambda + 2\theta\beta + \beta^2\lambda - 2\beta^2\right)\mu^2 - 3\mu^4 \end{bmatrix} \tag{26}$$

4.2 Quantile Function

The quantile function, x_q can be defined as

$$G(x_q) = q$$

$$\Rightarrow x_q = \theta + \beta\left(-\ln\left[1 - \left\{\frac{(\lambda+1) - \sqrt{(\lambda+1)^2 - 4\lambda q}}{2\lambda}\right\}\right]\right) \tag{27}$$

And when q=0.5, then median of the distribution of the distribution is obtained

$$\Rightarrow x_{0.5} = \theta + \beta\left(-\ln\left[1 - \left(\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 - 4\lambda q}}{2\lambda}\right)\right]\right)$$

$$= \theta + \beta\left(-\ln\left[\frac{(\lambda - 1) + \sqrt{\lambda^2 + 1}}{2\lambda}\right]\right) \tag{28}$$

4.3 Moment Generating Function

If X is a random variable with probability function $f(x)$ then moment generating function can be defined as

$$M_x(t) = E\left[e^{tx}\right]$$

$$= \int_{\theta}^{\infty} e^{tx} f(x) dx$$

$$= 1 + \sum_{m=1}^{\infty}\frac{t^m}{m!}\left\{\theta^m\left[1 - {}^mC_1\frac{\beta}{\theta}\left(\frac{\lambda}{2} - 1\right) + 2.\,{}^mC_2\left(\frac{\beta}{\theta}\right)^2\left(\frac{\lambda}{2} - 1\right) + ... + (-1)^m.m.\left(\frac{\beta}{\theta}\right)\left(\frac{\lambda}{2} - 1\right)\right]\right\} \tag{29}$$

4.4 Order Statistics

The pdf of r-th order statistics $X_{(r)}$ can be written as

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}\, g_X(x)\left[G_X(x)\right]^{r-1}\left[1 - G_X(x)\right]^{n-r}$$

Now, using the pdf, $g(x)$ and cdf, $G(x)$ of the developed distribution

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}\ g_X(x)[G_X(x)]^{r-1}[1-G_X(x)]^{n-r}$$

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]$$
$$\left[\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\right]^{r-1}$$
$$\left[\exp\left(-\frac{x-\theta}{\beta}\right)\left\{\lambda\exp\left(-\frac{x-\theta}{\beta}\right)-\lambda+1\right\}\right]^{n-r}$$

(30)

Density function of smallest order statistics

$$f_{X_{(1)}}(x)=\frac{n!}{(n-1)!}\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]\left[\begin{array}{l}\exp\left(-\frac{x-\theta}{\beta}\right)\\ \left\{1-\lambda+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\end{array}\right]^{n-1}$$

$$=\frac{n}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]\left[\exp\left(-\frac{x-\theta}{\beta}\right)\left\{1-\lambda+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\right]^{n-1}$$

(31)

Density function of largest order statistics

$$f_{X_{(n)}}(x)=\frac{n!}{(n-1)!}\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]\left[\begin{array}{l}\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\\ \left\{1+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\end{array}\right]^{n-1}$$

$$=\frac{n}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]\left[\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\right]^{n-1}$$

(32)

4.5 Reliability Analysis

Mathematically, the survival and hazard function are given by

Survival function, $S(x)=1-G(x)$

$$=\lambda\exp\left(-2\left(\frac{x-\theta}{\beta}\right)\right)-(\lambda-1)\exp\left(-\frac{x-\theta}{\beta}\right)$$

(33)

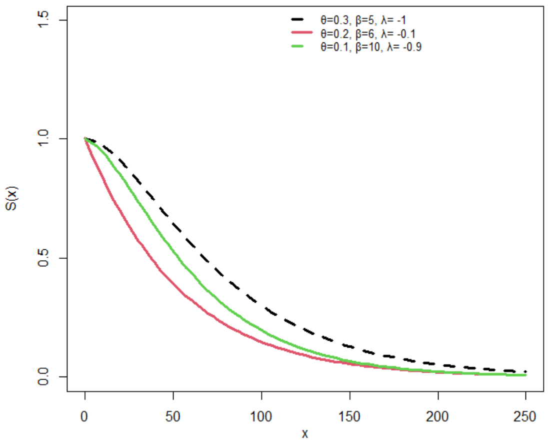

Fig. 5. Plot for survival function when λ takes negative values

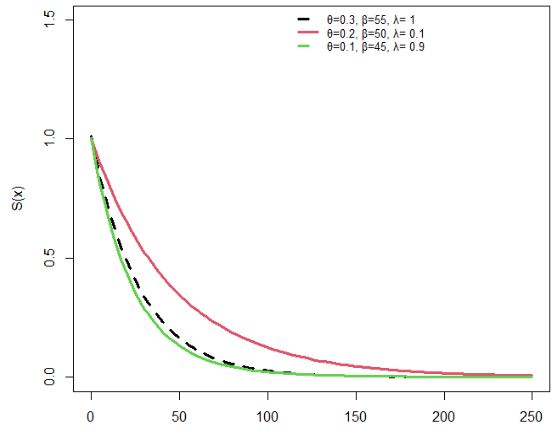

Fig. 6. Plot for survival function when λ takes positive values

$$\textit{Hazard function}, H(x) = \frac{g(x)}{1-G(x)}$$

$$= \frac{\frac{1}{\beta}\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]}{\left[1-\lambda+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]}$$

$$= \frac{\frac{1}{\beta}\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]}{\left[1-\lambda+\lambda\eta\exp\left(-\frac{x-\theta}{\beta}\right)\right]} \tag{34}$$

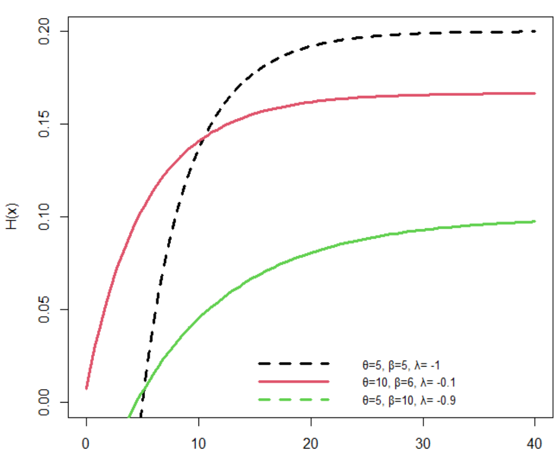

Fig. 7. plot of hazard function when λ takes negative values

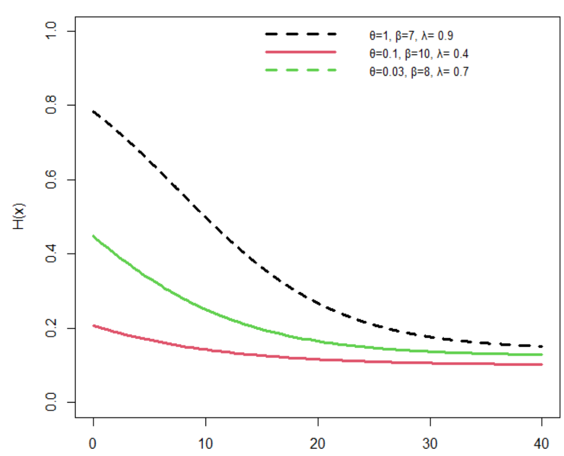

Fig. 8. plot of hazard function when λ takes positive values

$$Mills\ \ Ratio\ = \frac{\left[1-\lambda+\lambda\eta\exp\left(-\frac{x-\theta}{\beta}\right)\right]}{\frac{1}{\beta}\left[1-\lambda+2\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right]} \tag{35}$$

$$And \qquad Reverse\ \ Hazard\ \ Rate\ Function = \frac{f(x)}{F(x)}$$

$$= \frac{\frac{1}{\beta}\exp\left(-\frac{x-\theta}{\beta}\right)\left[1-\lambda+2\lambda\exp\left\{-\frac{x-\theta}{\beta}\right\}\right]}{\left\{1-\exp\left(-\frac{x-\theta}{\beta}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x-\theta}{\beta}\right)\right\}} \tag{36}$$

4.6 Bonferroni Curve, Lorenz Curve and Indices

The Bonferroni Index (BI) and Bonferroni Curve (BC), Lorenz Curve (LC) and Gini Index (GI) can be obtained as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right]$$

$$B(P) = \frac{1}{P}\left[1-\frac{1}{\mu}\left\{q\,e^{\left(-\frac{q-\theta}{\beta}\right)} + \beta\,e^{\left(-\frac{q-\theta}{\beta}\right)} - \lambda\,q\,e^{\left(-\frac{q-\theta}{\beta}\right)} + \beta\,e^{\left(-\frac{q-\theta}{\beta}\right)} + \lambda\left(q\,e^{\left(-2\left(\frac{q-\theta}{\beta}\right)\right)} + \frac{\beta}{2}e^{\left(-2\left(\frac{q-\theta}{\beta}\right)\right)}\right)\right\}\right] \tag{37}$$

And
$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right]$$

$$L(P) = 1-\frac{1}{\mu}\left\{q\,e^{\left(-\frac{q-\theta}{\beta}\right)} + \beta\,e^{\left(-\frac{q-\theta}{\beta}\right)} - \lambda\left(q\,e^{\left(-\frac{q-\theta}{\beta}\right)} + \beta\,e^{\left(-\frac{q-\theta}{\beta}\right)}\right) + \lambda\left(q\,e^{\left(-2\left(\frac{q-\theta}{\beta}\right)\right)} + \frac{\beta}{2}e^{\left(-2\left(\frac{q-\theta}{\beta}\right)\right)}\right)\right\} \tag{38}$$

The Bonferroni and Gini indices are obtained as

$$B = 1-\int_0^1 B(p)dp \quad and \quad G = 1-2\int_0^1 L(p)dp \tag{39}$$

4.7 Mean Deviation about Mean and Median

The mean deviation about the mean = μ and about the *median* =*M*can be derived by following equations

$$\delta_1(x) = \int_0^\infty |x-mean|\,f(x)dx \qquad and \qquad \delta_2(x) = \int_0^\infty |x-Median|\,f(x)dx$$

$$\delta_1(x) = 2\mu F(\mu) - 2\left[\begin{array}{l}\beta e^{\frac{\theta}{\beta}}\left(1-e^{-\frac{\mu}{\beta}}\right) - \mu e^{-\left(\frac{\mu-\theta}{\beta}\right)} - \lambda\left\{\beta e^{\frac{\theta}{\beta}}\left(1-e^{-\frac{\mu}{\beta}}\right) - \mu e^{-\left(\frac{\mu-\theta}{\beta}\right)}\right\} \\ +2\lambda\left\{-\mu e^{-2\left(\frac{\mu-\theta}{\beta}\right)} - \frac{\beta}{2}e^{-2\left(\frac{\mu-\theta}{\beta}\right)} + \frac{\beta}{2}e^{\frac{2\theta}{\beta}}\right\}\end{array}\right] \tag{40}$$

$$\delta_2(x) = \mu - 2\left[\begin{array}{l}\beta e^{\frac{\theta}{\beta}}\left(1-e^{\frac{-M}{\beta}}\right) - \mu e^{-\left(\frac{M-\theta}{\beta}\right)} - \lambda\left\{\beta e^{\frac{\theta}{\beta}}\left(1-e^{-\frac{M}{\beta}}\right) - Me^{-\left(\frac{M-\theta}{\beta}\right)}\right\} + \\ 2\lambda\left\{-Me^{-2\left(\frac{M-\theta}{\beta}\right)} - \frac{\beta}{2}e^{-2\left(\frac{M-\theta}{\beta}\right)} + \frac{\beta}{2}e^{\frac{2\theta}{\beta}}\right\}\end{array}\right] \tag{41}$$

5. ESTIMATION OF PARAMETERS

Here the parameters are estimated by the method of Maximum Likelihood Estimation

5.1 Random Number Generation

By the method of inversion we can generate random number as given by

$$G(x_u) = u$$

$$\Rightarrow (1+\lambda)\left\{1-\exp\left(-\frac{x_u-\theta}{\beta}\right)\right\} - \lambda\left[1-\exp\left(-\frac{x_u-\theta}{\beta}\right)\right]^2 = u$$

$$\Rightarrow x_u = \theta + \beta\left(-\ln\left[1-\left\{\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda q}}{2\lambda}\right\}\right]\right) \tag{42}$$

Where $u \sim U(0,1)$ then define it as uniform distribution.

5.2 Maximum Likelihood Estimation

Let $X_{1,}\ X_2 \ldots X_n$ be a sample size of $"n"$ from the Transmuted exponential (TE) distribution, the likelihood function is given by

$$L(x_1, x_{2.}, \ldots, x_n / \theta, \beta, \lambda)$$

$$= \left(\frac{1}{\beta}\right)^n e^{-\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\beta}\right)} \prod_{i=1}^{n}\left\{1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\lambda}\right)}\right\}$$

Taking log both sides, we get

$$\log L = n\log\left(\frac{1}{\beta}\right) - \sum_{i=1}^{n}\left(\frac{x_i-\theta}{\beta}\right) + \sum_{i=1}^{n}\log\left\{1-\lambda+2\lambda e^{-\frac{x_i-\theta}{\beta}}\right\}$$

$$= -n\log\beta - \sum_{i=1}^{n}\left(\frac{x_i-\theta}{\beta}\right) + \sum_{i=1}^{n}\log\left\{1-\lambda+2\lambda e^{-\frac{x_i-\theta}{\lambda}}\right\}$$

Now, differentiating $\log L$ with respect to $\theta\ , \beta\ and\ \lambda$, we get

$$\frac{d(\log L)}{d\theta} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{2\lambda \frac{1}{\beta} e^{-\left(\frac{x_i-\theta}{\beta}\right)}}{\left[1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right]} \qquad (43)$$

$$\frac{d(\log L)}{d\beta} = \frac{-n}{\beta} + \sum_{i=1}^{n}\left(\frac{x_i-\theta}{\beta^2}\right) + \sum_{i=1}^{n} \frac{2\lambda\left(\frac{x_i-\theta}{\beta^2}\right) e^{-\left(\frac{x_i-\theta}{\beta}\right)}}{\left[1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right]}$$

$$= -\frac{1}{\beta}\sum_{i=1}^{n}\left[1-\left(\frac{x_i-\theta}{\beta^2}\right)\right] + \frac{2\lambda}{\beta}\sum_{i=1}^{n} \frac{\left(\frac{x_i-\theta}{\beta}\right) e^{-\left(\frac{x_i-\theta}{\beta}\right)}}{\left[1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right]} \qquad (44)$$

$$\frac{d(\log L)}{d\lambda} = \sum_{i=1}^{n} \frac{-1+2e^{-\left(\frac{x_i-\theta}{\beta}\right)}}{\left[1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right]}$$

$$= \sum_{i=1}^{n} \frac{2e^{-\left(\frac{x_i-\theta}{\beta}\right)}-1}{\left[1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right]} \qquad (45)$$

From (43), (44) and (45), the second derivative of log likelihood function can be obtained as

$$\frac{d^2(\log L)}{d\theta^2} = \frac{2\lambda}{\beta^2}\sum_{i=1}^{n} \frac{e^{-\left(\frac{x_i-\theta}{\beta}\right)}.A}{\left\{1-\lambda+2de^{-\left(\frac{x_i-\theta}{\beta}\right)}\right\}^2}$$

Where, $A = \left\{1-\lambda+2de^{-\left(\frac{x_i-\theta}{\beta}\right)}\right\} - 2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}$

$$\frac{d^2(\log L)}{d\beta^2} = \frac{-n}{\beta^2} + \frac{2\lambda}{\beta^2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\theta}{\beta}\right)}}{B} + \frac{2\lambda}{\beta}\sum_{i=1}^{n}\frac{C}{B^2}$$

Where,

$$B = \left\{1-\lambda+2\lambda e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right\}$$

$$C = e^{-\left(\frac{x_i-\theta}{\beta}\right)}\left\{\frac{B}{\beta} - e^{-\left(\frac{x_i-\theta}{\beta}\right)}\right\}$$

$$\frac{d^2(\log L)}{d\lambda^2} = \sum_{i=1}^{n} \frac{\left\{2e^{-\left(\frac{x_i-\theta}{\beta}\right)} - 1\right\}^2}{B^2}$$

And similarly, $\frac{d^2(\log L)}{d\theta d\beta}$, $\frac{d^2(\log L)}{d\theta d\lambda}$, $\frac{d^2(\log L)}{d\beta d\lambda}$ can be obtained and equating to zero, the maximum likelihood estimates $\hat{\theta}, \hat{\beta}$ *and* $\hat{\lambda}$ of parameters θ, β *and* λ can be obtained.

Approximate $100(1-\alpha)\%$ two sided confidence intervals for θ, β, λ are respectively, given by

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}\left(\hat{\theta}\right)} \quad , \hat{\beta} \pm z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}\left(\hat{\beta}\right)} \quad and \quad \hat{\lambda} \pm z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}\left(\hat{\lambda}\right)}$$

Where z_α is the upper $\alpha - th$ percentiles of the standard normal distribution. Using R we can easily compute Hessian matrix and it's inverse and hence the values of the standard error and asymptotic confidence intervals.

We can use the LR test statistic to check whether the transmuted exponential distribution for the given data set is statistically superior to the exponential distribution transmuted exponential distribution where the base distribution was one parameter exponential distribution. Hypothesis test of the type $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ can be performed using a LR test. In this case, the LR test statistic for testing H_1 *versus* H_0 is $\omega = 2\left(L\left(\hat{\theta}; x\right) - L\left(\hat{\theta}_0; x\right)\right)$, where $\hat{\theta}$ *and* $\hat{\theta}_0$ are the MLE under H_1 *and* H_0, respectively. The statistic ω is asymptotically $(n \to \infty)$ distribute as χ_k^2, where k is the length of the parameter vector θ of interest. The LR test rejects H_0 *if* $\omega \rangle \chi_{k:\alpha}^2$, *where* $\chi_{k:\alpha}^2$ denotes the upper 100α % quantile of the χ_k^2 [13].

6. APPLICATION

In this section, the applicability of the three parameter transmuted exponential distribution has been studied. In order to compare the developed distribution with other distributions, Some criterion like -2log(L), AIC (Akaike Information Criterion) and BIC (Baysian Information Criterion) are considered on the real data. From these criterions, the value of lacking of fit of data can be calculated for several distributions. Less the value of lacking of fit of the data will give better model than others.

Data: The data set represents the monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010.This data set is selected as it shows the same pattern as the density function of the developed distribution.The data was extracted from Nassar and Nada (2011) [16].Summary of the data set is given below

Table 1. Descriptive statistics of the selected data

Min	4.10
1st Quartile	8.45
Median	10.60
Mean	13.49
3rd Quartile	16.85
Max	39.20
Variance	64.83
Standard deviation	8.05
Skewness	1.57
Kurtosis	2.08

From Table 1., we have concluded that the given data set is positively skewed and the peak of the curve of the given data is not similar to the mesokurtic curve since it is less than 3.

Table 2. Comparison

Model	Estimates					Goodness of fit criteria		
	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	-2ℓ	AIC	BIC
Three parameter TED	0.10	0.10	-	-	-0.90	-16145.44	-16139.44	-16133.21
TGIWD	-	0.02	2.26	0.03	-0.90	302.42	310.42	318.73
TEED	-	5.44	0.16	-	0.42	381.20	387.20	393.43
Two parameter TED	-	9.31	-	-	-0.90	400.70	404.70	408.86
ED	-	13.49	-	-	-	425.01	427.01	429.09

7CONCLUSION

In this paper, a new generalized distribution is proposed by the transmutation technique. It is termed as three parameter TED. Some statistical properties are studied. Parameters are estimated by the method of Maximum Likelihood Estimation (MLE). The applicability of the proposed distribution is studied by means of real set of data.

From this study, we may come to conclusion that the three parameter transmuted exponential distribution provides better fit than one parameter exponential distribution (ED) and two parameter transmuted exponential distribution (TED), transmuted generalized inverse Weibull distribution (TGIWD) and transmuted exponentiated exponential distribution (TEED). Since it has lower value of -2ℓ= -16145.44, AIC= -16139.44 and BIC= -16133.21than that of other mentioned distributions.

Some future works related to this studies are given below

1. To study the applicability of this developed distribution with various real data set.
2. To study the comparison of parameter estimation methods between maximum likelihood estimation procedure and TL moments and L moments procedure for this developed distribution
3. It is feasible to calculate the percentage points of order statistics using various values of the parameter.

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