Short Research Article

The set of rationale numbers is countably infinite – a simple proof

Abstract

This research note presents a very simple proof of the interesting fact that the set **Q** of rationale numbers is still *countably* infinite as is the set of natural and integer numbers. The proof is based on several innovative ideas and neither relies on Cantor's well-known diagonalization approach nor on the non-trivial Cantor-Schroeder-Bernstein Theorem.

In addition, we present a new proposal for a simple injective function ***f***: **Q** → **Z**, which allows one to encode rationals in a highly efficient manner and at the same time it can be understood much more easily (even by non-mathematicians). Moreover, also the inverse function $\boldsymbol{f}^{-1}$ can be derived in an extremely simple manner. Nevertheless, the growth of length is only logarithmic if we compare the resulting length of ***f***(*r*=*p*/*q*) with the value of *p*, while the length of *q* has no impact at all on the length of ***f***(*r*). Our approach also allows us to introduce a total ordering for the set of rationale numbers in a straight-forward manner.

Keywords: *Cardinality of rationals, elementary injective mapping from **Q** to **Z**, simplification of Cantor's proof, total-ordering of rationals.*

Introduction

1. PROOF OF COUNTABILITY OF THE SET Q.

Typically, the fact that the set of rationale numbers has *countably* infinite elements is proven by means of constructing a bijective mapping between the set **Q** of rationale numbers and the set **N** of natural numbers or the set **Z** of integers, (cf. for example Cantor's method (called: "diagonalization method" or "zig-zag method"), see, e.g., [1] and [2]). An alternate proof of the countability of the set **Q** can be achieved by defining two injective mappings ***f***: **Q** → **Z** as well as a mapping ***g***: **Z** → **Q** in combination with the non-trivial Cantor-Schroeder-Bernstein Theorem [3].

In this paper we want to present a proof of the countability of **Q** which is significantly simpler as it requires only one injective mapping, namely ***f***: **Q** → **Z**. Our proof makes use of the very helpful fact that by finding an injective mapping ***f*** from **Q** into **Z** we easily see that ***f***: **Q** → ***f***(**Q**) is even a bijection between the sets **Q** and ***f***(**Q**)={***f***(x)| x∈**Q**}.

Moreover, to the best of our knowledge, all injective mappings ***f***: **Q** → **Z** proposed until now possess serious weaknesses, like e.g. the injective mapping ***f***, which encodes a ratio *r* in such a way that the value of the numerator implies a corresponding number of "8"-digits (if *r*>0) or of "7"-digits (if *r*<0) followed by "1"-digits according to the value of the denominator. Example: 3/5 is encoded by 88811111. Though, here, encoding and decoding are really straight-forward, we realize easily that the length of the resulting integer number is growing exponentially with the lengths of numerator/ denominator of the ratio being encoded! Despite this quite horrible fact, renowned mathematicians, when this author consulted them (cf. [4]) argued that this mapping function $\boldsymbol{f}$ is considered by them as being the best of the proposals published up to now.

Consulting further mathematicians (all renowned experts in Number/Set Theory) led to the result that, from their point of view, the simplest injective function published up to now to encode a ratio $r=p/q$ with $p=p_1p_2\ldots p_n$, $q=q_1q_2\ldots q_m$ by mapping r onto the integer number $\boldsymbol{f}(r)=(p_1p_2\ldots p_n X q_1q_2\ldots q_m)_{11}$, where we have to view each digit as being in Base 11.
Example: $r=(3)_{10}/(5)_{10}$ implies $\boldsymbol{f}(r)=(3X5)_{11}$, where X denotes $(10)_{10}$.
Here, the problem of "length explosion" is clearly fixed, but leaving the conventional decimal system and making use of the Base 11 number system (unknown to most non-mathematicians) seems to be quite unacceptable. An additional problem is the considerably high expenditure which is required to achieve both the encoding and the decoding (e.g., finding the ratio r to a given $\boldsymbol{f}(r)$ typically will be very tedious – even a computer could be required for treating large numbers).

This is why we have searched for an injective mapping $\boldsymbol{f}$: $\mathbf{Q} \rightarrow \mathbf{Z}$, which does not possess the disadvantages of the corresponding proposals for $\boldsymbol{f}$ published up to now. In particular, our new proposal should satisfy the two following requirements:

R1. The output (i.e. the length of $\boldsymbol{f}(r)$) should not become much longer than the input (i.e. the lengths of numerator and of denominator of r).

R2. Both, the encoding function $\boldsymbol{f}$ as well as the decoding (i.e. the inverse function $\boldsymbol{f}^{-1}$) should be so simple that they can be understood extremely easily – even by non-mathematicians.

Theorem 1. *The set **Q** contains a countably infinite number of elements.*

Proof of Theorem 1.

STEP1:

As a first fundamental step of the proof let us construct an *injective* mapping $\boldsymbol{f}$: $\mathbf{Q} \rightarrow \mathbf{Z}$, which maps an arbitrary rationale number b in a unique manner onto an integer number $z = \boldsymbol{f}(b) \in \mathbf{Z}$. In particular, the function $\boldsymbol{f}$ should satisfy all our requirements R1 and R2 (cf. above). Indeed, we were able to find such a function which will be presented now.
So, let us consider an element $b \in \mathbf{Q}$, arbitrarily chosen and then fixed, $b \neq 0$. As b does represent a ratio, we can write b in the following way $b = sgn(b) \bullet \frac{p}{q}$, where $p = p_1p_2p_3\ldots p_n$ denotes the numerator (consisting of n digits p_i, i=1,2,3,…,n) and $q = q_1q_2q_3\ldots q_m$ denotes the denominator (consisting of m digits) and sgn indicates the sign, where $sgn(x) = 0$, for $x=0$ and $sgn(x) = x/|x|$, for $x\neq 0$.
To achieve a unique encoding of a rationale number (i.e. a ratio) we suppose, without loss of generality, that p and q have no common divisor larger than 1 and are non-negative. First, we assume $b\neq 0$ and we construct $\boldsymbol{f}$ by using an auxiliary number h_b , which starts with exactly n "1"s, $n \geq 1$. So, the number of "1"s indicates the length of the numerator (also n). The "1"s at the beginning of h_b are followed by just one "0", which is then followed by the n digits

which constitute the numerator p and these digits are then directly followed by the m digits corresponding to the denominator q.

To summarize, h_b has the following form:

$$h_b = 111\ldots10p_1p_2p_3 \ldots p_nq_1q_2q_3 \ldots q_m \,.$$

Based on our auxiliary number h_b, for arbitrarily chosen elements $b \in \mathbf{Q}$, $b \neq 0$, we now can directly obtain the function $\boldsymbol{f}$ we are looking for. The case b=0 can be covered in a straight-forward way by defining $\boldsymbol{f}(0)=0$. So, the complete definition of $\boldsymbol{f}$ is:

$$\boldsymbol{f}(b) = \begin{cases} h_b\,, & \text{for } b>0. \\ 0\,, & \text{for } b=0. \\ -h_b\,, & \text{for } b<0. \end{cases}$$

It is very easy to determine the original number b being mapped on a given number $\boldsymbol{f}(b)$. For example, if $\boldsymbol{f}(b)=0$ we can conclude b=0. In all other cases, we obtain h_b by means of $sgn(f(b)) \bullet \boldsymbol{f}(b) = h_b$, i.e., we obtain h_b easily by just eliminating the sign of $\boldsymbol{f}(b)$. By knowing h_b we know the number of digits of the numerator (indicated by the number of "1"s at the beginning of h_b) and by looking at the n successive digits following the first "0" of h_b we know the complete numerator. The rest of the digits of h_b represent the denominator. The sign of b is identical to the one of $\boldsymbol{f}(b)$.

Therefore, $\boldsymbol{f}$ is a (very simple) injective mapping of **Q** onto **Z**, which implies: $|\mathbf{Z}| \geq |\mathbf{Q}|$.

STEP 2:

We see that $\boldsymbol{f}$ resp. $\boldsymbol{f}^{-1}$ (the inverse of function $\boldsymbol{f}$) represent a bijection between **Q** and $\boldsymbol{f}(\mathbf{Q})=\{\boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbf{Q}\}$, i.e. $\boldsymbol{f}\colon \mathbf{Q} \rightarrow \boldsymbol{f}(\mathbf{Q})$ and $\boldsymbol{f}^{-1}\colon \boldsymbol{f}(\mathbf{Q}) \rightarrow \mathbf{Q}$. Therefore, **Q** and **f(Q)** possess the same cardinality

STEP 3:

The cardinality of $\boldsymbol{f}(\mathbf{Q})$ is countably infinite because it contains an infinite number of elements and it is a subset of a set containing a countably infinite number of elements (namely the set **Z**). Therefore, because of the existing bijective mapping between **Q** and $\boldsymbol{f}(\mathbf{Q})$, it is proven that **Q** is countably infinite, too.

q.e.d.

It is quite remarkable that our proof of Theorem 1 can easily be generalized to the case that we replace **Q** by an arbitrary set X possessing an infinite number of elements and the set **Z** is replaced by a set Y which we assume to possess a countably infinite number of elements. If we still are able to provide an injective mapping $\boldsymbol{f}$: X $\rightarrow$ Y, an argumentation according to the proof of Theorem 1, still proves in this generalized case, too, that X possesses a countably infinite number of elements – a rather general result, indeed.

Lemma 1. *Let M be a set possessing a countably infinite number of elements and let M_s be an arbitrary subset of M with an infinite number of elements. Then, M_s also possesses a countably infinite number of elements.*

Proof of Lemma 1.

Let the set M possess a countably infinite number of elements. Then, the elements of M are countable (per definitionem). Every (strict) subset of $M_s \subseteq M$ results by eliminating some of the elements of M. Anyway, the elements of M_s still remain countable. Therefore, M_s can only become a finite set or if the set M_s is assumed to possess an infinite number of elements (cf. assumption in Lemma 1) it is proven that M_s is a countably infinite set.

q.e.d.

Remark: The astonishing simplicity of the proof of this Lemma results from the strong assumption underlying Lemma 1, namely the set M is assumed to be countably infinite. If, however, M would be allowed to be uncountably infinite, a subset $M_s \subseteq M$ containing an infinite number of elements, in principle, could be either countably or uncountably infinite. Proving that M_s and M still remain equipotent (having the same cardinality), then will require additional strong assumptions and it will become much more complicated. □

Let us now give two examples to illustrate how the original number *b* can be determined for a given ***f***(*b*).

Example 1. For ***f***(*b*) = 11025410 we observe that h_b = 11025410. Thus, the numerator has two digits because h_b has two "1"s in front of the first "0". So, it is easy to see that h_b = 11025410 corresponds to ratio *b* = 25/410.

Boundary between numerator and denominator

Example 2. For ***f***(*b*) = 1025410 we observe that h_b = 1025410. Thus, the numerator now has only one digit because h_b has only a single "1" in front of the first "0". So, it is easy to see that h_b = 1025410 corresponds to ratio *b* = 2/5410.

Boundary between numerator and denominator

Some readers may be slightly concerned by the fact that integers resulting from the mapping ***f***(*b*) of the rational $b = \frac{p}{q}$ onto **Z** might become quite large numbers (e.g., if *n*, i.e. the number of digits of *p*, is rather large). Anyway, there exists a straight-forward solution to eliminate this potential problem. We propose to use the notation $\langle n \rangle p_1 \ldots p_n q_1 \ldots q_m$ to represent the integer $11\ldots10 p_1 \ldots p_n q_1 \ldots q_m$
(*n* times digit "1" at the beginning of this integer) and, therefore, the representation of ***f***(*b*) becomes much more compact for large values of n. Already for *n*>2 our proposed notation will reduce the number of symbols required to represent *f*(*b*), namely the symbols 〈 , 〉, and the digits which are used. Encoding now is even significantly less cumbersome than before because the only (trivial) task remaining is to determine *n*, i.e. the length of *p*. And, decoding ***f***(*b*) to determine *b* is completely trivial now.

Example 3. Using the simplifying, new notation for $b = \frac{123}{1234}$ we obtain ***f***(*b*) = $\langle 3 \rangle 1231234$.

Remark: Besides, the design of the injective function ***f*** represents a nice example for the fact that different scientific disciplines (here, Computer Science and Mathematics) can enrich each other. Due to his long-term experience as a professor of Computer Science focusing on the scientific areas of Data Communication and Computer Networking the author became acquainted with an important approach to efficiently organize the communication between a sender S and a receiver R which exchange signals being interpreted by the receiver as a sequence of ”0”- and ”1”-bits. If no need for data exchange exists the sender just sends ”1”-bits only. In order to structure the communication between S and R, the sender terminates the sequence of ”1”-bits being sent to R and is sending a ”0”-bit. The receipt of a ”0”-bit after a (perhaps long) sequence of ”1”-bits tells R that, directly after receiving the ”0”-bit, receipt of the message to be sent from S to R now starts. Similar to this example from Data Communication the ”0”-digit directly after a sequence of ”1”-digits indicates in the encoding represented by ***f***, that the digits of the numerator are beginning after this first ”0”-digit. We see that a principle which is successfully applied in Computer Science can be successfully used in Mathematics, too.

2. REPRESENTATION OF THE SET OF RATIONALE NUMBERS BY A TOTALLY-ORDERED SET.

Based on the mapping function ***f***, introduced by us, we can obtain a total-ordering of the set of rationale numbers, whose common divisor is not larger than 1. For this purpose, we define an ordering relation **<•** (in words “smaller after being encoded“) which has the property, that either b_A **<•** b_B or b_B **<•** b_A for two arbitrary ratios $b_A \in \mathbf{Q}$ and $b_B \in \mathbf{Q}$, $b_A \neq b_B$. To obtain the solution (regarding the relation **<•**) we compare the integer numbers ***f***(b_A) and ***f***(b_B).
We now define the ordering relation as follows:

$$b_A <\bullet\ b_B \quad \Leftrightarrow \quad \boldsymbol{f}(b_A) < \boldsymbol{f}(b_B)$$
$$b_B <\bullet\ b_A \quad \Leftrightarrow \quad \boldsymbol{f}(b_B) < \boldsymbol{f}(b_A)\ .$$

Therefore, the relation **<•** implies a total ordering for the set of rationale numbers for which numerator and denominator do not possess a common divisor larger than 1, because $b_A \neq b_B$ implies that $\boldsymbol{f}(b_A) \neq \boldsymbol{f}(b_B)$ also holds and thus one of both ratios (after being encoded) is smaller than the other.

The assumption that the ratios being compared possess numerator and denominator without a common divisor larger than 1 is required to make sure that there is a unique result when we order both ratios. As an example we choose $b_A = \frac{1}{10}$ and $b_B = \frac{3}{10}$. Then, we get b_A **<•** b_B because 10110 < 10310. However, if we represent b_A by $b_A = \frac{10}{100}$, we suddenly would get b_B **<•** b_A because 10310 < 11010100. Therefore, we assume that $\frac{1}{10}$ for which numerator and denominator do not possess a common divisor larger than 1 does represent the ratio b_A with the unique consequence b_A **<•** b_B .

3. INVERSION OF THE INJECTIVE MAPPING ONTO A SUBSET OF THE INTEGERS.

Unlike a lot of other injective functions ***f***: $\mathbf{Q} \rightarrow \mathbf{Z}$, suggested up to now, for our function ***f*** it is extremely easy to characterize the elements of ***f***(**Q**). To demonstrate this shortly, let $\mathbf{Z_0^*} := \mathbf{Z^*} \cup \{\mathbf{0}\}$, where **Z*** comprises that set of integer numbers which consists of the following numbers $x \in \mathbf{Z}$

$$x = +/-\ x_1 x_2 \ldots x_r x_{r+1} x_{r+2}\ \ldots\ x_{2r+1} x_{2r+2}\ \ldots\ x_s\ ,$$

where $r \geq 1$, $x_k = 1$ for $k \leq r$, $x_{r+1} = 0$, $x_{r+2} \neq 0$, $x_{2r+2} \neq 0$ and $s \geq 2r+2$.

Moreover, we assume that $p=x_{r+2}\ldots x_{2r+1}$ and $q=x_{2r+2}\ldots\ x_s$ do not possess a common divisor larger than 1. We easily see that $\boldsymbol{f}(\mathbf{Q}) = \mathbf{Z_0}^*$. So, $\boldsymbol{f}$ can be used to obtain a bijective mapping between **Q** and $\mathbf{Z_0}^*$.

Example 4. We choose x = -11107891234 as an arbitrary element of **Z***. Then x, in a unique manner is mapped onto the rational -789/1234.

Example 5. We now choose an arbitrary ratio, e.g., b = 12/347. Then, in a unique manner, b is mapped onto the integer number x = 11012347. We observe that $x\in\mathbf{Z}^*$, because all conditions are fulfilled which are required for the elements of set **Z*** , i.e. in particular: $r=2\geq 1$, $x_k=1$ for $k \leq 2$, $x_3=0$, $x_4=1\neq 0$, $x_6=3\neq 0$ and $s=8 \geq 6=2r+2$.

Declarations

- **Ethical approval**
  not applicable
- **Availability of data and materials**
  not applicable

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