Original Research Article

Classical Estimation of the Parameters of Exponentiated Power Lindley-Logarithmic Distribution

Abstract

In this article, some non-bayesian estimation procedures of the Exponentiated Power Lindley-Logarithmic Distribution for modeling real life data are considered. A simulation study is carried out to determine the best among others using the root meas squared error values. Real life data are used to substantiate the result from the simulation study. It was discovered from simulation study and real data that the weighted least squares estimation procedure is the best method for estimating the parameters of the Exponentiated Power Lindley-Logarithmic distribution.

Keywords— Flexibility, Survival functions, Hazard functions, Estimation

1 Introduction

Many authors have tried to determine best estimation procedures for the parameters of certain distributions. It is important since certain methods of estimation do not guaranty good estimates in all situations. A common indicator of a good estimate is how small its standard error is. Anabike et al [1] used some classical methods and bayesian method to estimate the parameters of Zubair-Exponential distribution. Dey et al [2] explored different estimation procedures for estimating Dagum distribution.

Lindley [3] introduced a mixture model from the exponential and gamma densities. The superiority of the Lindley to other noble life time distributions such as exponential, gamma, Weibull, beta and Akash distributions has been discussed in Ghitany et el [4]. These distributions are referred to as baseline distributions when compared to the generalized forms. Some distributions in the class of Lindley are Chris-Jerry distribution proposed by Onyekwere and Obulezi [5], Ishita distribution by Shanker and Shukla [6], Rani distribution by Shanker [7], Sujatha distribution by Shanker et al [8], Pranav distribution by Shukla [9],Odoma distribution by Odom and Ijeomah [10] and Shukla distribution by Shukla et al [6]. Baseline distributions are often generalized using exponentiation methods proposed by Mudholker and Srivastava [11]. This exponentiation method has been shown to provide better fit and more flexibility than its baseline distribution (see for instance, Nadarajah and Kotz [12]. A further improved version of exponentiation is the Power Lindley distribution proposed by Ghitany et al [13] in which its exponentiation was proposed by Warahena-Liyanage and Pararai [14]. Although, the Lindley, Power Lindley and Eponentiated Power Lindley distributions have found wide application in life time modeling, they however, cannot handle complementary risk problem.

This study considers the estimation of the parameters of the Exponentiated Power Lindley-Logarithmic distribution (EPLLD) using some non-Bayesian procedures. The rest of the article is organized as follows: in section two, we specify the model, section three discusses some classical parameter estimation procedures of the EPLLD. In section four, we carry out simulation study of EPLLD, in section five, numerical applications are explored while conclusion is presented in section six.

2 Model Specification

Ghitany et al [4] gave the cumulative distribution function (cdf) and the probability density function of the Lindley distribution respectively by

$$F(x) = 1 - \left[\frac{1+\beta+\beta x}{1+\beta}\right] e^{-\beta x} \tag{1}$$

and

$$f(x)=\frac{\beta^2}{1+\beta}(1+x)e^{-\beta x};\quad x,\quad \beta>0 \tag{2}$$

Later on, Ghitany et al [13] defined the cdf and the pdf of the Power Lindley distribution respectively as

$$F(x)=1-\left[\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right]e^{-\beta x^{\alpha}} \tag{3}$$

and

$$f(x)=\frac{\alpha\beta^2}{1+\beta}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\beta x^{\alpha}};\quad x,\alpha,\beta>0 \tag{4}$$

Warahena-Liyanage and Pararai [14] defined the cdf and pdf of the Exponentiation Power Lindley distribution with cdf and pdf, respectively by

$$F(X)=\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega} \tag{5}$$

and

$$f(x)=\frac{\alpha\beta^2\omega}{1+\beta}x^{\alpha-1}\left(1+x^{\alpha}\right)e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1};\quad x,\alpha,\beta,\omega>0 \tag{6}$$

Now, suppose that the random variable, , has the Exponentiated Power Lindley distribution with cdf and pdf in equation (5) and (6). Let $X_1, X_2, ..., X_N$ be independent and identically distributed random variables from the Exponentiated Power Lindley distribution. Suppose to be discrete and follows the zero-truncated logarithmic distribution defined by Noack [15] by the probability mass function (pmf) of the form

$$P(N=n)=\frac{\lambda^n}{-n\ln(1-\lambda)},\quad n=1,2,...,\quad 0<\lambda<1 \tag{7}$$

Let $X_{(n)}=max(X_1,X_2,...,X_N)$ which is the n^{th} order statistic of the sequence $X_1, X_2, ..., X_N$. Following Pararai et al [16], the cdf of the random variable $X_{(n)}|N=n$, can be expressed by

$$F_{X_{(n)}|N=n}(x)=\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n};\quad x,\alpha,\beta,\omega>0,\quad n\geq 0 \tag{8}$$

Equation (8) is the cdf of the Exponentiated Power Lindley distribution with parameters α, β and ωn. The corresponding pdf is obtained by differentiating equation (8) with respect to x, which yields

$$f_{X_{(n)}|N=n}(x)=\frac{\alpha\beta^2\omega n}{1+\beta}x^{\alpha-1}\left(1+x^{\alpha}\right)e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n} \tag{9}$$

The cdf of the proposed Exponentiated Power Lindley-Logarithmic Distribution (EPLLD) is the marginal cdf of $X_{(n)}$ obtained from equation 8).

The marginal cdf of $X_{(n)}$ is the same as the cdf of the proposed distribution EPLLD. The marginal cdf of $X_{(n)}$ is expressed as

$$F_{EPLL}(x)=\sum_{n=1}^{\infty}P(N=n)F_{X_{(n)}|N=n}(x) \tag{10}$$

Making appropriate substitutions, we have

$$\begin{aligned}F_{EPLL}(x)&=\frac{\lambda^n}{-n\ln(1-\lambda)}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega n}\\&=\frac{1}{-\ln(1-\lambda)}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}^{n}\end{aligned} \tag{11}$$

Let $y=\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}$, notice that $-\ln(1-y)=\sum_{n=1}^{\infty}\frac{y^n}{n};\quad -1\leq y\leq 1$. Again, notice that for $0<\lambda<1$; $\alpha,\beta,\omega>0$ $\beta,\alpha,\omega>0$, y will lie in the interval $0<y<1$, which is the sub-interval $[-1,1]$. It follows that we can rewrite equation (11) as follows

$$F_{EPLL}(x)=F_{EPLL}(x)=\frac{\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)};\quad x,\alpha,\beta,\omega>0,\quad 0<\lambda<1 \tag{12}$$

Equation (12) is the cdf of the proposed distribution EPLLD. Differentiating with respect to x gives the pdf as

$$f_{EPLL}(x)=\frac{\alpha\beta^2\lambda\omega x^{\alpha-1}(1+x^{\alpha})e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\alpha-1}}{-\ln(1-\lambda)\left(1+\beta\right)\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}} \tag{13}$$

where α is the shape parameter (flexibility measure), β is the scale parameter, ω is the shape parameter (extreme value measure) and λ is the shape parameter (competing risk measure).

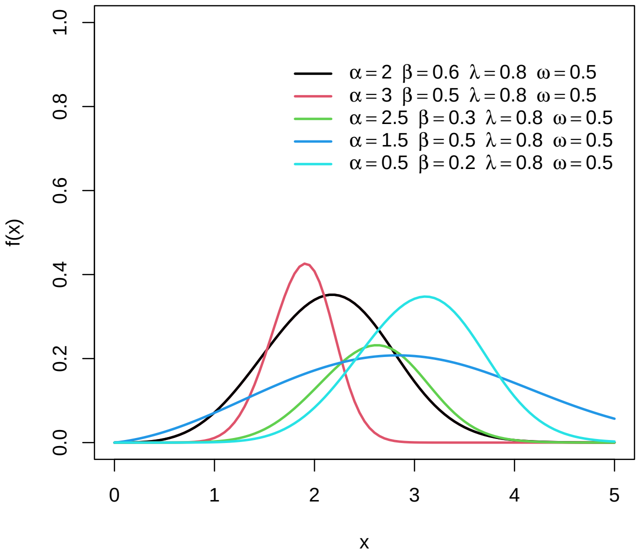

Fig 1. pdf of EPLLD

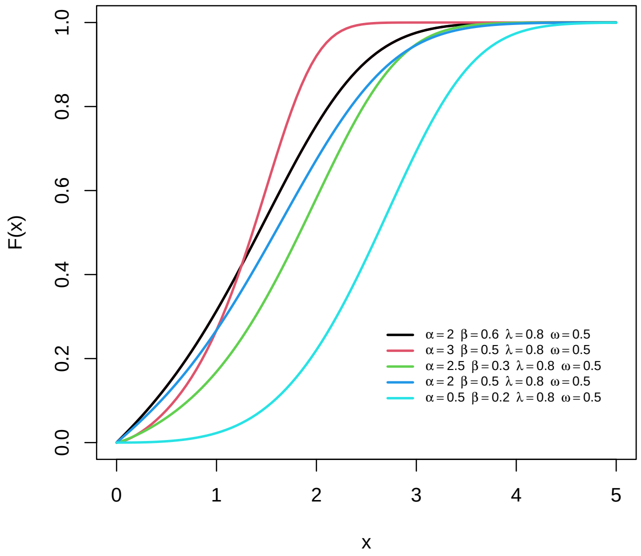

Fig2.cdf of EPLLD

Definition 2.1. Let $X \sim \text{EPLLD}(\alpha,\beta,\lambda,\omega)$, then the survival and hazard rate functions are given as

$$S_{EPLL}(x) = 1 - \frac{\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)} = \frac{\log_e(1-\lambda)-\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega}\right\}}{\log_e(1-\lambda)} \tag{14}$$

and

$$h_{EPLL}(x) = \frac{\alpha\beta^2\lambda\omega x^{\alpha-1}(1+x^{\alpha})e^{-\beta x^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1}}{-(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}\right]A_1(x,\alpha,\beta,\lambda,\omega)}; \quad x>0, \quad \alpha,\beta,\omega>0, \quad 0<\lambda<1 \tag{15}$$

where $A_1(x,\alpha,\beta,\lambda,\omega) = -\log_e(1-\lambda)(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right\}^{\omega}\right]$

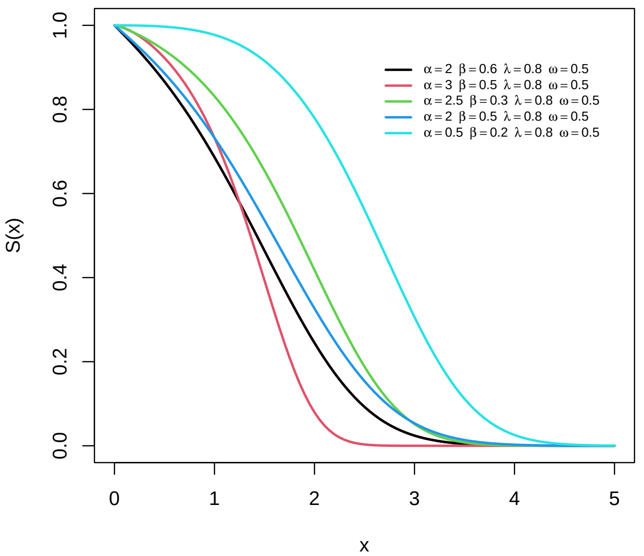

Fig3.survival function of EPLLD

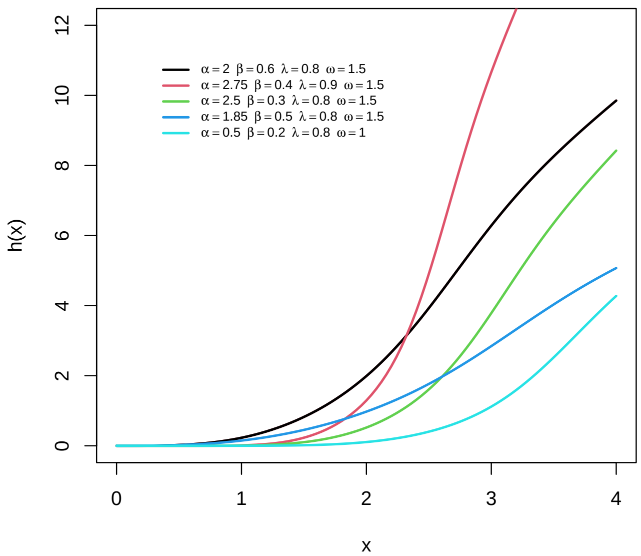

Fig4. hazard rate function of EPLLD

3 Some Classical Parameter Estimation Algorithms

Definition 3.1 (Maximum Likelihood Estimation)**.** Let $x_1, x_2, ..., x_n$ be independent random sample of size n which is distributed according to EPLLD, then the likelihood function is obtained as follows

$$\ell = \prod_{i=1}^{n} f(x_i;\psi) \tag{16}$$

where ψ is the vector of parameters and $f(x_i;\psi)$ is the pdf of the proposed EPLLD.

$$\left.\begin{aligned}
\Rightarrow \log_e \ell = \sum_{i=1}^{n} \log_e f(x_i) = \sum_{i=1}^{n} \log_e \left\{ \frac{\alpha\beta^2\lambda x_i^{\alpha-1}(1+x_i^{\alpha})e^{-\beta x_i^{\alpha}}\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1}}{-\log_e(1-\lambda)(1+\beta)\left[1-\lambda\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}^{\omega}\right]} \right\} \\
\log_e \ell = n\log_e \alpha + 2n\log_e \beta + n\log_e \lambda + n\log_e \omega - n\log_e(1+\beta) - n\log_e(-\log_e(1-\lambda)) \\
-\beta\sum_{i=1}^{n} x_i^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log_e x_i + \sum_{i=1}^{n}\log_e(1+x_i^{\alpha}) + (\omega-1)\sum_{i=1}^{n}\log_e\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right] \\
-\sum_{i=1}^{n}\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}
\end{aligned}\right\} \tag{17}$$

Differentiating partially with respect to the parameters where $L = \log_e \ell$

$$\left.\begin{aligned}
\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \beta\sum_{i=1}^{n} x_i^{\alpha}\log_e x_i + \sum_{i=1}^{n}\log_e x_i + \sum_{i=1}^{n}\frac{x_i^{\alpha}\log_e x_i}{(1+x_i^{\alpha})} + (\omega-1)\sum_{i=1}^{n}\frac{\beta\left(1+\beta+\beta x_i^{\alpha}\right)x_i^{\alpha}\log_e x_i e^{-\beta x_i^{\alpha}} - \beta x_i^{\alpha}\log_e x_i}{(1+\beta)\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]} \\
-\sum_{i=1}^{n}\frac{\lambda\omega\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1}\beta\left(1+\beta+\beta x_i^{\alpha}\right)x_i^{\alpha}\log_e x_i e^{-\beta x_i^{\alpha}} - \beta x_i^{\alpha}\log_e x_i}{(1+\beta)\left[1-\lambda\left(1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right)^{\omega-1}\right]}
\end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial \beta} = \frac{2n}{\beta} - \frac{n}{1+\beta} - \sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\frac{x_i^{\alpha}(1+\beta)(1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}} - (1+\beta)(1+x_i^{\alpha})e^{-\beta x_i^{\alpha}} + (1+\beta++\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}} \\
-\sum_{i=1}^{n}\frac{\lambda\omega\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega-1} x_i^{\alpha}(1+\beta)(1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} - \sum_{i=1}^{n}\frac{(1+x_i^{\alpha})(1+\beta)e^{-\beta x_i^{\alpha}} - (1+\beta+\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}}}{(1+\beta)^2\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}}
\end{aligned}\right\} \tag{19}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{(1-\lambda)\log_e(1-\lambda)} + \sum_{i=1}^{n}\frac{\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}}{\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} \tag{20}$$

$$\frac{\partial L}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n}\log_e\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right] + \sum_{i=1}^{n}\frac{\lambda\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}^{\omega}\log_e\left\{1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right\}}{\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x_i^{\alpha}}{1+\beta}\right)e^{-\beta x_i^{\alpha}}\right]^{\omega}\right\}} \tag{21}$$

The maximum likelihood estimate ψ is obtained by solving the non-linear systems of equation $U(\psi) = 0$. Since the resulting systems of equation are not in closed form, the solutions are obtained using Newton-Raphson's iterative algorithm implemented in R.

Definition 3.2 (Least Squares Estimation (LSE))**.** The Least Squares Estimation was proposed by Swain et al [17] to estimate the parameters of Beta distribution. Using the deductions from the work of Swain et al [17], we write

$$E[F(x_{i:n}|\alpha,\beta,\lambda,\omega)] = \frac{i}{n+1}.$$

$$V[F(x_{i:n}|\alpha,\beta,\lambda,\omega)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The least squares estimates $\hat{\alpha}_{LSE}$, $\hat{\beta}_{LSE}$, $\hat{\lambda}_{LSE}$ and $\hat{\omega}_{LSE}$ of the parameters α,β,λ and ω are obtained by minimizing the function $L(\alpha,\beta,\lambda,\omega)$ with respect to α,β,λ and ω

$$L(\alpha,\beta,\lambda,\omega) = \underset{(\alpha,\beta,\lambda,\omega)}{\arg\min}\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{i}{n+1}\right]^2. \tag{22}$$

The estimates are obtained by solving the following non-linear equations

$$\left.\begin{aligned}
\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega)-\frac{i}{n+1}\right]^2\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega)-\frac{i}{n+1}\right]^2\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega)-\frac{i}{n+1}\right]^2\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega)-\frac{i}{n+1}\right]^2\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega)=0
\end{aligned}\right\} \tag{23}$$

where

$$\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega)=-\frac{n\lambda\omega}{(1+\beta)(\log_e(1-\lambda)}\frac{\left\{1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}^{\omega-1}\left\{\beta x^\alpha\log_e xe^{-\beta x^\alpha}-\beta(1+\beta+\beta x^\alpha)x^\alpha\log_e x\right\}}{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right]^\omega} \tag{24}$$

$$\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega)=-\frac{n\lambda\omega}{(1+\beta)^2\log_e(1-\lambda)}\frac{\left\{1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}^{\omega-1}\left\{(1+\beta)\left[(1+x^\alpha)e^{-\beta x^\alpha}-x^\alpha(1+\beta+\beta x^\alpha)e^{-\beta x^\alpha}\right]-(1+\beta+\beta x^\alpha)e^{-\beta x^\alpha}\right\}}{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right]^\omega} \tag{25}$$

$$\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega)=\frac{n}{\lambda\log_e(1-\lambda)}-\frac{1}{(\lambda-1)\left(\log_e(1-\lambda)\right)^2}\log_e\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right]^\omega\right\}^n \tag{26}$$

$$\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega)=-\frac{\lambda n\left\{1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}^\omega\log_e\left\{1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right\}}{\log_e(1-\lambda)\left\{1-\lambda\left[1-\left(\frac{1+\beta+\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}\right]^\omega\right\}} \tag{27}$$

Definition 3.3 (Weighted Least Squares Estimation (WLSE))**.** The weighted least squares estimates $\hat{\alpha}_{WLSE},\hat{\beta}_{WLSE},\hat{\lambda}_{WLSE}$ and $\hat{\omega}_{WLSE}$ of EPLLD distribution parameters α,β,λ and ω are obtained by minimizing the function $W(\alpha,\beta,\lambda,\omega)$ with respect to α,β,λ and ω

$$W(\alpha,\beta,\lambda,\omega)=\underset{(\alpha,\beta,\lambda,\omega)}{\arg\min}\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]^2. \tag{28}$$

Solving the following non-linear equation yields the estimate

$$\left.\begin{aligned}
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega)=0\\
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega)=0
\end{aligned}\right\} \tag{29}$$

where $\Delta_1(x.|\alpha,\beta,\lambda,\omega),\Delta_2(x.|\alpha,\beta,\lambda,\omega),\Delta_3(x.|\alpha,\beta,\lambda,\omega)$ and $\Delta_4(x.|\alpha,\beta,\lambda,\omega)$ is as defined in equations (24 - 27) respectively.

Definition 3.4 (Maximum Product Spacing Estimators (MPSE))**.** A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the EPLLD is given as follows

$$Gs(\alpha,\beta,\lambda,\omega|data)=\left(\prod_{i=1}^{n+1}D_l(x_i,\alpha,\beta,\lambda,\omega)\right)^{\frac{1}{n+1}}, \tag{30}$$

where $D_l(x_i,\alpha,\beta,\lambda,\omega)=F(x_i;\alpha,\beta,\lambda,\omega)-F(x_{i-1};\alpha,\beta,\lambda,\omega)$, $i=1,2,3,...,n$

Similarly, one can also choose to maximize the function

$$H(\alpha,\beta,\lambda,\omega)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\alpha,\beta,\lambda,\omega). \tag{31}$$

By taking the first derivative of the function $H(\vartheta)$ with respect to α,β,λ and ω, and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial\alpha}=0,\frac{\partial H(\phi)}{\partial\beta}=0,\frac{\partial H(\phi)}{\partial\lambda}=0$ and $\frac{\partial H(\phi)}{\partial\omega}=0$ where $\phi=(\alpha,\beta,\lambda,\omega)$, we obtain the value of the parameter estimates.

Definition 3.5 (Cramér-von-Mises Estimation (CVME)). The Cramér-von-Mises estimates $\hat{\alpha}_{CVME}, \hat{\beta}_{CVME}, \hat{\lambda}_{CVME}$ and $\hat{\omega}_{CVME}$ of the EPLLD distribution parameters λ, and θ are obtained by minimizing the function $C(\alpha,\beta,\lambda,\omega)$ with respect to λ, and θ

$$C(\alpha,\beta,\lambda,\omega) = \underset{(\alpha,\beta,\lambda,\omega)}{\arg\min}\left\{\frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{2i-1}{2n}\right]^2\right\}. \tag{32}$$

The estimates are obtained by solving the following non-linear equations

$$\left.\begin{aligned}
\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{2i-1}{2n}\right)\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega) = 0\\
\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{2i-1}{2n}\right)\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega) = 0\\
\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{2i-1}{2n}\right)\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega) = 0\\
\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{2i-1}{2n}\right)\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega) = 0
\end{aligned}\right\} \tag{33}$$

where $\Delta_1(x.|\alpha,\beta,\lambda,\omega)$ and $\Delta_2(x.|\alpha,\beta,\lambda,\omega)$ is as defined in equations (24 - 27) respectively.

Definition 3.6 (Anderson-Darling Estimation (ADE)). The Anderson-Darling estimates $\hat{\alpha}_{ADE}, \hat{\beta}_{ADE}, \hat{\lambda}_{ADE}$ and $\hat{\omega}_{ADE}$ of the EPLLD parameters α,β,λ and ω are obtained by minimizing the function $A(\alpha,\beta,\lambda,\omega)$ with respect to α,β,λ and ω

$$A(\alpha,\beta,\lambda,\omega) = \underset{(\alpha,\beta,\lambda,\omega)}{\arg\min}\sum_{i=1}^{n}(2i-1)\left\{\ln F(x_{i:n}|\alpha,\beta,\lambda,\omega) + \ln\left[1 - F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)\right]\right\}. \tag{34}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\left.\begin{aligned}
\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} - \frac{\Delta_1(x_{n+1-i:n}|\lambda,\theta)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} - \frac{\Delta_2(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} - \frac{\Delta_3(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} - \frac{\Delta_4(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0
\end{aligned}\right\} \tag{35}$$

where $\Delta_1(x.|\alpha,\beta,\lambda,\omega)$ and $\Delta_2(x.|\alpha,\beta,\lambda,\omega)$ is as defined in equations (24 - 27) respectively.

Definition 3.7 (Right-Tailed Anderson-Darling Estimation (RTADE)). The Right-Tailed Anderson-Darling estimates $\hat{\lambda}_{RTADE}$ and $\hat{\theta}_{RTADE}$ of the EPLLD parameters α,β,λ and ω are obtained by minimizing the function $R(\alpha,\beta,\lambda,\omega)$ with respect to α,β,λ and ω

$$R(\alpha,\beta,\lambda,\omega) = \underset{(\alpha,\beta,\lambda,\omega)}{\arg\min}\left\{\frac{n}{2} - 2\sum_{i=1}^{n}F(x_{i:n}|\alpha,\beta,\lambda,\omega) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\left[1 - F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)\right]\right\}. \tag{36}$$

The estimates can be obtained by solving the following set of non-linear equations

$$\left.\begin{aligned}
-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
-2\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
-2\sum_{i=1}^{n}\frac{\Delta_3(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0\\
-2\sum_{i=1}^{n}\frac{\Delta_4(x_{i:n}|\alpha,\beta,\lambda,\omega)}{F(x_{i:n}|\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_4(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}{1-F(x_{n+1-i:n}|\alpha,\beta,\lambda,\omega)}\right] = 0
\end{aligned}\right\} \tag{37}$$

where $\Delta_1(x.|\alpha,\beta,\lambda,\omega), \Delta_2(x.|\alpha,\beta,\lambda,\omega), \Delta_3(x.|\alpha,\beta,\lambda,\omega)$ and $\Delta_4(x.|\alpha,\beta,\lambda,\omega)$ is as defined in equations (24 - 27) respectively.

The estimates given in equations (18 - 21) (23), (29), (31), (33), (35) and (37) are obtained using **optim()** function in R with the Newton-Raphson iterative algorithm.

4 Simulation study

In this subsection, we simulate data for the EPLLD to compare the performance of the Non-Bayesian estimation methods discussed in the previous section. we generate 1000 data from the EPLLD by considering the initial parameter values as

- $\alpha = 0.50, \beta = 0.15, \lambda = 0.70$ and $\omega = 1.30$
- $\alpha = 0.75, \beta = 0.05, \lambda = 0.70$ and $\gamma = 1.30$
- $\alpha = 0.75, \beta = 0.15, \lambda = 0.70$ and $\omega = 1.50$
- $\alpha = 0.50, \beta = 0.25, \lambda = 0.50$ and $\omega = 1.50$

and sample sizes $n = 25, 50, 75, 100$. For each estimate $\hat{\phi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\omega})$, we compute the Root Mean Squared Error(RMSE) as

$$RMSE(\hat{\phi}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\phi}_i - \phi)^2}.$$

we deploy the Newton-Raphson algorithm for finding the desired estimates.

Table 1: Average estimated Biases and RMSEs of different estimation methods for EPLLD distribution at different sample sizes n and different values of the parameters $\alpha = 0.50, \beta = 0.15, \lambda = 0.70$ and $\omega = 1.30$

Method	Parameters	RMSE			
		$n = 25$	$n = 50$	$n = 75$	$n = 100$
MLE	α	0.0140	0.0073	0.0124	0.0162
	β	0.2730	0.2100	0.1926	0.1872
	λ	6.2452	2.0508	0.7461	0.4270
	ω	4.0343	1.4197	1.2130	1.0567
MPSE	α	0.0178	0.0094	0.0070	0.0081
	β	0.1742	0.1083	0.0882	0.0893
	λ	4.5315	1.0538	0.6950	0.4191
	ω	4.8523	0.7989	0.5001	0.4855
LSE	α	0.0215	0.0063	0.0016	0.0022
	β	0.1387	0.0577	0.0354	0.0225
	λ	2.9169	1.2808	0.7007	0.3813
	ω	2.6539	0.7845	0.3833	0.1922
WLSE	α	0.0184	0.0078	0.0012	0.0021
	β	0.1115	0.0484	0.0242	0.0189
	λ	2.8443	0.9044	0.6480	0.3236
	ω	2.0437	0.5339	0.2071	0.1226
CVME	α	0.0099	0.0043	0.0004	0.0053
	β	0.1328	0.0633	0.0386	0.0215
	λ	2.9556	1.1865	0.6699	0.4182
	ω	3.9850	0.9764	0.4287	0.2118
ADE	α	0.0121	0.0061	0.0048	0.0040
	β	0.1179	0.0609	0.0453	0.0401
	λ	2.1555	0.6416	0.3724	0.2247
	ω	1.9770	0.5399	0.3105	0.2164
RTADE	α	0.0392	0.0247	0.0225	0.0182
	β	0.1789	0.0903	0.0703	0.0556
	λ	5.8661	1.2522	0.4532	0.3205
	ω	6.7013	1.2911	0.7412	0.4791

Table 2: Average estimated Biases and RMSEs of different estimation methods for EPLLD distribution at different sample sizes n and different values of the parameters$\alpha = 0.75, \beta = 0.05, \lambda = 0.70$ and $\omega = 1.30$

Method	Parameters	RMSE			
		$n = 25$	$n = 50$	$n = 75$	$n = 100$
MLE	α	0.0075	0.0141	0.0178	0.0142
	β	0.0997	0.0731	0.0630	0.0540
	λ	7.9656	1.0731	0.8401	0.4817
	ω	2.5861	0.9262	0.7037	0.5573
MPSE	α	0.0424	0.0194	0.0181	0.0133
	β	0.0838	0.0566	0.0489	0.0388
	λ	102.3872	0.9687	0.5478	0.3360
	ω	3.9016	0.6297	0.4603	0.3248
LSE	α	0.0311	0.0130	0.0134	0.0062
	β	0.0619	0.0289	0.0200	0.0125
	λ	2.4513	1.0055	0.6276	0.3786
	ω	1.5192	0.5937	0.3350	0.1611
WLSE	α	0.0201	0.0031	0.0056	0.0012
	β	0.0511	0.0193	0.0111	0.0085
	λ	2.6228	0.8788	0.6043	0.3400
	ω	1.3363	0.4074	0.1820	0.1007
CVME	α	0.0045	0.0006	0.0113	0.0052
	β	0.0526	0.0249	0.0209	0.0146
	λ	2.5895	0.9231	0.6546	0.3300
	ω	1.5238	0.6047	0.3791	0.2221
ADE	α	0.0150	0.0058	0.0161	0.0088
	β	0.0489	0.0227	0.0191	0.0138
	λ	2.4088	0.6103	0.4590	0.2425
	ω	1.2108	0.3841	0.2758	0.1627
RTADE	α	0.0391	0.0320	0.0258	0.0220
	β	0.0686	0.0364	0.0223	0.0208
	λ	4.0499	1.3183	0.6173	0.3526
	ω	3.0252	0.9355	0.4703	0.3689

Table 3: Average estimated Biases and RMSEs of different estimation methods for EPLLD distribution at different sample sizes n and different values of the parameters $\alpha = 0.75, \beta = 0.15, \lambda = 0.70$ and $\omega = 1.50$

Method	Parameters	RMSE			
		$n = 25$	$n = 50$	$n = 75$	$n = 100$
MLE	α	0.0214	0.0250	0.0233	0.0324
	β	0.2550	0.2220	0.1929	0.1843
	λ	9.9049	1.4208	1.0158	0.4419
	ω	4.5971	1.9253	1.4719	1.3818
MPSE	α	0.0237	0.0022	0.0066	0.0040
	β	0.1461	0.1066	0.0955	0.0888
	λ	7.3464	1.3981	0.7647	0.4481
	ω	3.1202	0.9289	0.7138	0.6308
LSE	α	0.0305	0.0229	0.0154	0.0116
	β	0.1382	0.0721	0.0483	0.0353
	λ	2.4236	1.0244	0.6341	0.3609
	ω	3.2009	0.9750	0.5395	0.3340
WLSE	α	0.0228	0.0134	0.0122	0.0049
	β	0.1062	0.0534	0.0423	0.0270
	λ	3.1246	0.7927	0.4884	0.3210
	ω	2.2334	0.5882	0.3871	0.2070
CVME	α	0.0138	0.0125	0.0116	0.0078
	β	0.1346	0.0659	0.0516	0.0360
	λ	2.8934	1.1713	0.6167	0.3615
	ω	3.6252	1.0176	0.6551	0.3871
ADE	α	0.0212	0.0109	0.0084	0.0102
	β	0.1139	0.0575	0.0427	0.0378
	λ	2.5174	0.6195	0.4527	0.2319
	ω	2.2131	0.6017	0.3561	0.2742
RTADE	α	0.0329	0.0172	0.0214	0.0153
	β	0.1380	0.0700	0.0587	0.0428
	λ	4.1561	0.9959	0.6006	0.3182
	ω	4.0366	1.2197	0.7147	0.4637

Table 4: Average estimated Biases and RMSEs of different estimation methods for EPLLD distribution at different sample sizes n and different values of the parameters $\alpha = 0.50, \beta = 0.25, \lambda = 0.50$ and $\omega = 1.50$

Method	Parameters	RMSE			
		$n = 25$	$n = 50$	$n = 75$	$n = 100$
MLE	α	0.0021	0.0040	0.0004	0.0103
	β	0.3561	0.2861	0.2607	0.2696
	λ	9.2858	2.9596	2.0970	1.2926
	ω	4.1433	1.6366	1.3166	1.3424
MPSE	α	0.0013	0.0232	0.0253	0.0138
	β	0.2049	0.1262	0.1167	0.1377
	λ	5.3821	2.1532	1.3559	1.0668
	ω	3.0892	0.9345	0.6433	0.7147
LSE	α	0.0199	0.0165	0.0130	0.0094
	β	0.2024	0.1149	0.0873	0.0589
	λ	5.6869	1.7196	1.0067	0.5287
	ω	4.9296	1.4852	0.7829	0.3918
WLSE	α	0.0170	0.0076	0.0036	0.0042
	β	0.1691	0.0804	0.0552	0.0463
	λ	4.7861	1.4155	0.7277	0.5398
	ω	3.4541	0.8226	0.3739	0.2458
CVME	α	0.0097	0.0097	0.0053	0.0057
	β	0.1977	0.1093	0.0743	0.0576
	λ	6.0222	1.5835	1.0730	0.5748
	ω	4.5785	1.4089	0.7011	0.4212
ADE	α	0.0089	0.0082	0.0036	0.0093
	β	0.1602	0.0928	0.0643	0.0668
	λ	3.5265	1.0484	0.5869	0.3884
	ω	2.7839	0.7370	0.3824	0.3213
RTADE	α	0.0306	0.0313	0.0227	0.0206
	β	0.2263	0.1505	0.1025	0.0854
	λ	7.5669	1.8295	0.8682	0.5288
	ω	6.1531	2.1222	0.9456	0.6294

The following conclusions can be established from Tables (1-4) in the simulation study

- Because the range of the values of RMSE for the parameters of the EPLLD is quite small, the results of Tables (1-4) demonstrate that the EPLLD is stable.
- In some cases, we see a drop in the RMSE for all estimates as the sample size rises.
- This demonstrates that multiple estimating strategies produce accurate Bias and RMSE findings for large sample sizes.
- The Weighted Least Squares estimation (WLSE) approach is the accurate method to estimate the EPLLD parameters.
- As the sample size increases, all estimators RMSE values fall, demonstrating improved accuracy in model parameter estimation.

5 Numerical Application

We, apply the EPLLD to some real data sets. The first data set is the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. This data was studied by Owoloko et al [18].

Table 5: The life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.9120	0.9836
1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630	1.7746	1.8475	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.1330	2.2100
2.2460	2.2878	2.3203	2.3470	2.3513	2.4951	2.5260	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.0960	

Next, we estimate the parameters of the EPLLD using the data in Table 5.

Table 6: Estimation of EPLLD Parameters using some Classical Methods using life of fatigue fracture of Kevlar 373/epoxy

Method	Parameter	Estimate	Standard Error
	a	0.9623	0.3417
MLE	b	0.9869	0.8234
	c	-0.1030	2.4222
	d	1.5149	0.8010
	a	0.7937	0.2868
MPS	b	1.0526	0.7639
	c	0.1366	1.6858
	d	1.4800	0.7356
	a	0.7937	7.8227
LSE	b	1.7956	33.8276
	c	0.6737	14.3996
	d	2.9200	70.8107
	a	0.7937	**0.1702**
WLSE	b	1.7956	**0.7480**
	c	0.3716	**0.6202**
	d	3.4370	**1.9607**
	a	0.8177	6.4946
CVME	b	1.7661	27.8762
	c	0.6813	12.8241
	d	2.8466	54.8040
	a	0.9042	0.7775
ADE	b	1.2301	2.1558
	c	0.1765	3.6939
	d	1.9487	3.1927
	a	0.6212	2.0479
RTADE	b	2.0986	12.7458
	c	-0.7251	34.7267
	d	6.5973	56.1617

From Table 6, the weighted Least Squares estimator (WLSE) gives the best estimates since its standard errors are the least among others.

The next application is on the survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal [19], Anabike et al [1] and Onyekwere and Obulezi [5].

Table 7: The survival times of guinea pigs injected with different amount of tubercle bacilli

10	33	44	56	59	72	74	77	92	93	96	100	100	102	105	107	107	108
108	108	109	112	113	115	116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196	197	202	213	215	216	222
230	231	240	245	251	253	254	255	278	293	327	342	347	361	402	432	458	555

Table 8: Estimation of EPLLD Parameters using some Classical Methods using the survival times of guinea pigs injected with different amount of tubercle bacilli

Method	Parameters	Estimate	Standard Error
MLE	α	1.0178	0.0899
	β	0.0125	0.0070
	λ	-0.3814	1.6426
	ω	1.7063	0.4970
MPSE	α	0.9458	0.1174
	β	0.0179	0.0138
	λ	-0.5478	2.4234
	ω	1.7413	0.6062
LSE	α	0.5295	0.6337
	β	0.3274	1.4924
	λ	-0.0312	7.2600
	ω	16.3290	106.7112
WLSE	α	0.7035	0.0322
	β	0.0914	0.0193
	λ	-2.2058	1.4038
	ω	5.5903	0.8694
CVME	α	0.6074	0.8847
	β	0.1682	1.0019
	λ	-4.9567	78.9160
	ω	9.8846	50.9643
ADE	α	0.9800	0.1225
	β	0.0158	0.0106
	λ	-1.4411	7.9960
	ω	2.1870	1.4503
RTADE	α	0.6465	0.7822
	β	0.1322	0.4218
	λ	-3.6024	40.6363
	ω	7.9022	5.2674

From Table 8, the Maximum Likelihood estimator (MLE), Maximum Product Spacing Estimator (MPSE) and the Weighted Least Squares Estimator (WLSE) are competitive in producing good estimates of the parameters of the EPLLD using data on the survival times of guinea pigs injected with different amount of tubercle bacilli. This is because their standard errors are minimum.

6 Conclusion

The estimation of the parameters of the Exponentiated Power Lindley-Logarithmic distribution has been studied in this paper. For the simulated data, the Weighted Least Squares Estimation (WLSE) is the best among others due to its minimum standard errors of the distributions parameters. The same applies to the data on life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. Again, the Weighted Least Squares Estimation (WLSE), Maximum Likelihood Estimation (MLE) and the Maximum Product Spacing Estimation (MPSE) were comparative strong in yielding the best estimates of the parameters of EPLLD due to their relative minimum values of their standard errors. Overall, it is deduced that the Weighted Least Squares Estimation is the best estimation procedure for the estimation of the parameters of the Exponentiated Power Lindley-Logarithmic distribution.

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