Convergence of Baskakov Durrmeyer operators in the reverse order of *q*-Analogue

Abstract: This research paper is an introduction to a new type of analogue named as $\mathbb{Q}$ -analogue for well-known Baskakov Durrmeyer operators. This new type of analogue is considered as reverse order of q-analogue. In this paper, we establish the direct approximation theorem, a weighted approximation theorem followed by the estimations of the rate of convergence of these new type of operators for functions of polynomial growth on the interval $[0,\infty)$.

Keywords: Baskakov Durrmeyer operators, Direct approximation theorem, Linear positive operators, Rate of convergence, Weighted-approximation.

2020AMS Subject Classification: 41A25, 41A30.

1. Introduction

"In the theory of approximation, the quantum calculus has been studied for a long time. Quantum calculus was started by the well-known mathematician Lupus [9] when he firstly proposed q-variant of the Bernstein polynomials. T. Kim gave his valuable contribution on q-type of polynomial" [7], [8]. "In the same notions higher valued types of results on q-analogue of linear positive operators were obtained by Garg S. [5], Sharma-Garg [16] and many other authors [13] etc". After that several papers have been dealt with (p, q)-calculus (post-quantum calculus), which is an advanced extension of quantum calculus. Mursaleen et al. [12] introduced "the Bernstein polynomials using (p, q)-calculus, which was further improved in" [10]. "(p, q)-calculus was further studied by the classical work" of Sadjang [14], Sahai-Yadav [15]. "A lot of work on (p, q)-version of linear positive operators" has been published in Acar et al. [1] [2], Aral-Gupta [4], V. Gupta [6], Mursaleen et al. [11] etc. Now we define new type of variant, i.e., $\mathbb{Q}$-variant, where $0 < \mathbb{Q} \leq 1$. This variant $(\mathbb{Q})$ can be considered as the reverse order of q-variant. First, we give some notations and formulae regarding $\mathbb{Q}$-variant as-

$$[n]_{\mathbb{Q}} = \frac{\mathbb{Q}^n - 1}{\mathbb{Q} - 1}, \qquad n = 0,1,2,\dots, [0]_{\mathbb{Q}} = 0$$

$$[n]_{\mathbb{Q}}! = \prod_{k=1}^{n} [k]_{\mathbb{Q}}, \qquad n \geq 1, \qquad [0]_{\mathbb{Q}}! = 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Q}} = \frac{[n]_{\mathbb{Q}}!}{[k]_{\mathbb{Q}}!.\,[n-k]_{\mathbb{Q}}!}, \qquad 0 \leq k \leq n$$

$$D_{\mathbb{Q}} f(x) = \frac{f(\mathbb{Q}x) - f(x)}{(\mathbb{Q}-1)x}, \qquad x \neq 0$$

$$(x \oplus y)_{\mathbb{Q}}^n = (x+y)(\mathbb{Q}x+y)(\mathbb{Q}^2x+y)\dots\dots(\mathbb{Q}^{n-1}x+y),$$

$$(x \ominus y)_{\mathbb{Q}}^{n} = (x-y)(\mathbb{Q}x-y)(\mathbb{Q}^2x-y)\ldots\ldots(\mathbb{Q}^{n-1}x-y),$$

$$B_{\mathbb{Q}}(m,n) = \mathbb{Q}^{\frac{n}{2}} \int_0^{\infty/A} \frac{x^{m-1}}{(1\oplus x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}t, \qquad m,n \in N$$

$$\Gamma_{\mathbb{Q}}(n+1) = \frac{(\mathbb{Q}\ominus 1)_{\mathbb{Q}}^{n}}{(\mathbb{Q}-1)^n} = [n]_{\mathbb{Q}}!, \qquad 0 < \mathbb{Q} < 1$$

Preposition 1: The relation between $\mathbb{Q}$ -Beta and $\mathbb{Q}$ -Gamma functions can be defined as-

$$B_{\mathbb{Q}}(m,n) = \mathbb{Q}^{-\frac{m(m+1)}{2}} \frac{\Gamma_{\mathbb{Q}} m \Gamma_{\mathbb{Q}} n}{\Gamma_{\mathbb{Q}}(m+n)}$$

Preposition 2: The $\mathbb{Q}$ -integration by parts is defined as-

$$\int_a^b g(x) D_{\mathbb{Q}} h(x) d_{\mathbb{Q}}x = g(b)h(b) - g(a)h(a) - \int_a^b h(x) D_{\mathbb{Q}} g(x) d_{\mathbb{Q}}x$$

To approximate Lebesgue integrable function on the interval $[0,\infty)$, Agrawal-Thamar [3] introduced the following operators, which is an extension of Srivastava-Gupta operators [17],

$$\chi_n(f,x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t) f(t)dt + s_{n,0}(x)f(0) \quad \ldots.(1.1)$$

where $s_{n,v}(x) = \begin{bmatrix} n+v-1 \\ v \end{bmatrix} \frac{x^v}{(1+x)^{n+v}}$.

Now we introduce a new type of variant $\mathbb{Q}$-analogue of genuine Baskakov-Durrmeyer operators for $[0,\infty)$ and the operators are defined as-

$$\chi_n^{\mathbb{Q}}(f,x) = [n-1]_{\mathbb{Q}} \sum_{v=1}^{\infty} s_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{(n-1)^2+v} \int_0^{\infty} s_{n,v}^{\mathbb{Q}}(t) f(t)dt + s_{n,v}^{\mathbb{Q}}(x) f(0) \quad \ldots(1.2)$$

where $s_{n,v}^{\mathbb{Q}}(x) = \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{x^{n-1}}{(1\oplus x)_{\mathbb{Q}}^{n+v}}$.

It can be noted here, if we put $\mathbb{Q} = 1$, we get well known Baskakov Durrmeyer operators (1.1).

2. Auxiliary Results

In this section, we establish some basic results to prove our main theorems.

Lemma 1. For $x \in [0,\infty)$ and $0 < \mathbb{Q} \leq 1$, we have

$$\chi_n^{\mathbb{Q}}(1,x) = 1, \qquad \chi_n^{\mathbb{Q}}(t,x) = \frac{[n]_p}{[n-2]_{\mathbb{Q}}} x,$$

$$\chi_n^{\mathbb{Q}}(t^2,x) = \frac{[n]_{\mathbb{Q}}[n+1]_{\mathbb{Q}}}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} x^2 + \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{\mathbb{Q}^{-n+4}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} x$$

Lemma 2. For $0 < \mathbb{Q} \leq 1$ we have the following explicit formulae for the central moments

1. $\chi_n^{\mathbb{Q}}(t-x,x) = \left(\frac{[n]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} - 1\right)x$
2. $\chi_n^{\mathbb{Q}}((t-x)^2,x) = \left(\frac{[n]_{\mathbb{Q}}[n+1]_{\mathbb{Q}}}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} - \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} + 1\right)x^2 + \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{\mathbb{Q}^{-n+4}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}x$

Remark 1: For $0 < \mathbb{Q} \leq 1$, we may have $\lim_{n\to\infty}[n]_{\mathbb{Q}} = \frac{1}{1-\mathbb{Q}}$.
To find the convergence of the mentioned operators (1.2), we consider $\mathbb{Q} \equiv \mathbb{Q}_n$ such that $0 < \mathbb{Q}_n \leq 1$ and for sufficiently large *n*, $\mathbb{Q}_n \to 1$, $[n]_{\mathbb{Q}_n} \to \infty$.

3. Main Results

Definition 1: Let $C_{x^2}[0,\infty)$ be the class of all functions *f* defined on some positive real axis and satisfying $|f(x)| \leq C(1+x^2)$, where C is a positive constant depending on *f*. Here $C_{x^2}[0,\infty)$ is a subspace of the space of all functions $f \in C_{x^2}^*[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The class $C_{x^2}^*[0,\infty)$ is endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty}\frac{|f(x)|}{1+x^2}$$

Theorem 3.1. For $0 < \mathbb{Q} \leq 1$, let $\mathbb{Q} \equiv \mathbb{Q}_n$ such that $0 < \mathbb{Q}_n \leq 1$ and for sufficiently large *n*, $\mathbb{Q}_n \to 1$ then for each $f \in C_{x^2}^*[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|\chi_n^{\mathbb{Q}_n}(f(t),x) - f\right\|_{x^2} = 0. \qquad \text{... (3.1)}$$

Proof. To prove this theorem, it is sufficient to show that

$$\lim_{n\to\infty}\left\|\chi_n^{\mathbb{Q}}(t^k,x) - x^k\right\|_{x^2} = 0, \quad k = 0,1,2. \qquad \text{... (3.2)}$$

For $k = 0$, we have $\chi_n^{\mathbb{Q}_n}(1,x) = 1$. Therefore relation (3.1) is true for $k = 0$. Now, for $k = 1$, from Lemma 1 and above definition

$$\left\|\chi_n^{\mathbb{Q}_n}(t,x) - x\right\|_{x^2} = \sup_{x\in[0,\infty)}\frac{\left|\chi_n^{\mathbb{Q}_n}(t,x) - x\right|}{1+x^2}$$

$$\leq \left\{\frac{[n]_{\mathbb{Q}_n}}{\mathbb{Q}_n[n-2]_{\mathbb{Q}_n}} - 1\right\}\sup_{x\in[0,\infty)}\frac{x}{1+x^2} = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left\|\chi_n^{\mathbb{Q}_n}(t,x) - x\right\|_{x^2} = 0$$

Hence (3.1) is also true for $k = 1$. To check for $k = 2$, we proceed as

$$\left\|\chi_n^{\mathbb{Q}_n}(t^2,x) - x^2\right\|_{x^2} = \sup_{x\in[0,\infty)}\frac{\left|\chi_n^{\mathbb{Q}_n}(t^2,x) - x^2\right|}{1+x^2}$$

$$\leq \left\{\frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - 1\right\} \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}$$

$$+ \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^{-n+4}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} \sup_{x\in[0,\infty)} \frac{x}{1+x^2}$$

$$= \frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - 1$$

$$\Rightarrow \qquad \lim_{n\to\infty} \left\| \chi_n^{\mathbb{Q}_n}(t^2,x) - x^2 \right\|_{x^2} = 0$$

by using Remark 1. Hence (3.2) is proved for all $k = 0,1,2$ and so the theorem.

Definition 2: Let $C_B[0,\infty)$ be the space of all real valued uniformly continuous and bounded function f on the interval $[0,\infty)$. For $f \in C_B[0,\infty)$ the Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\|; g \in C_B^2[0,\infty)\}.$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$. By Devore and Lorentz [6], there exists
an absolute constant $A > 0$ such that

$$K_2(f,\delta) = A\omega_2(f,\sqrt{\delta})$$

where ω_2 is the second order modulus of continuity defined by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<|h|<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

and the usual modulus of continuity is given by

$$\omega(f,\sqrt{\delta}) = \sup_{0<|h|<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|$$

Theorem 3.2. Let $f \in C_B[0,\infty)$ and $x \in [0,\infty)$ then there exists a constant $A > 0$ such that

$$\left\| \chi_n^{\mathbb{Q}_n}(f(t),x) - f \right\|_{x^2} \leq A\omega_2\left(f, \sqrt{\delta_n^{\mathbb{Q}_n}(x)}\right)$$

where

$$\delta_n^{\mathbb{Q}_n}(x) = \left(\frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n[n-2]_{\mathbb{Q}_n}} + 1\right)x^2 + \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^{-n+4}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} x$$

Proof. Let $g \in C_B^2[0,\infty)$, then by Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-m)g''(m)dm, \quad t \in [0,\infty)$$

$$\Rightarrow \qquad \chi_n^{\mathbb{Q}_n}(g(t),x) - g(x) = \chi_n^{\mathbb{Q}_n}\left(\int_x^t (t-m)g''(m)dm, x\right)$$

$$\Rightarrow \quad \left|\chi_n^{\mathbb{Q}_n}(g(t),x) - g(x)\right| \leq \chi_n^{\mathbb{Q}_n}((t-x)^2,x)\|g''\|$$

$$= \left[\left(\frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n[n-2]_{\mathbb{Q}_n}} + 1\right)x^2 + \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^{-n+4}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} x\right] \|g''\|$$

From Lemma 1, we can conclude that

$$\chi_n^{\mathbb{Q}_n}(f,x) \le \|f\|$$

therefore, we consider that

$$\left|\chi_n^{\mathbb{Q}_n}(f(t),x) - f\right| \le \left|\chi_n^{\mathbb{Q}_n}(f-g,x) - (f-g)(x)\right| + \left|\chi_n^{\mathbb{Q}_n}(g(t),x) - g(x)\right|$$

$$\le 2\|f-g\| + \left[\begin{array}{c}\left(\frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n{}^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n[n-2]_{\mathbb{Q}_n}} + 1\right)x^2 \\ + \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n{}^{-n+4}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}}x\end{array}\right]\|g''\|$$

Taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and applying the Peetre's K-functional, we get the required result, i.e.

$$\left\|\chi_n^{\mathbb{Q}_n}(f(t),x) - f\right\|_{x^2} \le A\omega_2\left(f, \sqrt{\delta_n^{\mathbb{Q}_n}(x)}\right)$$

Hence the proof of theorem has been completed.

4. Conclusion

By this paper we have introduced a new type of analogue of linear positive operators. This new study will give a new direction in the study of summation integral type operators in approximation theory. These operators also can be used for several type of statistical distribution functions and other functions such as Szasz, Beta and Baskakov basis functions etc. Researchers can obtain furthermore results with interest.

Conflict of Interest: I have no conflict of Interest.

Data Availability Statement: All data generated or analyzed during this study are included in the current article and its supplementary information.

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