

Original Research Article

Convergence of Baskakov Durrmeyer operators in the reverse order of q -Analogue

Abstract: This research paper is an introduction to a new type of analogue named as \mathbb{Q} -analogue for well-known Baskakov Durrmeyer operators. This new type of analogue is considered as reverse order of q -analogue. In this paper we establish the direct approximation theorem, a weighted approximation theorem followed by the estimations of the rate of convergence of these new type of operators for functions of polynomial growth on the interval $[0, \infty)$.

Keywords: Baskakov operators, Direct approximation theorem, Rate of convergence, Reverse order of q -variant, Weighted-approximation

2020AMS Subject Classification: 41A25, 41A30.

1. Introduction

In the theory of approximation, the quantum calculus has been studied for a long time. Quantum calculus was started by the well-known mathematician Lupus [8] when he first proposed the q -variant of the Bernstein polynomials. T. Kim gave his contribution on q -type of polynomial in [6], [7]. In the same notions higher valued types of results on q -analogue of linear positive operators were obtained by Garg S. [4], Sharma-Garg [18] and many other authors [15], [19] etc. After that several papers have been dealt with (p, q) -calculus (post-quantum calculus), which is an advanced extension of quantum calculus. Mursaleen et al. [12], introduced the Bernstein polynomials using (p, q) -calculus, which was further improved in [11]. (p, q) -calculus was introduced by the classical work of Sahai-Yadav [17]. A lot of work on (p, q) -version of linear positive operators has been published in Acer et al. [1] [2], Aral-Gupta [3], Gupta [5], Mursaleen et al. [10], Sadjang [16] etc. Now we define new type of variant, i.e., \mathbb{Q} -variant, where $0 < \mathbb{Q} \leq 1$. This variant (\mathbb{Q}) can be considered as the reverse order of q -variant. Some notations and formulae regarding \mathbb{Q} -variant are given as-

$$[n]_{\mathbb{Q}} = \frac{\mathbb{Q}^n - 1}{\mathbb{Q} - 1}, \quad n = 0, 1, 2, \dots, [0]_{\mathbb{Q}} = 0$$

$$[n]_{\mathbb{Q}}! = \prod_{k=1}^n [k]_{\mathbb{Q}}, \quad n \geq 1, \quad [0]_{\mathbb{Q}}! = 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Q}} = \frac{[n]_{\mathbb{Q}}!}{[k]_{\mathbb{Q}}! [n-k]_{\mathbb{Q}}!}, \quad 0 \leq k \leq n$$

$$D_{\mathbb{Q}}f(x) = \frac{f(\mathbb{Q}x) - f(x)}{(\mathbb{Q} - 1)x}, \quad x \neq 0$$

$$(x \oplus y)_{\mathbb{Q}}^n = (x + y)(\mathbb{Q}x + y)(\mathbb{Q}^2x + y) \dots (\mathbb{Q}^{n-1}x + y),$$

$$(x \ominus y)_{\mathbb{Q}}^n = (x - y)(\mathbb{Q}x - y)(\mathbb{Q}^2x - y) \dots (\mathbb{Q}^{n-1}x - y),$$

$$B_{\mathbb{Q}}(m, n) = \mathbb{Q}^{\frac{n}{2}} \int_0^{\infty/A} \frac{x^{m-1}}{(1 \oplus x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}t, \quad m, n \in \mathbb{N}$$

$$\Gamma_{\mathbb{Q}}(n + 1) = \frac{(\mathbb{Q} \ominus 1)_{\mathbb{Q}}^n}{(\mathbb{Q} - 1)^n} = [n]_{\mathbb{Q}}!, \quad 0 < \mathbb{Q} < 1$$

Proposition 1: The relation between \mathbb{Q} -Beta and \mathbb{Q} -Gamma functions can be defined as-

$$B_{\mathbb{Q}}(m, n) = \mathbb{Q}^{-\frac{m(m+1)}{2}} \frac{\Gamma_{\mathbb{Q}}m\Gamma_{\mathbb{Q}}n}{\Gamma_{\mathbb{Q}}(m+n)}$$

Proposition 2: The \mathbb{Q} -integration by parts is defined as-

$$\int_a^b g(x)D_{\mathbb{Q}}h(x)d_{\mathbb{Q}}x = g(b)h(b) - g(a)h(a) - \int_a^b h(x)D_{\mathbb{Q}}g(x)d_{\mathbb{Q}}x$$

To approximate Lebesgue integrable function on the interval $[0, \infty)$, Agrawal and Thamar [4] introduced the following operators, which is an extension of Srivastava-Gupta operators [18],

$$\chi_n(f, x) = (n - 1) \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t)f(t)dt + s_{n,0}(x)f(0)$$

where $s_{n,v}(x) = \left[\begin{matrix} n+v-1 \\ v \end{matrix} \right] \frac{x^v}{(1+x)^{n+v}}$.

Now we introduce a new type of variant \mathbb{Q} -analogue of genuine Baskakov-Durrmeyer operators for $[0, \infty)$ and the operators are defined as-

$$\chi_n^{\mathbb{Q}}(f, x) = [n - 1]_{\mathbb{Q}} \sum_{v=1}^{\infty} s_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{(n-1)^2+v} \int_0^{\infty} s_{n,v}^{\mathbb{Q}}(t)f(t)dt + s_{n,v}^{\mathbb{Q}}(x)f(0)$$

where $s_{n,v}^{\mathbb{Q}}(x) = \left[\begin{matrix} n+v-1 \\ v \end{matrix} \right]_{\mathbb{Q}} \frac{x^{n-1}}{(1 \oplus x)_{\mathbb{Q}}^{n+v}}$.

It can be noted here, if we put $\mathbb{Q} = 1$, we get well known Baskakov Durrmeyer operators.

2. Auxiliary Results

In this section, we establish some basic results to prove our main theorems.

Lemma 1. For $x \in [0, \infty)$ and $0 < \mathbb{Q} \leq 1$, we have

$$\chi_n^{\mathbb{Q}}(1, x) = 1, \quad \chi_n^{\mathbb{Q}}(t, x) = \frac{[n]_{\mathbb{Q}}}{[n - 2]_{\mathbb{Q}}} x,$$

$$\chi_n^{\mathbb{Q}}(t^2, x) = \frac{[n]_{\mathbb{Q}}[n + 1]_{\mathbb{Q}}}{\mathbb{Q}^2[n - 2]_{\mathbb{Q}}[n - 3]_{\mathbb{Q}}} x^2 + \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{\mathbb{Q}^{-n+4}[n - 2]_{\mathbb{Q}}[n - 3]_{\mathbb{Q}}} x$$

Lemma 2. For $0 < \mathbb{Q} \leq 1$ we have the following explicit formulae for the central moments

1. $\chi_n^{\mathbb{Q}}(t - x, x) = \left(\frac{[n]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} - 1 \right) x$
2. $\chi_n^{\mathbb{Q}}((t - x)^2, x) = \left(\frac{[n]_{\mathbb{Q}}[n+1]_{\mathbb{Q}}}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} - \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} + 1 \right) x^2 + \frac{[n]_{\mathbb{Q}}[2]_{\mathbb{Q}}}{\mathbb{Q}^{-n+4}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} x$

Remark1: For $0 < \mathbb{Q} \leq 1$, we may have $\lim_{n \rightarrow \infty} [n]_{\mathbb{Q}} = \frac{1}{1-\mathbb{Q}}$.

To find the convergence of the mentioned operators, we consider $\mathbb{Q} \equiv \mathbb{Q}_n$ such that $0 < \mathbb{Q}_n \leq 1$ and for sufficiently large n , $\mathbb{Q}_n \rightarrow 1$, $[n]_{\mathbb{Q}_n} \rightarrow \infty$.

3. Main Results

Definition 1: Let $C_{x^2}[0, \infty)$ be the class of all functions f defined on some positive real axis and satisfying $|f(x)| \leq C(1 + x^2)$, where C is a positive constant depending on f . Here $C_{x^2}[0, \infty)$ is a subspace of the space of all functions $f \in C_{x^2}^*[0, \infty)$ for which $\lim_{|x| \rightarrow \infty} \frac{f(x)}{1+x^2}$ is finite. The class $C_{x^2}^*[0, \infty)$ is endowed with the norm

$$\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}$$

Theorem 1. For $0 < \mathbb{Q} \leq 1$, let $\mathbb{Q} \equiv \mathbb{Q}_n$ such that $0 < \mathbb{Q}_n \leq 1$ and for sufficiently large n , $\mathbb{Q}_n \rightarrow 1$ then for each $f \in C_{x^2}^*[0, \infty)$, we have

$$\lim_{n \rightarrow \infty} \|\chi_n^{\mathbb{Q}_n}(f(t), x) - f\|_{x^2} = 0.$$

Proof. To prove this theorem, it is sufficient to show that

$$\lim_{n \rightarrow \infty} \|\chi_n^{\mathbb{Q}}(t^k, x) - x^k\|_{x^2} = 0, \quad k = 0, 1, 2. \quad (3.1)$$

For $k = 0$, we have $\chi_n^{\mathbb{Q}_n}(1, x) = 1$. Therefore relation (3.1) is true for $k = 0$. Now, for $k = 1$, from Lemma 1 and above definition

$$\begin{aligned} \|\chi_n^{\mathbb{Q}_n}(t, x) - x\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|\chi_n^{\mathbb{Q}_n}(t, x) - x|}{1 + x^2} \\ &\leq \left\{ \frac{[n]_{\mathbb{Q}_n}}{[\mathbb{Q}_n][n-2]_{\mathbb{Q}_n}} - 1 \right\} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} = 0 \\ &\Rightarrow \lim_{n \rightarrow \infty} \|\chi_n^{\mathbb{Q}_n}(t, x) - x\|_{x^2} = 0 \end{aligned}$$

Hence (3.1) is also true for $k = 1$. Again, to check $k = 2$, we proceed as

$$\|\chi_n^{\mathbb{Q}_n}(t^2, x) - x^2\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|\chi_n^{\mathbb{Q}_n}(t^2, x) - x^2|}{1 + x^2}$$

$$\begin{aligned}
&\leq \left\{ \frac{[n]_{\mathbb{Q}_n} [n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n^2 [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} - 1 \right\} \sup_{x \in [0, \infty)} \frac{x^2}{1+x^2} \\
&\quad + \frac{[n]_{\mathbb{Q}_n} [2]_{\mathbb{Q}_n}}{\mathbb{Q}_n^{-n+4} [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \\
&= \frac{[n]_{\mathbb{Q}_n} [n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n^2 [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} - 1 \\
\Rightarrow &\quad \lim_{n \rightarrow \infty} \left\| \chi_n^{\mathbb{Q}_n}(t^2, x) - x^2 \right\|_{x^2} = 0
\end{aligned}$$

by using Remark 1. Hence the theorem is proved for all $k = 0, 1, 2$.

Definition 2: Let $C_B[0, \infty)$ be the space of all real valued uniformly continuous and bounded function f on the interval $[0, \infty)$. For $f \in C_B[0, \infty)$ the Peetre's K-functional is defined as

$$K_2(f, \delta) = \inf\{\|f - g\| + \delta \|g''\|; g \in C_B^2[0, \infty)\}.$$

where $\delta > 0$ and $C_B^2[0, \infty) = \{g \in C_B[0, \infty); g', g'' \in C_B[0, \infty)\}$. By Devore and Lorentz [6], there exists

an absolute constant $A > 0$ such that

$$K_2(f, \delta) = A\omega_2(f, \sqrt{\delta})$$

where ω_2 is the second order modulus of continuity defined by

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < |h| < \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

and the usual modulus of continuity is given by

$$\omega(f, \sqrt{\delta}) = \sup_{0 < |h| < \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|$$

Theorem 2. Let $f \in C_B[0, \infty)$ and $x \in [0, \infty)$ then there exists a constant $A > 0$ such that

$$\left\| \chi_n^{\mathbb{Q}_n}(f(t), x) - f \right\|_{x^2} \leq A\omega_2\left(f, \sqrt{\delta_n^{\mathbb{Q}_n}(x)}\right)$$

where

$$\delta_n^{\mathbb{Q}_n}(x) = \left(\frac{[n]_{\mathbb{Q}_n} [n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n^2 [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n} [2]_{\mathbb{Q}_n}}{\mathbb{Q}_n [n-2]_{\mathbb{Q}_n}} + 1 \right) x^2 + \frac{[n]_{\mathbb{Q}_n} [2]_{\mathbb{Q}_n}}{\mathbb{Q}_n^{-n+4} [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} x$$

Proof. Let $g \in C_B^2[0, \infty)$, then by Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-m)g''(m)dm, \quad t \in [0, \infty)$$

$$\Rightarrow \chi_n^{\mathbb{Q}_n}(g(t), x) - g(x) = \chi_n^{\mathbb{Q}_n}\left(\int_x^t (t-m)g''(m)dm, x\right)$$

$$\Rightarrow \left| \chi_n^{\mathbb{Q}_n}(g(t), x) - g(x) \right| \leq \chi_n^{\mathbb{Q}_n}((t-x)^2, x) \|g''\|$$

$$= \left[\left(\frac{[n]_{\mathbb{Q}_n} [n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n^2 [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n} [2]_{\mathbb{Q}_n}}{\mathbb{Q}_n [n-2]_{\mathbb{Q}_n}} + 1 \right) x^2 + \frac{[n]_{\mathbb{Q}_n} [2]_{\mathbb{Q}_n}}{\mathbb{Q}_n^{-n+4} [n-2]_{\mathbb{Q}_n} [n-3]_{\mathbb{Q}_n}} x \right] \|g''\|$$

From Lemma 1, we can conclude that

$$\chi_n^{\mathbb{Q}_n}(f, x) \leq \|f\|.$$

Now we consider

$$\begin{aligned} \left| \chi_n^{\mathbb{Q}_n}(f(t), x) - f \right| &\leq \left| \chi_n^{\mathbb{Q}_n}(f - g, x) - (f - g)(x) \right| + \left| \chi_n^{\mathbb{Q}_n}(g(t), x) - g(x) \right| \\ &\leq 2\|f - g\| + \left[\left(\frac{[n]_{\mathbb{Q}_n}[n+1]_{\mathbb{Q}_n}}{\mathbb{Q}_n^2[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} - \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n[n-2]_{\mathbb{Q}_n}} + 1 \right) x^2 \right. \\ &\quad \left. + \frac{[n]_{\mathbb{Q}_n}[2]_{\mathbb{Q}_n}}{\mathbb{Q}_n^{-n+4}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} x \right] \|g''\|_9 \end{aligned}$$

Taking the infimum on the right hand side over all $g \in C_B^2[0, \infty)$ and applying the Peetre's K -functional, we get the required result, i.e.

$$\left\| \chi_n^{\mathbb{Q}_n}(f(t), x) - f \right\|_{x^2} \leq A\omega_2 \left(f, \sqrt{\delta_n^{\mathbb{Q}_n}(x)} \right)$$

Data Availability Statement: All data generated or analyzed during this study are included in the current article and its supplementary information.

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