Original Research Article

Exponentiated Power Lindley-Logarithmic Distribution and its Applications

Abstract

This article proposes a new distribution call the Exponentiated Power Lindley-Logarithmic Distribution for modeling real life data. The quantile function was derived. Maximum likelihood procedure was used for finding the estimate of the parameters of the distribution. The model provides a better fit to lifetime data sets than other competing models.

Keywords— Flexibility, Survival functions, Hazard functions

1 Introduction

In order to improve the modeling of life data, Lindley [1] introduced a mixture model from the exponential and gamma densities. The superiority of the Lindley to other noble life time distributions such as exponential, gamma, Weibull, beta and Akash distributions has been discussed in Ghitany et el [2]. These distributions are referred to as baseline distributions when compared to the generalized forms. Some distributions in the class of Lindley are Chris-Jerry distribution proposed by Onyekwere and Obulezi [3], Ishita distribution by Shanker and Shukla [4], Rani distribution by Shanker [5], Sujatha distribution by Shanker et al [6], Pranav distribution by Shukla [7],Odoma distribution by Odom and Ijeomah [8] and Shukla distribution by Shukla et al [4]. Baseline distributions are often generalized using exponentiation methods proposed by Mudholker and Srivastava [9]. This exponentiation method has been shown to provide better fit and more flexibility than its baseline distribution (see for instance, Nadarajah and Kotz [10]. A further improved version of exponentiation is the Power Lindley distribution proposed by Ghitany et al [11] in which its exponentiation was proposed by Warahena-Liyanage and Pararai [12]. Although, the Lindley, Power Lindley and Eponentiated Power Lindley distributions have found wide application in life time modeling, they however, cannot handle complementary risk problem. Anabike et al [13] stuidied inference on Zubair-Exponential distribution. This study proposes a new distribution known as Exponentiated Power Lindley-Logarithmic (EPLL) distribution for modeling real life data with complementary risk problem in life time data. The rest of the article is organized as follows: section two present the methodology for generating the EPLL distribution, section three discuses the mathematical properties of EPLL distribution, section four contains parameter estimation of EPLL, simulation experiment and comparison of EPLL distribution with other well known life time distribution, while conclusion is presented in section five.

2 Derivation of the Exponentiated Power Lindley-Logarithmic Distribution (EPLLD)

Ghitany et al [2] gave the cumulative distribution function (cdf) and the probability density function of the Lindley distribution respectively by

$$F(x) = 1 - \left[\frac{1+\beta+\beta x}{1+\beta}\right] e^{-\beta x} \tag{1}$$

and

$$f(x) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x}; \quad x, \quad \beta > 0 \tag{2}$$

Later on, Ghitany et al [11] defined the cdf and the pdf of the Power Lindley distribution respectively as

$$F(x) = 1 - \left[\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right] e^{-\beta x^{\alpha}} \tag{3}$$

and

$$f(x) = \frac{\alpha\beta^2}{1+\beta}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\beta x^{\alpha}}; \quad x,\alpha,\beta > 0 \tag{4}$$

Warahena-Liyanage and Pararai [12] defined the cdf and pdf of the Exponentiation Power Lindley distribution with cdf and pdf, respectively by

$$F(X) = \left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega} \tag{5}$$

and

$$f(x) = \frac{\alpha\beta^2\omega}{1+\beta}x^{\alpha-1}\left(1+x^{\alpha}\right)e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1}; \quad x,\alpha,\beta,\omega > 0 \tag{6}$$

Now, suppose that the random variable, , has the Exponentiated Power Lindley distribution with cdf and pdf in equation (5) and (6). Let $X_1, X_2, ..., X_N$ be independent and identically distributed random variables from the Exponentiated Power Lindley distribution. Suppose to be discrete and follows the zero-truncated logarithmic distribution defined by Noack [14] by the probability mass function (pmf) of the form

$$P(N=n) = \frac{\lambda^n}{-n\ln(1-\lambda)}, \quad n = 1,2,..., \quad 0 < \lambda < 1 \tag{7}$$

Let $X_{(n)} = max(X_1, X_2, ..., X_N)$ which is the n^{th} order statistic of the sequence $X_1, X_2, ..., X_N$. Following Pararai et al [15], the cdf of the random variable $X_{(n)}|N=n$, can be expressed by

$$F_{X_{(n)}|N=n}(x) = \left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n}; \quad x,\alpha,\beta,\omega > 0, \quad n \geq 0 \tag{8}$$

Equation (8) is the cdf of the Exponentiated Power Lindley distribution with parameters α, β and ωn. The corresponding pdf is obtained by differentiating equation (8) with respect to x, which yields

$$f_{X_{(n)}|N=n}(x) = \frac{\alpha\beta^2\omega n}{1+\beta}x^{\alpha-1}\left(1+x^{\alpha}\right)e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n} \tag{9}$$

The cdf of the proposed Exponentiated Power Lindley-Logarithmic Distribution (EPLLD) is the marginal cdf of $X_{(n)}$ obtained from equation 8).

The marginal cdf of $X_{(n)}$ is the same as the cdf of the proposed distribution EPLLD. The marginal cdf of $X_{(n)}$ is expressed as

$$F_{EPLL}(x) = \sum_{n=1}^{\infty} P(N=n)F_{X_{(n)}|N=n}(x) \tag{10}$$

Making appropriate substitutions, we have

$$\begin{aligned} F_{EPLL}(x) &= \frac{\lambda^n}{-n\ln(1-\lambda)}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n} \\ &= \frac{1}{-\ln(1-\lambda)}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}^{n} \end{aligned} \tag{11}$$

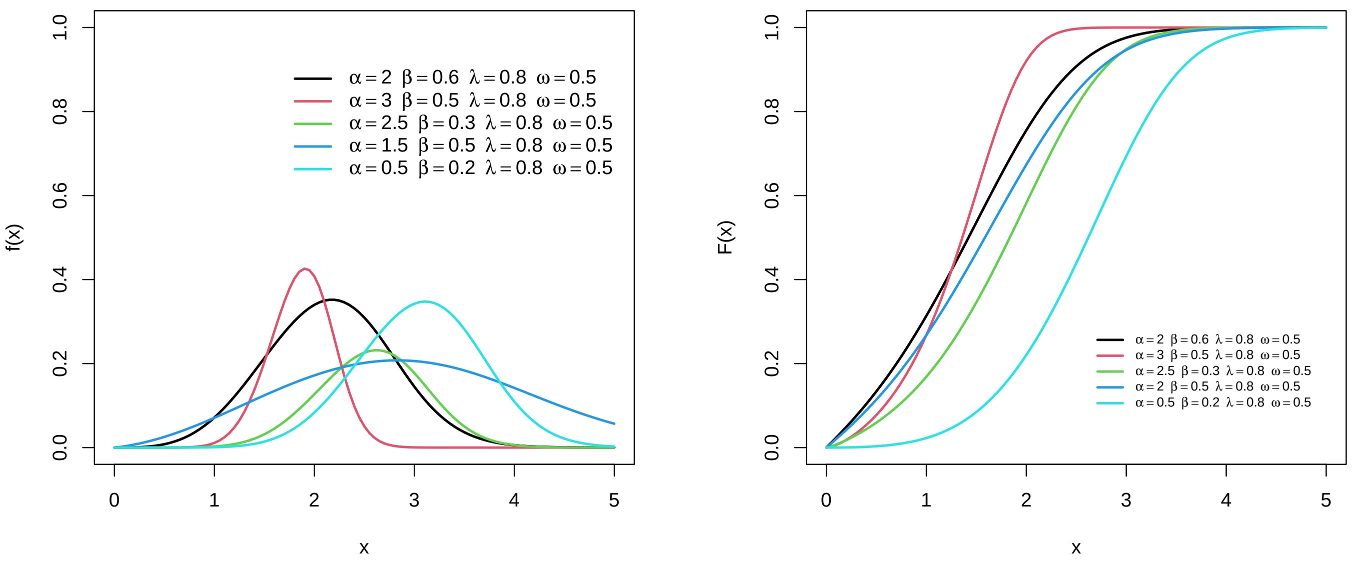
Let $y = \lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}$, notice that $-\ln(1-y) = \sum_{n=1}^{\infty}\frac{y^n}{n}; \quad -1 \leq y \leq 1$. Again, notice that for $0 < \lambda < 1$; $\alpha, \beta, \omega > 0$ $\beta, \alpha, \omega > 0$, y will lie in the intervsl $0 < y < 1$, which is the sub-interval $[-1, 1]$. It follows that we can rewrite equation (11) as follows

$$F_{EPLL}(x) = F_{EPLL}(x) = \frac{\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)}; \quad x,\alpha,\beta,\omega > 0, \quad 0 < \lambda < 1 \tag{12}$$

Equation (12) is the cdf of the proposed distribution EPLLD. Differentiating with respect to x gives the pdf as

$$f_{EPLL}(x) = \frac{\alpha\beta^2\lambda\omega x^{\alpha-1}(1+x^{\alpha})e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\alpha-1}}{-\ln(1-\lambda)\left(1+\beta\right)\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}} \tag{13}$$

where α is the shape parameter (flexibility measure), β is the scale parameter, ω is the shape parameter (extreme value measure) and λ is the shape parameter (competing risk measure).



(a) pdf of EPLLD **(b)** cdf of EPLLD

FIG 1 Graphical representation of EPPLD

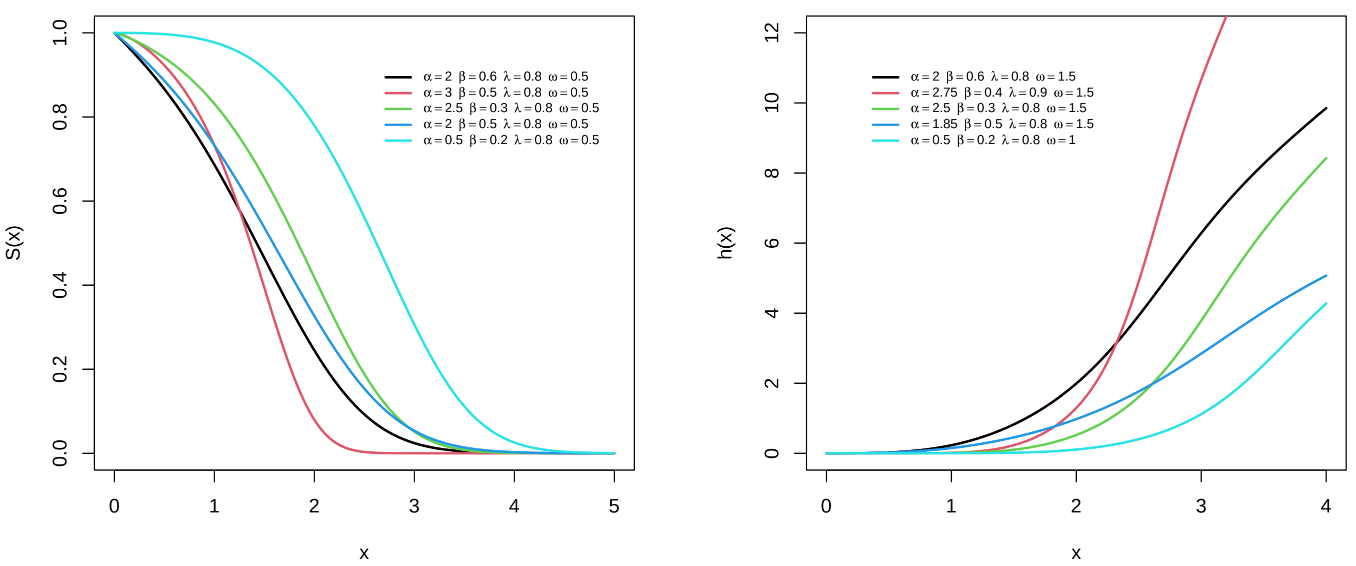
Definition 2.1. Let $X \sim \text{EPLLD}(\alpha,\beta,\lambda,\omega)$, then the survival and hazard rate functions are given as

$$S_{EPLL}(x) = 1 - \frac{\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)} = \frac{\log_e(1-\lambda)-\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)} \tag{14}$$

and

$$h_{EPLL}(x) = \frac{\alpha\beta^2\lambda\omega x^{\alpha-1}(1+x^{\alpha})e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1}}{-(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}\right]A_1(x,\alpha,\beta,\lambda,\omega)}; \quad x>0, \quad \alpha,\beta,\omega>0, \quad 0<\lambda<1 \tag{15}$$

where $A_1(x,\alpha,\beta,\lambda,\omega) = -\log_e(1-\lambda)(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}^{\omega}\right]$



(a) survival function of EPLLD **(b)** hazard rate function of EPLLD

FIG 2 Survival and hazard rate function

3 Statistical Properties of EPLLD

Definition 3.1 (Quantile Function)**.** Let cdf F be continuous and monotonically increasing, then the quantile function of the EPLLD

$$Q(p) = \left[-\frac{1}{\beta} - 1 - \frac{1}{\beta} W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \right]^{\frac{1}{\alpha}} \tag{16}$$

where $W(.)$ is the negative branch of the Lambert-W function. It follows therefore that

$$p = -1 - \beta - \beta [Q(p)]^{\alpha} \tag{17}$$

See Corless et al [16] for details on Lambert-W function.

Proof. Given the cdf of the EPLLD Quantile function is obtained by

$$F_{EPLL}(Q(p)) = p$$

$$\frac{\log_e \left\{ 1 - \lambda \left[1 - \left(\frac{1+\beta+\beta[Q(p)]^{\alpha}}{1+\beta} \right) e^{-\beta[Q(p)]^{\alpha}} \right]^{\omega} \right\}}{\log_e(1-\lambda)} = p \tag{18}$$

$$\Rightarrow \left(\frac{1+\beta+\beta[Q(p)]^{\alpha}}{1+\beta} \right) e^{-\beta[Q(p)]^{\alpha}} = 1 - \left[\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right]^{\frac{1}{\omega}} \tag{19}$$

Let $z(p) = -1 - \beta - \beta[Q(p)]^{\alpha}$. It follows that

$$z(p) e^{z(p)} = -(1+\beta) e^{-(1+\beta)} \left[1 - \left(\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right]$$

and the solution for $z(p)$ is

$$z(p) = W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\}; \quad for \quad 0 < p < 1 \tag{20}$$

It follows that

$$-1 - \beta - \beta[Q(p)]^{\alpha}$$

$$\Rightarrow -\beta[Q(p)]^{\alpha} = 1 + \beta + W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \tag{21}$$

Therefore

$$Q(p) = \left[-\frac{1}{\beta} - 1 - \frac{1}{\beta} W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{p \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \right]^{\frac{1}{\alpha}} \tag{22}$$

□

Definition 3.2. From equation 22, we derive the first three quantiles of the EPLLD as follows

$$\left. \begin{aligned} Q(\tfrac{1}{4}) &= \left[-\frac{1}{\beta} - 1 - \frac{1}{\beta} W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{\frac{1}{4} \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \right]^{\frac{1}{\alpha}} \\ M = Q(\tfrac{1}{2}) &= \left[-\frac{1}{\beta} - 1 - \frac{1}{\beta} W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{\frac{1}{2} \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \right]^{\frac{1}{\alpha}} \\ Q(\tfrac{3}{4}) &= \left[-\frac{1}{\beta} - 1 - \frac{1}{\beta} W \left\{ -(1+\beta) e^{(1+\beta)} \left[1 - \left(\frac{1 + e^{\frac{3}{4} \log_e(1-\lambda)}}{\lambda} \right)^{\frac{1}{\omega}} \right] \right\} \right]^{\frac{1}{\alpha}} \end{aligned} \right\} \tag{23}$$

Where M is the median.

4 Parameter Estimation

Definition 4.1 (Maximum Likelihood Estimation). Let $x_1, x_2, ..., x_n$ be independent random sample of size n which is distributed according to EPLLD, then the likelihood function is obtained as follows

$$\ell = \prod_{i=1}^{n} f(x_i; \psi) \tag{24}$$

where ψ is the vector of parameters and $f(x_i; \psi)$ is the pdf of the proposed EPLLD.

$$\left.\begin{aligned}
\Rightarrow \log_e \ell = \sum_{i=1}^{n} \log_e f(x_i) = \sum_{i=1}^{n} \log_e \left\{ \frac{\alpha\beta^2\lambda x_i^{\alpha-1}(1+x_i^{\alpha})e^{-\beta x_i^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1}}{-\log_e(1-\lambda)(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}^{\omega}\right]} \right\} \\
\log_e \ell = n\log_e \alpha + 2n\log_e \beta + n\log_e \lambda + n\log_e \omega - n\log_e(1+\beta) - n\log_e(-\log_e(1-\lambda)) \\
-\beta\sum_{i=1}^{n} x_i^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log_e x_i + \sum_{i=1}^{n}\log_e(1+x_i^{\alpha}) + (\omega-1)\sum_{i=1}^{n}\log_e\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right] \\
-\sum_{i=1}^{n}\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}
\end{aligned}\right\} \tag{25}$$

Differentiating partially with respect to the parameters where $L = \log_e \ell$

$$\left.\begin{aligned}
\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \beta\sum_{i=1}^{n} x_i^{\alpha}\log_e x_i + \sum_{i=1}^{n}\log_e x_i + \sum_{i=1}^{n}\frac{x_i^{\alpha}\log_e x_i}{(1+x_i^{\alpha})} + (\omega-1)\sum_{i=1}^{n}\frac{\beta\left(1+\beta+\beta x_i^{\alpha}\right)x_i^{\alpha}\log_e x_i e^{-\beta x_i^{\alpha}} - \beta x_i^{\alpha}\log_e x_i}{(1+\beta)\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]} \\
-\sum_{i=1}^{n}\frac{\lambda\omega\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1}\beta\left(1+\beta+\beta x_i^{\alpha}\right)x_i^{\alpha}\log_e x_i e^{-\beta x_i^{\alpha}} - \beta x_i^{\alpha}\log_e x_i}{(1+\beta)\left[1-\lambda\left(1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right)^{\omega-1}\right]}
\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial \beta} = \frac{2n}{\beta} - \frac{n}{1+\beta} - \sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\frac{x_i^{\alpha}(1+\beta)(1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}} - (1+\beta)(1+x_i^{\alpha})e^{-\beta x_i^{\alpha}} + (1+\beta++\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}} \\
-\sum_{i=1}^{n}\frac{\lambda\omega\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1} x_i^{\alpha}(1+\beta)(1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} - \sum_{i=1}^{n}\frac{(1+x_i^{\alpha})(1+\beta)e^{-\beta x_i^{\alpha}} - (1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}}
\end{aligned}\right\} \tag{27}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{(1-\lambda)\log_e(1-\lambda)} + \sum_{i=1}^{n}\frac{\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}}{\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} \tag{28}$$

$$\frac{\partial L}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n}\log_e\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right] + \sum_{i=1}^{n}\frac{\lambda\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}^{\omega}\log_e\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}}{\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} \tag{29}$$

The maximum likelihood estimate ψ is obtained by solving the non-linear systems of equation $U(\psi) = 0$. Since the resulting systems of equation are not in closed form, the solutions are obtained using Newton-Raphson's iterative algorithm implemented in R.

5 Application

In this section, two real data sets are introduced to check the performance of the EPLLD distribution.

The survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal [17] will be used for this application. This real data set is given as

10	33	44	56	59	72	74	77	92	93	96	100	100	102	105	107	107	108
108	108	109	112	113	115	116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196	197	202	213	215	216	222
230	231	240	245	251	253	254	255	278	293	327	342	347	361	402	432	458	555

We fit Exponentiated Power Lindley-Logarithmic Distribution (EPLLD) to the survival times of Guinea pigs injected with different amounts of tubercle bacilli and compare its goodness of fit with those of the Weibull distribution, Power Lindley Distribution (PLD), Lindley Distribution (LD) and Exponentiated Power Lindley Distribution (EPLD). The analytical

Table 1: The Analytical Measures of Fitness and MLE for the parameters of the fitted models using Guinea Pigs data

Dist.	Par.	Estimate	Std. Error	LL	AIC	BIC	K-S	CVM	AD	P-value
EPLLD	a	0.8613	0.3433	-94.2900	196.58	205.6866	0.0893	0.0813	0.5168	0.5826
	b	1.8802	1.2901							
	c	0.4511	0.9206							
	d	4.21752	4.2998							
Weibull	α	1.82542	0.15871	-95.7898	195.5796	200.1329	0.1048	0.1681	1.0074	0.3817
	β	1.99615	0.13631							
PLD	a	1.5344	0.1214	-96.0508	196.1015	200.6548	0.1043	0.1745	1.0543	0.387
	b	0.5998	0.0760							
LD	θ	0.8683	0.0766	-106.9285	215.8569	218.1336	0.2467	0.8792	4.8678	0.0023
EPLD	a	1.01977	0.23582	-93.9693	193.9387	200.7687	0.0874	0.0861	0.5317	9e-16
	b	1.3708	0.5003							
	d	2.8347	1.6033							

measures of fitness are the log-likelihood (LL), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the Kolmogrov-Smirnov (K-S) statistic, the Cramer von mises (CVM) and the Anderson-Darling (AD) statistic. The model that has the largest p-value which must be greater than $\alpha = 0.05$ and the least K-S statistic is the best.

From table 1, we see that the proposed distribution EPLLD has the largest p-value with the minimum K-S statistics hence a better model for the survival times of guinea pigs injected with different amount of tubercle bacilli.

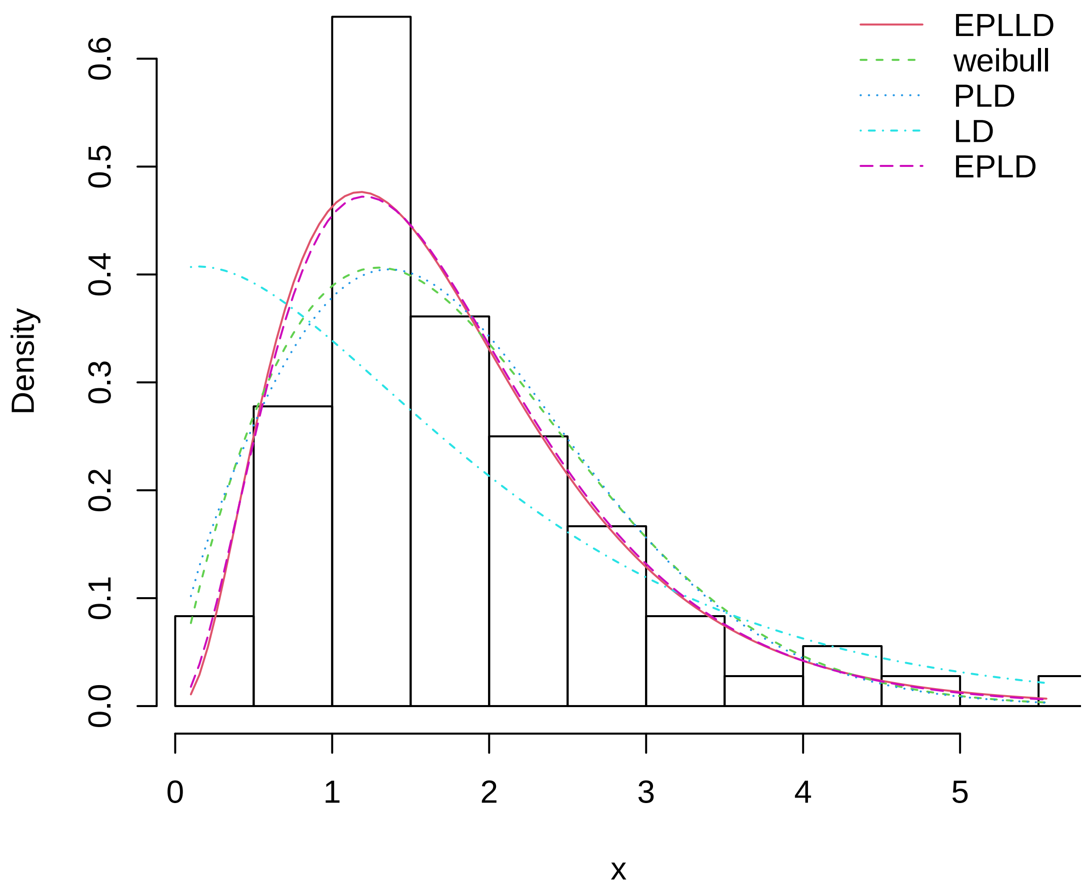


Figure 3: Fitted density plots of the distributions using the survival times of Guinea Pigs

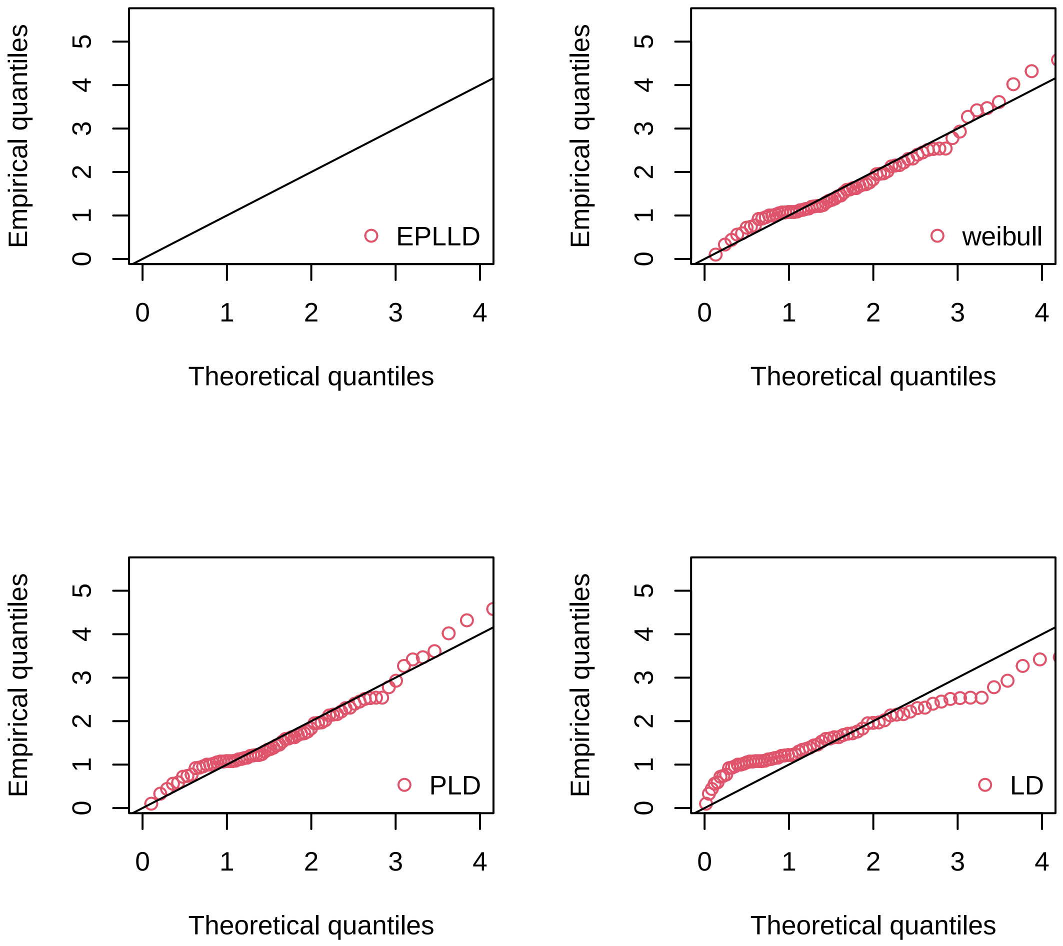


Figure 4: Q-Q plots of the EPLLD, Weibull, PLD and LD using the survival times of Guinea Pigs

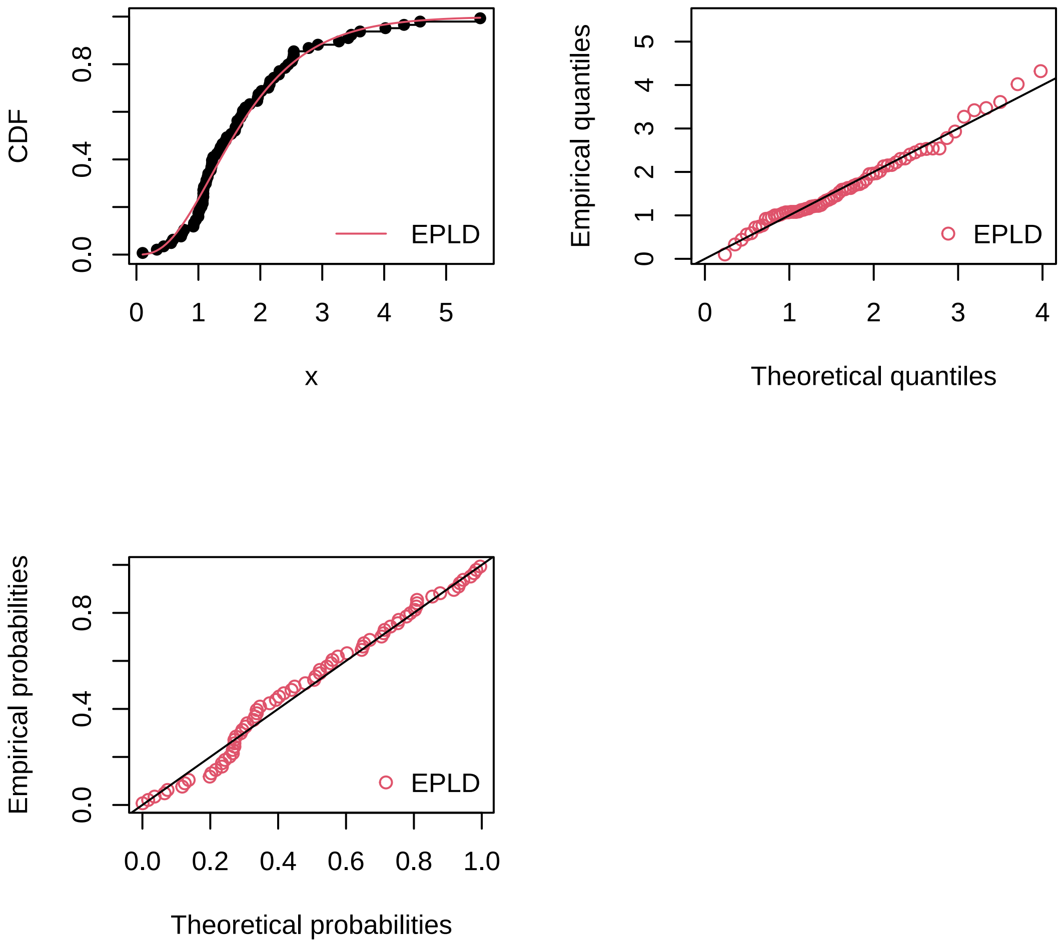


Figure 5: Fitted CDF, Q-Q plot and Probabilities of the EPLD using the survival times of Guinea Pigs

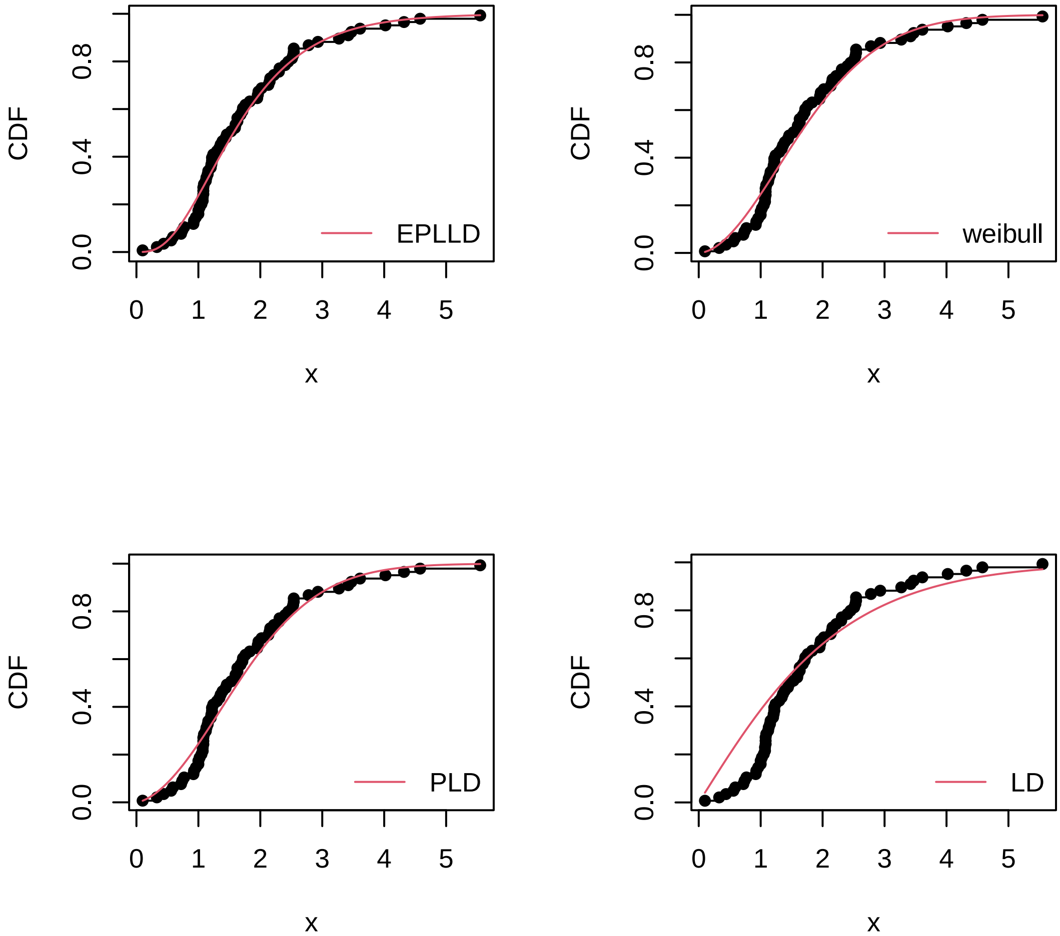


Figure 6: Fitted CDF of EPLLD, Weibull, PLD, LD and EPLD using the survival times of Guinea Pigs

From the figures in 3, 4, 5 and 6, its obvious that the proposed EPLLD has a better fit than the competing distributions based on the survival times of guinea pigs injected with different amount of tubercle bacilli.

The next application is on the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. This data was studied by Owoloko et al [18].

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.9120	0.9836
1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630	1.7746	1.8475	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.1330	2.2100
2.2460	2.2878	2.3203	2.3470	2.3513	2.4951	2.5260	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.0960	

We fit Exponentiated Power Lindley-Logarithmic Distribution (EPLLD) to the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed and compare its goodness of fit with those of the Zubair-Exponential distribution (ZED), Kumaraswamy-Weibull Distribution (KWD)(PLD), Lindley Distribution (LD) and Exponentiated Power Lindley Distribution (EPLD).

From table 2, we see that the proposed distribution EPLLD has the largest p-value with the minimum K-S statistics hence a better model for life of fatigue fracture of Kevlar 373/epoxy. The model also perform better since it has the minimum value of the various criteria used.

Table 2: The Analytical Measures of Fitness and MLE for the parameters of the fitted models using the life of fatigue fracture of Kevlar 373/epoxy

Dist.	Par.	Estimate	Std. Error	LL	AIC	BIC	K-S	CVM	AD	P-value
EPLLD	a	0.9614	0.3389	-121.8748	251.7495	261.0725	0.0996	0.1027	0.6146	0.6036
	b	0.9888	0.8168							
	c	-0.1004	2.3881							
	d	1.5177	0.8010							
ZED	α	-0.6653	2.4804	-122.6466	249.2932	253.9547	0.0881	0.0870	0.5885	4.36e-11
	θ	0.6666	0.3677							
KWD	a	1.8754	2.0690	-122.0763	252.1527	261.4756	0.0970	0.1053	0.6349	1.068e-10
	b	4.4957	19.0745							
	c	0.8491	0.8456							
	l	0.2385	0.6421							
LD	θ	0.07947	0.0679	-123.684	249.368	251.6987	0.1156	0.2652	1.4756	0.2255
EPLD	a	0.9500	0.1935	-121.8757	249.7514	256.7436	0.0993	0.1028	0.6156	1.554e-15
	b	1.0205	0.3589							
	d	1.5357	0.6646							

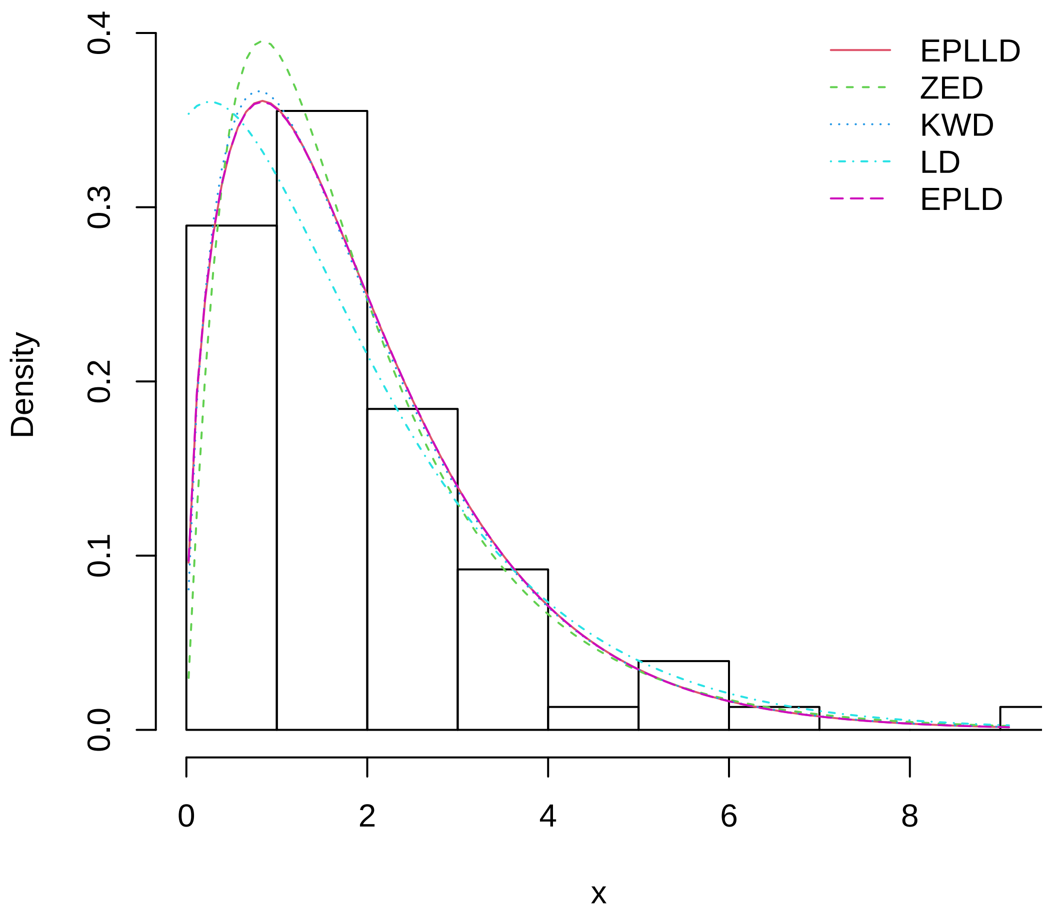


Figure 7: Fitted density plots of the distributions using the life of fatigue fracture of Kevlar 373/epoxy

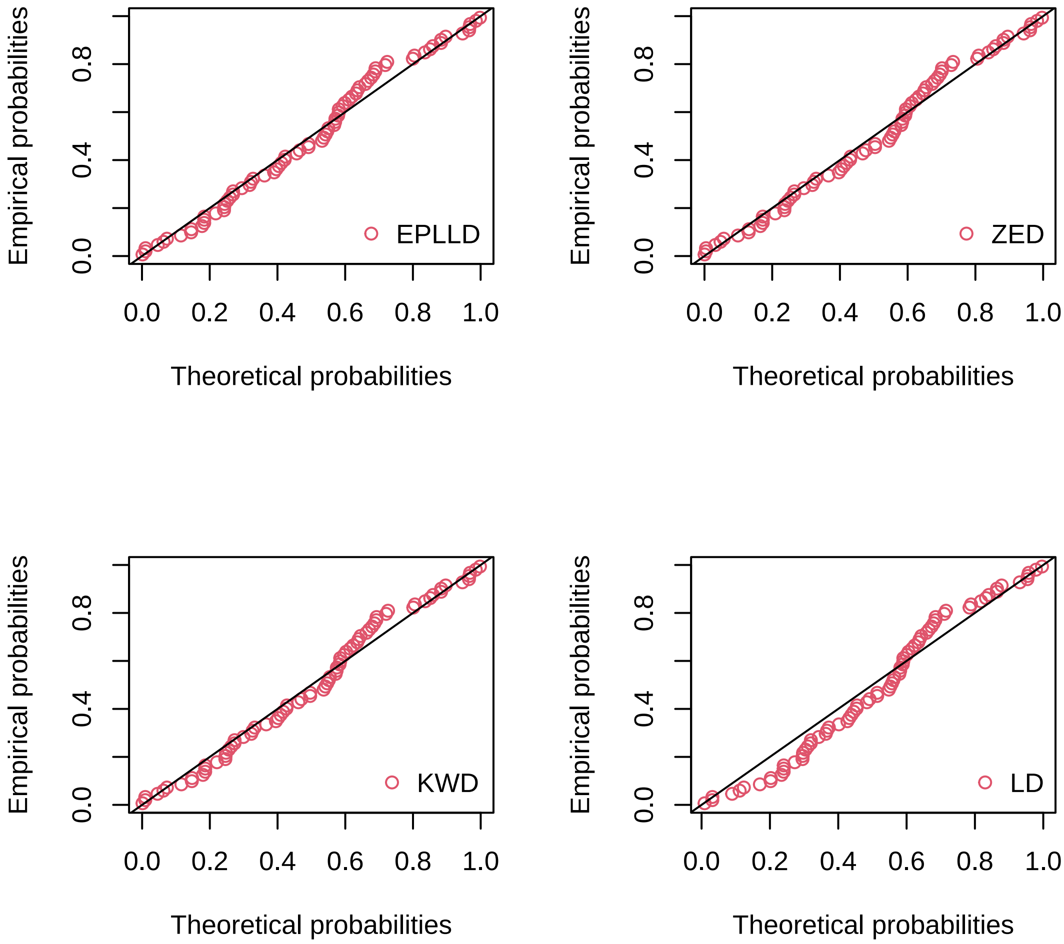


Figure 8: Q-Q plots of the EPLLD, ZED, KWD and LD using the life of fatigue fracture of Kevlar 373/epoxy

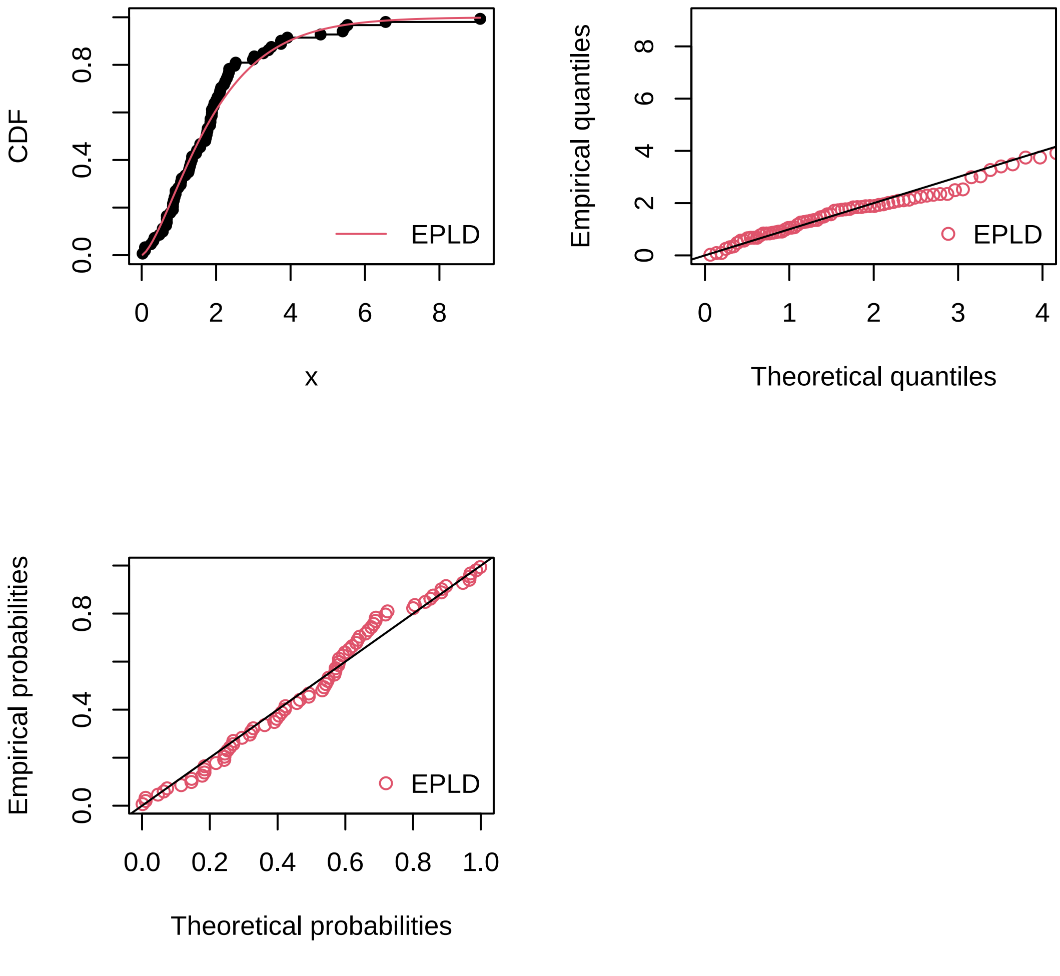


Figure 9: Fitted CDF, Q-Q plot and Probabilities of the EPLD using the life of fatigue fracture of Kevlar 373/epoxy

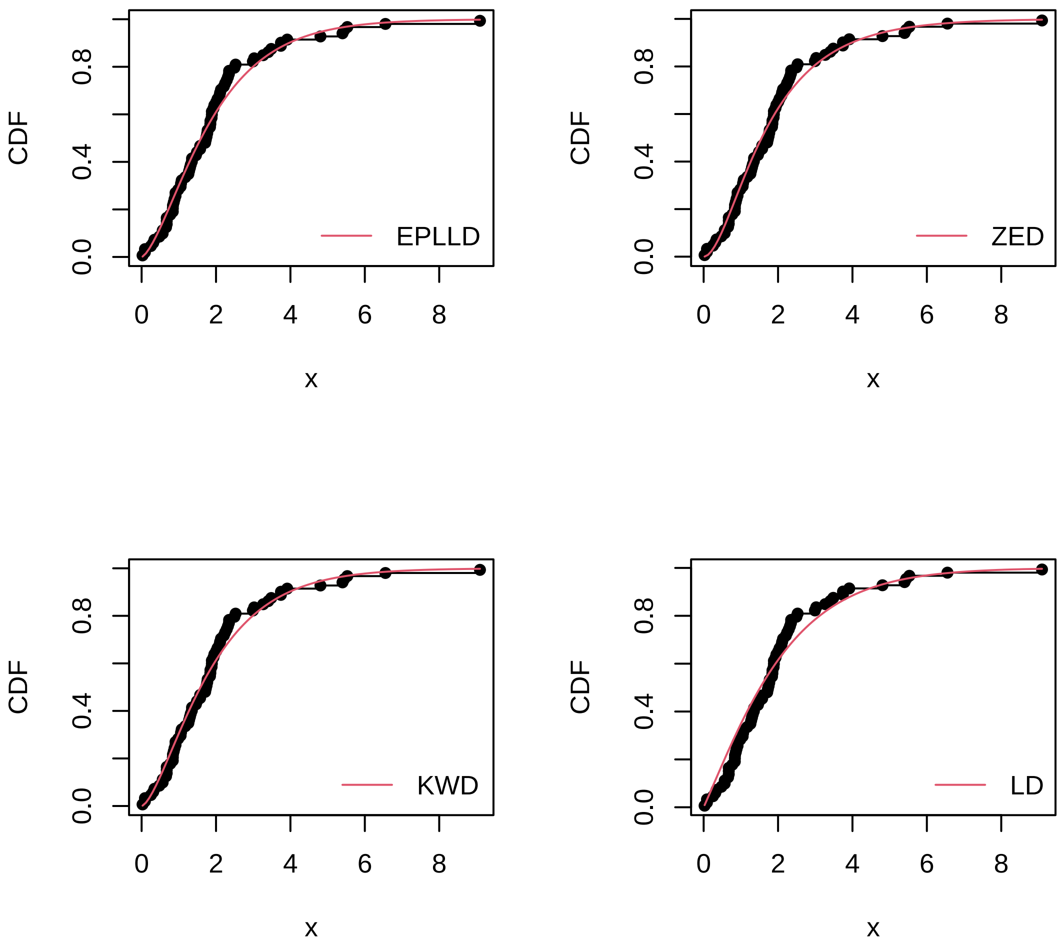


Figure 10: Fitted CDF of EPLLD, ZED, KWD, LD and EPLD using the life of fatigue fracture of Kevlar 373/epoxy

From the figures in 7, 8, 9 and 10, its obvious that the proposed EPLLD has a better fit than the competing distributions based on the life of fatigue fracture of Kevlar 373/epoxy .

6 Conclusion

For the second dataset, the Zubair-Exponential distribution, Kumaraswamy-Weibull distribution and the Exponentiated Power Lindley distribution proved to be very inadequate in fitting the data set. This is supported by their very low p-value of the K-S statistic. The other extended versions of the Lindley distribution namely: the Power Lindley (PL)distribution and the Exponentiated power Lindley (EPL)distribution provided an improvement on the Lindley distribution (LD) in fitting the first data set. Again, the proposed Exponentiated Power Lindley Logarithmic (EPLL) distribution proved to offer the best fit to both data sets. This is because it possesses the largest log-likelihood value as well as the highest p-value of the K-S statistics. Results in Tables 1 and 2 also show that the EPLLD outperformed the other sub-distributions since it has the largest loglikelihood value and the largest p-value of the K- S statistics. We can conclude from the foregoing that the proposed Exponentiated Power Lindley-Logarithmic distribution is an improvement on the Lindley distribution and some of its variants.

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