Original Research Article

CLOSED RELATIONS AND LYAPUNOV FUNCTIONS FOR DYNAMICAL POLYSYSTEMS

Abstract. This work follows the ideas of E. Akin in an attempt to ease the construction of strict Lyapunov functions for dynamical polysystems by means of closed relations. A "best hope" type of result is presented.

Keywords closed, relation, polysystem, Lyapunov

1. Introduction

The notion of *dynamical polysystem* appeared in the 1970's, being introduced by C. Lobry, [2]. It had the following meaning: a dynamical polysystem on a manifold M is a family

$$\mathcal{F}_{pc} = \{\mathcal{F}(\cdot, u) : u \in \mathcal{U}_{pc}\}$$

of smooth vector fields depending on a piecewise constant parameter u, called *input*. A similar meaning was given to dynamical polysystems in the work of J. Tsinias and N. Kalouptsidis, [3].

In this paper, a dynamical polysystem is regarded in a slightly more general way, as a family of continuous dynamical systems, all defined on the same metric space X, not necessarily by means of differential equations. The analogy between dynamical polysystems and control systems with piecewise constant inputs is quite natural. Intuitively, a motion in a dynamical polysystem means starting at a point $x \in X$, traveling for a time t_1 according to a dynamical system Φ_1, then switching to another dynamical system Φ_2 and traveling for a time t_2, and so forth. This work is following some ideas of ([1], 1993).

2. Definitions

Consider a family $\mathcal{F}$ of continuous dynamical systems, all defined on a metric space X. For any $\phi \in \mathcal{F}$ and $t \in \mathbb{R}$, $\phi_t(x) = \phi(t, x)$ defines a homeomorphism ϕ_t on X, having inverse ϕ_{-t}.

Definition 1. *Let $\mathcal{G}$ be the subgroup of $(\mathbb{R} \times Homeo(X), (+, \circ))$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \in \mathbb{R}\}$. The pair $(\mathcal{G}, X)$ is called a* ***dynamical polysystem*** *on X. The* ***accessibility semigroup*** *of*

$\mathcal{G}$, denoted by $\mathcal{S}$, is the subsemigroup of $\mathcal{G}$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \geq 0\}$. The pair $(\mathcal{S}, X)$ is called the ***accessibility polysystem*** *on X generated by $\mathcal{F}$.*

Remark 1. *An element of $\mathcal{G}$ has form*

$$g = (t, h) = (t_1 + t_2 + ... + t_k, \phi^1_{t_1} \circ \phi^2_{t_2} \circ ... \circ \phi^k_{t_k}), \tag{1}$$

with $t_i \in \mathbb{R}$ and $\phi^i \in \mathcal{F}$, for $0 \leq i \leq k$.

The polysystem $(\mathcal{G}, X)$ can be considered (and, in fact, is) a $\mathcal{G}$-dynamical system. In what follows, though, notions related to dynamical systems in general may be defined or approached differently, given the concern for regarding polysystems in close connection with continuous-time dynamical systems.

3. Preliminaries

This section follows the ideas of E. Akin in an attempt to ease the problem of finding strict Lyapunov functions for polysystems. In order to use these ideas, let us observe that a polysystem can be viewed as a closed relation, in the following sense. Define a closed relation on X by

$$f = \overline{\{(x, gx) \in X \times X : g \in \mathcal{S}_{[0,1]}\}}, \tag{2}$$

where $\mathcal{S}_{[0,1]}$ denotes all elements of $\mathcal{S}$ with time component between 0 and 1. Note that if $y = gx$, with $g \in \mathcal{S}$, then $(x, y) \in f^k$, for some positive integer k.

The facts about closed relations listed below can be found in [1].

Definition 2. *Let X be a metric space and f a closed relation on X.*

A ***Lyapunov function*** *for f is a continuous real-valued function L on X with the property that $L(x) \leq L(y)$ whenever $(x, y) \in f$.*

A point $x \in X$ is ***regular*** *for L if*

$$L(y_1) < L(x) < L(y_2) \text{ whenever } (y_1, x) \in f \text{ and } (x, y_2) \in f$$

and ***critical*** *for L if it is not regular.*

Denote by $|L|$ the set of critical points for L.

Also, $|f|$ denotes the ***cyclic set*** *of f, that is*

$$|f| := \{x \in X : (x, x) \in f\}$$

Definition 3. *Given a metric space X, a closed relation f on X, $x, y \in X$ and $\epsilon > 0$, an $\epsilon-$**chain** from x to y is a sequence of points in X, $x = x_0, x_1, ..., x_n = y$ with the property that*

$$d(x_{i+1}, f(x_i)) < \epsilon, \text{ for all } i \in \{0, ..., n-1\}.$$

Note that in the above definition $d(x_{i+1}, f(x_i))$ refers to the distance from a point to a set, which means, as usually, the infimum of distances from x_{i+1} to every point in $f(x_i)$.

Definition 4. *Given a closed relation f on a metric space X, define the*
***chain relation** $\mathcal{C}f$ associated to f, by*
$(x, y) \in \mathcal{C}f$ if for every $\epsilon > 0$, there exists an ϵ-chain from x to y.

Note that $\mathcal{C}f$ is a closed transitive relation containing f.

Theorem 1. *(Akin,* [1, pp. 33]*) If F is a closed transitive relation on a compact metric space X then there exists a Lyapunov function L for F with $|L| = |F|$.*

Corollary 1. *(Akin,* [1, pp. 34]*) If f is a closed relation on a compact metric space X then there exists a Lyapunov function L for f with $|L| = |\mathcal{C}f|$.*

4. Polysystems viewed as closed relations

Definition 5. *Let X be a metric space and $(\mathcal{S}, X)$ a polysystem, as defined in section 1. A **Lyapunov function** for the polysystem $(\mathcal{S}, X)$ is a continuous real-valued function L on X with $L(x) \leq L(gx)$ for every $x \in X$ and $g \in \mathcal{S}$.*

Remark 2. *If f is defined by 2 and L is a Lyapunov function for f then L is a Lyapunov function for the polysystem $(\mathcal{S}, X)$.*

Proof. Let L be a Lyapunov function for f, let $g \in \mathcal{S}$ and $x \in X$. Writing g as $g = g_1 g_2 ... g_k$, with $g_i \in \mathcal{S}_{[0,1]}$ for all $i \in \{1, 2, ..., k\}$, we have

$$L(gx) = L(g_1 g_2 ... g_k.x) \geq L(g_2 ... g_k.x) \geq ... \geq L(g_k.x) \geq L(x).$$

□

Definition 6. *Given $\epsilon > 0$ and $x, y \in X$, an ϵ-**chain** from x to y in the polysystem $(\mathcal{S}, X)$ is a sequence of pairs $(g_0, x_0), (g_1, x_1), ..., (g_k, x_k)$ in $(\mathcal{S}, X)$ with $x_0 = x, x_k = y$, $g_i \in \mathcal{S}_{[1,\infty)}$ for all i and $d(x_{i+1}, g_i.x_i) < \epsilon$ for all $i \in \{0, 1, ..., k\}$.*

Note that the requirement $g_i \in \mathcal{S}_{[1,\infty)}$ is needed to avoid triviality in constructing $\epsilon-$chains. Without it, any two points in X could be connected through an $\epsilon-$chain, using the mere continuity of actions by elements in $\mathcal{S}$ on X.

Finally, define a chain relation $\mathcal{C}$ for the polysystem $(\mathcal{S}, X)$, by

(3) $(x, y) \in \mathcal{C}$ if for every $\epsilon > 0$ there exists an $\epsilon-$*chain* from x to y,

(in the sense of polysystems).

Definition 7. *A point x in X is said to be **chain-recurrent** (in the sense of polysystems) if $x \in |\mathcal{C}|$, (that is, for every $\epsilon > 0$ there exists an ϵ-chain from x to x).*

Proposition 1. *If f is defined by 2 and $\mathcal{C}$ by 3 then $\mathcal{C} \subset \mathcal{C}f$.*

Proof. Let $(x, y) \in \mathcal{C}$. For $\epsilon > 0$ there exists an $\epsilon - chain$ (in the sense of polysystems) from x to y, $(g_0, x_0), (g_1, x_1), ..., (g_k, x_k)$. Every g_i in this chain can be written as

$$g_i = g_i^{j_1} g_i^{j_2} ... g_i^{j_{k_i}}$$

with $g_i^{j_l} \in \mathcal{S}_{[0,1]}$, for all l. We can construct then an $\epsilon - chain$ from x to y (in the sense of relations), as follows:

$$x = x_0, ..., g_{i-1}x_{i-1}, x_i, g_i^{j_{k_i}} x_i, g_i^{j_{k_i-1}} g_i^{j_{k_i}} x_i, ..., g_i^{j_1} g_i^{j_2} ... g_i^{j_{k_i}} x_i = g_i x_i, x_{i+1}, ..., ..., x_k.$$

It suffices to show now that $d(g_{i-1}x_{i-1}, f(x_i)) < \epsilon$ and $d(g_i^{j_{k_i}} x_i, f(x_i)) < \epsilon$. The first inequality is seen to be satisfied by noting that $d(g_{i-1}x_{i-1}, x_i) < \epsilon$ and $x_i \in f(x_i)$. The second one is true since $g_i^{j_{k_i}} x_i \in f(x_i)$ and so $d(g_i^{j_{k_i}} x_i, f(x_i)) = 0 < \epsilon$.

□

Theorem 2. *If $(\mathcal{S}, X)$ is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|L| = |\mathcal{C}f|$.*

Proof. The theorem follows from Corollary 1. □

Corollary 2. *If $(\mathcal{S}, X)$ is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|\mathcal{C}| \subset |L|$.*

From this Corollary we draw the conclusion that, in trying to obtain a strict Lyapunov function L for the polysystem $(\mathcal{S}, X)$, the most one can hope is that the critical points for L are precisely the chain-recurrent points in the polysystem.

References

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[3] J. Tsinias, N. Kalouptsidis, *Prolongations and Stability Analysis via Lyapunov Functions of Dynamical Polysystems*, Math. Systems Theory 20 (1987), no. 2-3, 215–233.