Original Research Article

CONVECTION – DIFFUSION EQUATION OF GALACTIC COSMIC RAYS (GCR) IN THE ATMOSPHERE AND ITS ANALYTICAL SOLUTION BY USING PARKER TRANSPORT EQUATION

Abstract:

This study depicts that to find the Analytical and numerical solutions of Diffusion – Convections Equations of Galactic Cosmic Rays (GCRs) by Finite Element method (FEM) and also to find the Energy Equation of GCRs by using a part of Parker's transport equation. This considers moreover centres on the exactness and acknowledgment of the FEM strategy by utilizing dissemination blunder, scattering mistake and add up to blunder investigation. The comes about are depicted both graphically and in a unthinkable frame, which essentially guarantees the method's legitimacy and the algorithm's proficiency to maintain the exactness, effortlessness, and nonlinear Convection-Diffusion Equation conditions. The proposed method may be connected for tackling any nonlinear convection diffusion equation. We concentrate on assaying the confluence and stability of the nonlinear parabolic partial differential equation. This study focuses on the delicacy and acceptance of FEM method by exercising dispersion error dissipation error, and total error analysis.

Keywords: Analytical Solutions, Numerical Solutions, Finite Element Method, Convection-Diffusion Equation, Parker's transport Equation.

INTRODUCTION

Enormous beams of GCR are high energy particles or clusters of particles that are move through space at about the speed of light. They start form the Sun, from exterior of the Sun powered framework in our claim system, and form removed galaxies. Modelling authentic life and industrial quandaries by applying partial differential equations (PDEs) is challenging for researchers and scientists. Bosen.G (1982) stated, a considerable number of quandaries arise from modelling nonlinear systems of differential equations. Researchers especially Kumar.A, et al.(2017) have endeavoured to solve these quandaries analytically or numerically utilizing different methods, Leonard B.P.,(1979, 1988) and equations to obtain higher precision levels. Present study addresses the one-dimensional Convection-Diffusion equation, Dehghan M, (2004), as this is a meaningful test to construct a novel discrete plan. As the base of Parker transport equation, we can evaluate the Analytical and Numerical solutions of Diffusion according to Pérez Guérrero JS (2009, 2013) and Convection equation of Galactic Cosmic Rays, Basdevant C et al. (1986)

The Parker Transport Equation, Sadiq Akter Lima et al. (2021) is to be written as

$$\frac{\partial \emptyset}{\partial t} + w.\frac{\partial \emptyset}{\partial x} = \mu\frac{\partial^2 \emptyset}{\partial x^2} + \frac{P}{3}\frac{\partial w}{\partial x}\frac{\partial \emptyset}{\partial P} + \frac{1}{P^2}\frac{\partial}{\partial P}\left(P^2 C_{PP}\frac{\partial \emptyset}{\partial P}\right) + \varepsilon_0\sigma(x) \qquad (1)$$

Now, from the above equation we consider the convection- diffusion (C-D) terms based on C. Zoppou (1999) and solve the analytical solution for Convection Diffusion (C-D) equation of GCR, Kumar.A, et al.(2017)

$$\frac{\partial \emptyset}{\partial t} = \mu\frac{\partial^2 \emptyset}{\partial x^2} - w.\frac{\partial \emptyset}{\partial x} \qquad (2)$$

$$t > 0$$

$$0 < x < K$$

Where μ is diffusion coefficient and w is the GCR velocity in the x-direction. Now by applying boundary conditions

$$\emptyset(0, x) = 0$$

$$\emptyset(t, K) = 1 \qquad (3)$$

By applying Initial conditions

$$\emptyset(0, x) = \begin{cases} 0, & 0 \le x \le K \\ 1, & x = K \end{cases} \qquad (4)$$

At present convert the partial differential equation (PDE) to pure PDE by transformation method

$$\emptyset(t, x) = D(t, x).u(t, x) \qquad (5)$$

Substitute the equation (5) in equation (1) we get

$$D_t u + D u_t = \mu[D_{xx} u + 2D_x u_x + D u_{xx}] - w(D_x u + D u_x) \qquad (6)$$

Dividing by D, we get

$$u_t = \mu u_{xx} + \mu u\left(\frac{D_{xx} - \frac{D_t}{\mu} - wD_x}{D}\right) - u_x\left(\frac{2\mu D_x - wD}{D}\right) \qquad (7)$$

To get the pure diffusion PDE we require

$$\mu u\left(\frac{D_{xx} - \frac{D_t}{\mu} - wD_x}{D}\right) = 0 \qquad (8)$$

$$u_x\left(\frac{2D_{xx} - wD}{D}\right) = 0 \qquad (9)$$

$$D_x - \frac{w}{2\mu} D = 0 \qquad (10)$$

The equation (10) has solution

$$D(t, x) = C(t).e^{\frac{w}{2\mu}x} \qquad (11)$$

Substitute the equation (10) in (8) we get $C(t) = C_1 e^{\frac{w^2}{4\mu}t}$ (12)

C_1 is a constant and we consider its value is 1 then we get

$$C(t) = e^{\frac{w^2}{4\mu}t} \qquad (13)$$

Substitute equation (13) in (11)

$$D(t, x) = e^{\left(\frac{w^2 t}{4\mu} + \frac{wx}{2\mu}\right)} \qquad (14)$$

By using equation (14) which gives PDE to solve

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \qquad (15)$$

By applying boundary condition from ∅ to u which leads to

$$\emptyset(t, 0) = 0$$

$$D(t, 0).u(t, 0) = 0$$

$$e^{\frac{w^2 t}{4\mu}} u(t, 0) = 0$$

$$u(t, 0) = 0 \qquad (16)$$

and also consider the conditions

$$\emptyset(t,K) = 1$$

$$\emptyset(t,K).u(t,K) = 1$$

$$e^{\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)}.u(t,K) = 1$$

$$u(t,K) = e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)} \tag{17}$$

Applying initial conditions

$$\emptyset(0,x) = \begin{cases} 0 & 0 \le x \le K \\ 1 & x = K \end{cases}$$

$$\emptyset(0,x).u(0,x) = \begin{cases} 0 & 0 \le x \le K \\ 1 & x = K \end{cases}$$

$$e^{\frac{wx}{2\mu}}.u(0,x) = \begin{cases} 0 & 0 \le x \le K \\ 1 & x = K \end{cases}$$

$$u(0,x) = \begin{cases} 0 & 0 \le x \le K \\ e^{-\frac{wx}{2\mu}} & x = K \end{cases} \tag{18}$$

By using variable separable method, the equation (15) become homogenous then let us consider

$$u(t,x) = \emptyset(t,x) + u_A(t,x) \tag{19}$$

Where u_A is the steady state solution and $\emptyset(t,x)$ satisfies the PDE with boundary conditions

$$u(t,x) = \emptyset(t,x) + \frac{x}{K}e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)} \tag{20}$$

Substitute equation (20) in (15) we get

$$\emptyset_t = \mu\emptyset_{xx} + P(t,x) \tag{21}$$

This PDE with homogeneous boundary conditions along source term is

$$P(t,x) = -\frac{d}{dt}u_A(t,x) \tag{22}$$

Now our intension is to find the value of $\emptyset(t,x)$, as we know the solution to diffusion is given by the following eigen function expansion

$$\emptyset(t,x) = \sum_{n=1}^{\infty}\beta_n(t)\sin\left(\sqrt{\gamma_n}x\right) \tag{23}$$

Where $\gamma_n = (\frac{n\pi}{K})^2$ are eigen values for n =1, 2, . . . and $\sin\left(\sqrt{\gamma_n}x\right)$ are eigen function. Substitute equation (23) in equation (21) in order to get an Ordinary Differential Equation to solve for $\beta_n(t)$ which gives

$$\sum_{n=1}^{\infty}\beta_n^{I}(t)\sin\left(\sqrt{\gamma_n}x\right) = \mu\sum_{n=1}^{\infty} -\beta_n(t)\gamma_n\sin\left(\sqrt{\gamma_n}x\right) + P(t,x) \tag{24}$$

Now we expand $P(t,x)$ in favour of eigen functions

$$P(t,x) = \sum_{n=1}^{\infty}q_n(t)\sin\left(\sqrt{\gamma_n}x\right) \tag{25}$$

By applying orthogonality, we get

$$\int_0^K P(t,x).\sin\left(\sqrt{\gamma_n}x\right)dx = q_n(t).\frac{K}{2} \tag{26}$$

But

$$\int_0^K P(t,x).\sin\left(\sqrt{\gamma_n}x\right)dx = \frac{(-1)^n w^2 e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)}}{4\mu\sqrt{\gamma_n}} \tag{27}$$

From (26) we can find

$$q_n(t) = \frac{(-1)^n w^2 e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)}}{2K\mu\sqrt{\gamma_n}} \tag{28}$$

$$\beta_n^I(t) + \mu\gamma_n\,\beta_n(t) = q_n(t) \tag{29}$$

To solve the above equation with the integrating factor $\sigma = e^{\mu\gamma_n t}$ then

$$\beta_n(t) = \int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau + C_n.\,e^{\mu\gamma_n t}e^{-\mu\gamma_n t} \tag{30}$$

Put equation (30) in (20) we lead to

$$u(t,x) = \frac{x}{K}e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)} + \sum_{n=1}^{\infty}\left[\int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau + C_n.\,e^{-\mu\gamma_n t}\right]\sin\left(\sqrt{\gamma_n}x\right) \tag{31}$$

By applying initial conditions t =0, the equation (31) becomes

$$u(0,x) - \frac{x}{K}e^{-\left(\frac{wK}{2\mu}\right)} = \sum_{n=1}^{\infty} C_n \sin\left(\sqrt{\gamma_n}x\right) \tag{32}$$

Applying orthogonality

$$\int_0^K u(0,x).\sin\left(\sqrt{\gamma_n}x\right)dx - \int_0^K \frac{x}{K}e^{-\left(\frac{wK}{2\mu}\right)}\sin\left(\sqrt{\gamma_n}x\right)dx = C_n.\frac{K}{2} \tag{33}$$

Since $u(0,x) = 0$ then C_n value becomes

$$C_n = \frac{2}{K}\,.\,\frac{(-1)^n e^{-\left(\frac{wK}{2\mu}\right)}}{\sqrt{\gamma_n}} \tag{34}$$

But

$$\int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau = \frac{2(-1)^n w^2 e^{-\left(\mu\gamma_n t+\frac{wK}{2\mu}\right)}\left(e^{\mu\gamma_n t-\frac{tw^2}{4\mu}}-1\right)}{n\pi\left(4\mu^2\gamma_n - w^2\right)} \tag{35}$$

$$u(t,x) = \frac{x}{K}e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n w^2 e^{-\left(\mu\gamma_n t+\frac{wK}{2\mu}\right)}\left(e^{\mu\gamma_n t-\frac{tw^2}{4\mu}}-1\right)}{n\pi\left(4\mu^2\gamma_n - w^2\right)} + \frac{2}{K}\,.\,\frac{(-1)^n e^{-\left(\frac{wK}{2\mu}\right)}}{\sqrt{\gamma_n}}\right].\,sin\left(\sqrt{\gamma_n}x\right) \tag{36}$$

Now we covert to $\emptyset(t,x)$, the final solution becomes

$$\emptyset(t,x) = e^{\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)}.\,u(t,x) \tag{37}$$

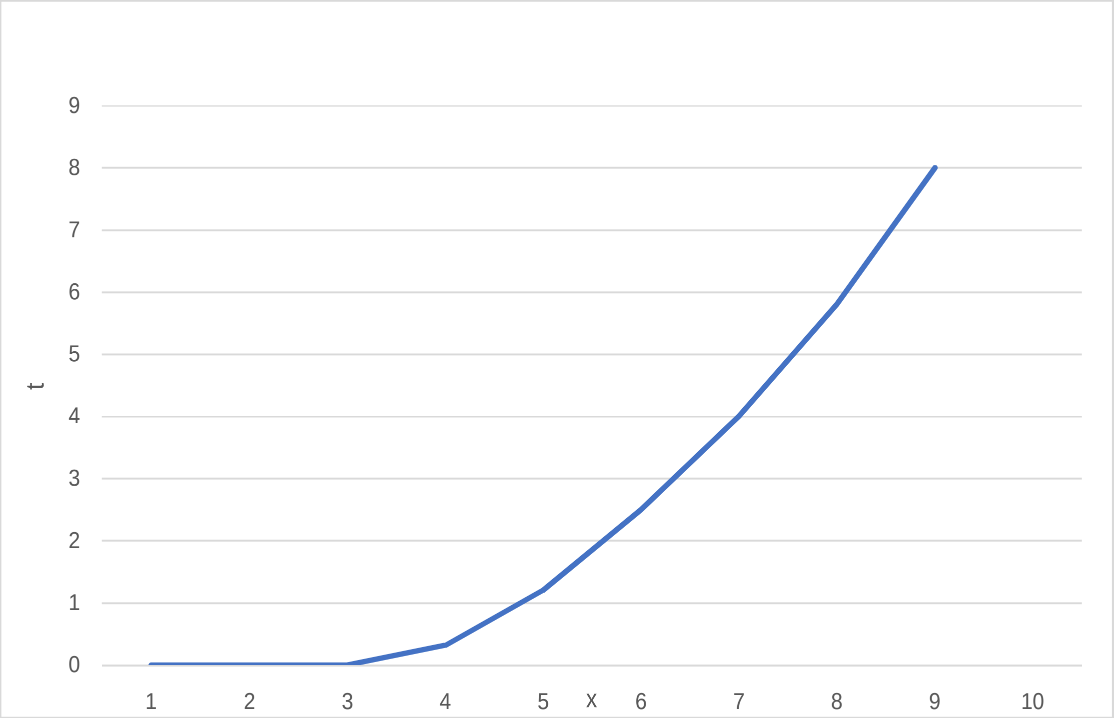

Figure 1: Analytical solution of Convection-Diffusion equation at 1st second

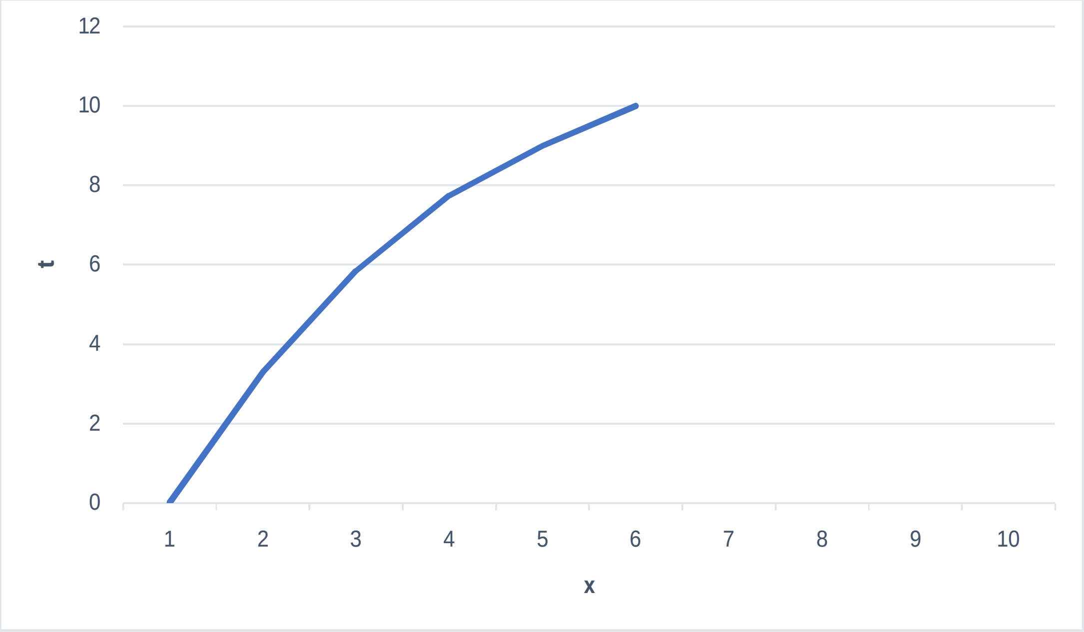

Figure 2: Analytical solution of Convection-Diffusion equation at last (30th) second

THE EQUATION OF ENERGY

By considering the law conservation of kinetic energy and integrating under the given limits

$$\int_0^1 [\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) + w\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) dx = \mu \int_0^1 \emptyset \frac{\partial^2 \emptyset}{\partial x^2} dx \quad \int_0^1 [\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) + w\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) dx = -\mu \int_0^1 \left(\frac{\partial \emptyset}{\partial x}\right)^2 dx \tag{38}$$

To evaluate the last term by using integral by parts then the energy equation reduces to

$$\frac{d}{dt}\int_0^1 \left(\frac{\emptyset^2}{2}\right) dx = -\mu \int_0^1 \left(\frac{\partial \emptyset}{\partial x}\right)^2 dx \tag{39}$$

This is the energy decay associated with diffusion, the time needed to reach zero velocity at each and every limit so integrate equation (2)

$$\int_0^1 \left(\frac{\partial \emptyset}{\partial t} + w\frac{\partial \emptyset}{\partial x}\right) dx = w\int_0^1 \frac{\partial^2 \emptyset}{\partial x^2} dx \tag{40}$$

Which results to the equation

$$\frac{d}{dt}\int_0^1 \emptyset dx = \frac{dU}{dt} = \mu[\frac{\partial \emptyset}{\partial x}(1,t) - \frac{\partial \emptyset}{\partial x}(0,t)] \tag{41}$$

U represents the average velocity. Successively we get

$$\mathrm{U(t)} = \mu \int_0^t [\frac{\partial \emptyset}{\partial x}(1,t) - \frac{\partial \emptyset}{\partial x}(0,t)] dt \tag{42}$$

FINITE ELEMENT METHOD

The Convection – Diffusion equation defined by Abdelkader Mojtabi (2015), takes the form of

$$\frac{\partial \emptyset}{\partial t} = \mu \frac{\partial^2 \emptyset}{\partial x^2} - w.\frac{\partial \emptyset}{\partial x} + f(C(x,t)),$$

$$(\mathrm{x, t}) \in B \equiv \Lambda(0,T], T > 0 \tag{43}$$

Where f($\emptyset$) is the equation source, by setting $\mu = 1$, $w = 1$ and f($\emptyset$) = - $\emptyset$ in the equation(43) with the domain $\Lambda \in [0,5]$, we get numerical simulation as follows

$$\frac{\partial \emptyset}{\partial t} - \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial \emptyset}{\partial x} = -\emptyset,$$

$$(x,t) \in B \equiv [0,5] \times (0,T], T > 0 \tag{44}$$

Along with this, by applying the boundary and initial condition we come across the equations (45 and (46) respectively

$$\emptyset(x,0) = e^{-x}, x \in [0,5] \tag{45}$$

$$\emptyset(0,t) = e^{t},$$

$$\emptyset(5,t) = e^{t-5},$$

$$t \in [0,T], \qquad x \in \partial B \tag{46}$$

The analytical solution of the Diffusion – Convection equation is

$$\emptyset(x,t) = e^{t-x} \tag{47}$$

The trial solution of equation (43) is

$$\emptyset(x,0) = \sum_{i=1}^{n} w(t).\Psi_j(x) \tag{48}$$

Here, comparing exact solution with approximate solutions of equation (44). In that case, Finite number of elements [is n = 5 and two linear shape functions are represented by

$$l_1(\xi) = \frac{1-\xi}{2},\ l_2(\xi) = \frac{1+\xi}{2};\ \xi \in [-1, +1] \qquad (49)$$

The convenient matrix form of equation (49) is given by

$Z_{i,j} = \alpha_{i,j} + \beta_{i,j} + \delta_{i,j};$

$S_{i,j} = \int \Psi_i(x)[\sum_{j=1}^{n} \Psi_j(x)]dx$

$\alpha_{i,j} = \int \frac{d\Psi_i(x)}{dx} . \frac{d\Psi_j(x)}{dx}\ dx$, F(t) = [$\frac{\partial \acute{\emptyset}}{\partial x} \Psi_j(x)$] $\beta_{i,j} = \int \Psi_i(x)[\sum_{j=1}^{n} \frac{d\Psi_j(x)}{dx}]\ dx$

$$\delta_{i,j} = \int \Psi_i(x)[\sum_{j=1}^{n} \Psi_j(x)]dx$$

We obtain approximate results at regular intervals in spatial distribution by numerical computation.

Table 1: Exact and Approximate solutions of (44) at h =Δt = 0.001

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0010	0.9903	1.0×10^{-2}
0.5	0.6846	0.6811	3.5×10^{-3}
1.0	0.3682	0.3719	3.6×10^{-3}
1.5	0.2519	0.2533	1.4×10^{-3}
2.0	0.1355	0.1348	7.0×10^{-4}
2.5	0.0922	0.0925	3.0×10^{-4}
3.0	0.0498	0.0501	3.0×10^{-6}
3.5	0.0341	0.0342	1.0×10^{-10}
4.0	0.0183	0.0183	1.0×10^{-10}
4.5	0.0126	0.0126	1.0×10^{-10}
5.0	0.0068	0.0068	1.0×10^{-10}

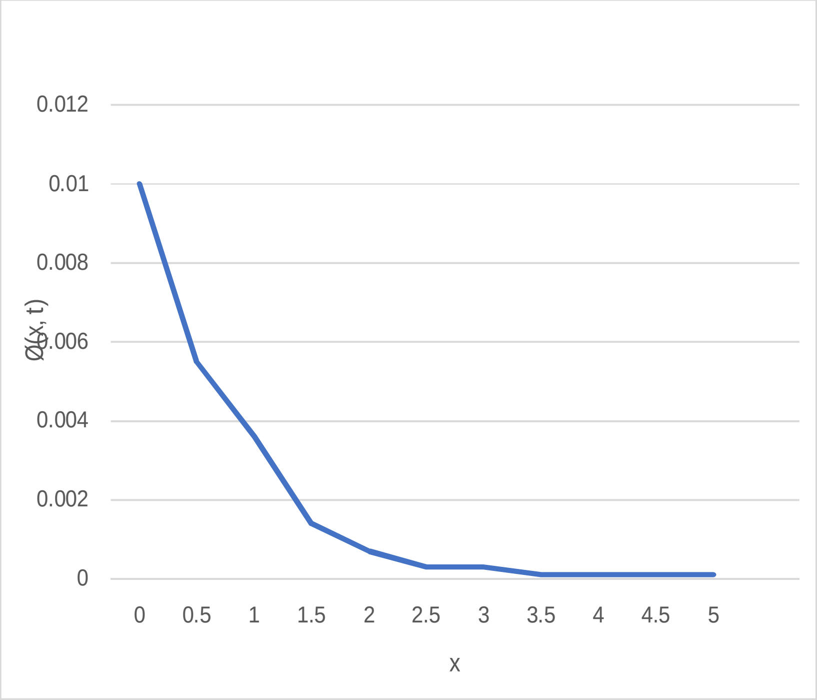

Figure 3: Graph of Exact and FEM solutions for Equation (44)

Table 2: Exact and Approximate solutions of (44) at h =Δt = 0.01

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0101	0.9026	10.0×10^{-2}
0.5	0.6909	0.6553	2.8×10^{-2}
1.0	0.3716	0.4079	3.6×10^{-2}
1.5	0.2542	0.2688	5.3×10^{-2}
2.0	0.1367	0.1297	7.0×10^{-3}
2.5	0.0935	0.0915	5.0×10^{-3}
3.0	0.0503	0.0532	2.9×10^{-3}
3.5	0.0344	0.0357	3.4×10^{-4}
4.0	0.0185	0.0181	4.0×10^{-4}
4.5	0.0127	0.0126	3.0×10^{-4}
5.0	0.0068	0.0070	2.0×10^{-4}

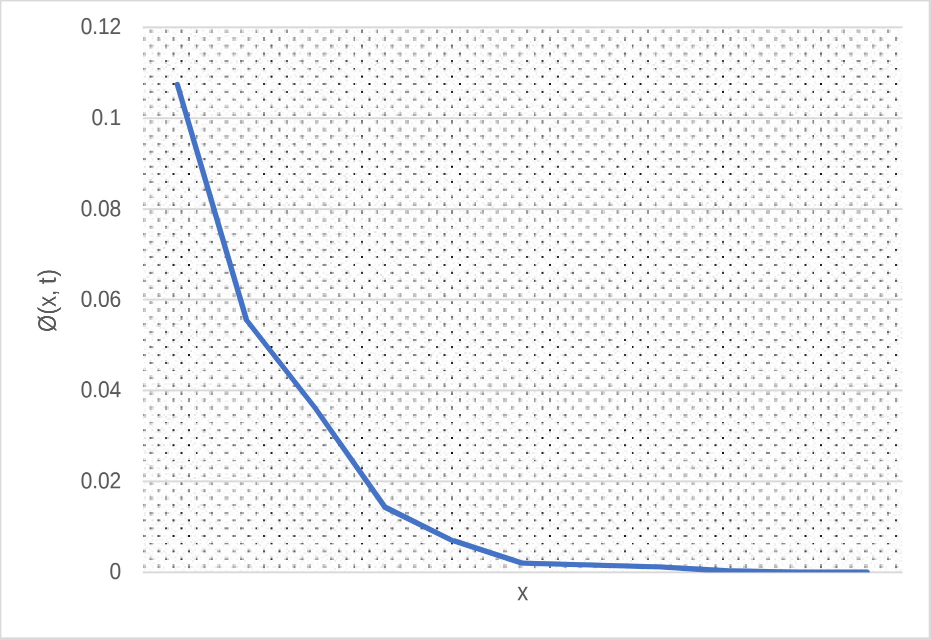

Figure 4: Graph of Exact and FEM solutions for Equation (44)

Table 3: Exact and Approximate solutions of (44) at h =Δt = 0.05

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0200	0.8052	2.1×10^{-1}
0.5	0.6977	0.6265	5.6×10^{-2}
1.0	0.3753	0.4478	7.2×10^{-2}
1.5	0.2067	0.2860	7.0×10^{-2}
2.0	0.1381	0.1241	1.4×10^{-3}
2.5	0.0945	0.0903	4.2×10^{-3}
3.0	0.0508	0.0565	5.8×10^{-3}
3.5	0.0348	0.0372	2.4×10^{-3}
4.0	0.0187	0.0179	8.0×10^{-4}
4.5	0.0128	0.0125	1.3×10^{-4}
5.0	0.0069	0.0074	5.0×10^{-4}

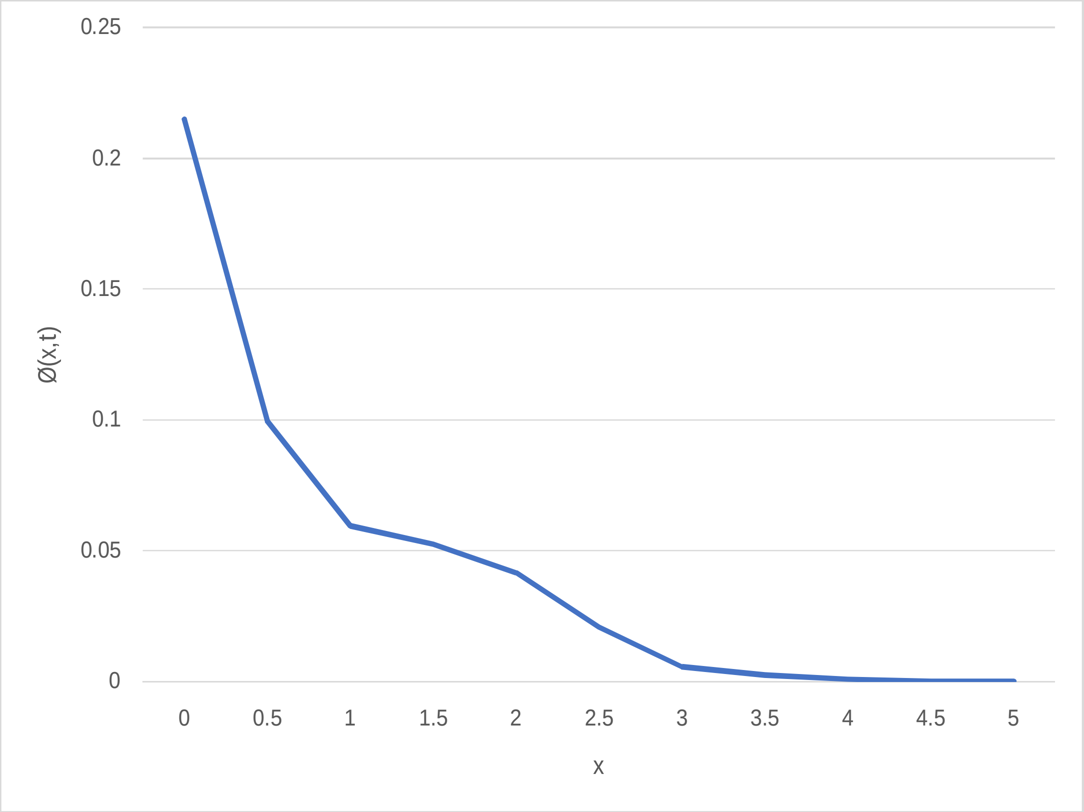

Figure 5: Graph of Exact and FEM solutions for Equation (44)

By combing the above three graphs we can get approximate similarity solutions of equation (44)

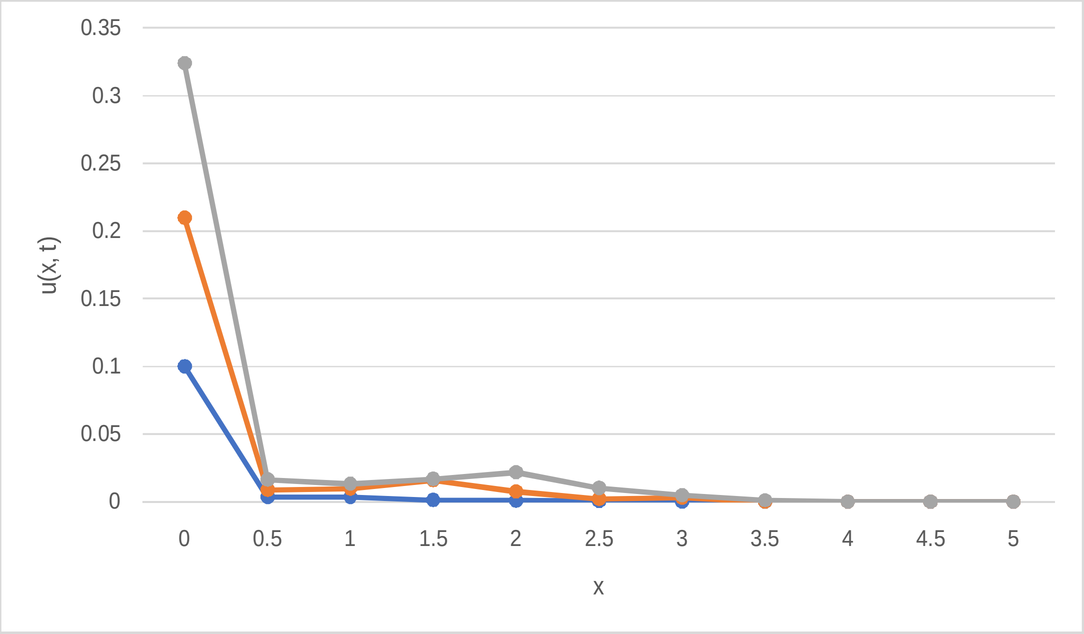

Figure 6: Similarity between exact and approximate solutions of equation (44)

CONCLUSION

From the above study one can understand the analytical solution to Diffusion and convection partial differential equation of GCR, the energy equation of Galactic Cosmic Rays (GCR) and by combining all the graphs we can get an idea that there must be rapid convergence in FEM than other numerical methods.

5. REFERENCES

Basdevant C, Deville M, Haldenwang P, Lacroix JM, Ouazzani J, Peyret R, 1986 “Spectral and finite difference solutions of the Burgers equation”, *Journal of Science Direct*, volume 14, Issue 1, pages 23-41; https://doi.org/10.1016/0045-7930(86)90036-8

Bosen G, 1982, “Method for the exact solution of a nonlinear diffusion–convection equation”. *Phys Rev Lett* ;1844–6, volume 49, Issue 25-20, https://doi.org/10.1103/ Phys Rev Lett49.1844

Dehghan M, 2004 “Weighted finite difference techniques for the one-dimensional advection–diffusion equation”. *Appl Math Comput: Science Direct Journal*, volume 147, issue 2, pages 307-319, https://doi.org/10.1016/S0096-3003(02)00667-7

Kumar A, Kumar Jaiswal D, Kumar N, 24 july 2017, “Analytical solutions to one-dimensional advection–diffusion equation with variable coefficients in semi-infinite media”.;380:330–7. *Science Direct journal*, volume 27, issue 14, pages R713 – R 715, https://doi.org/10.1016/ j. jhydrol. 2009. 11.008

Pérez Guérrero JS, Pimentel LCG, Skaggs TH, van Genuchten MTh., 2009, “Analytical solution of the advection–diffusion transport equation using a change-of variable and integral transform technique”. *International Journal of Heat Mass Transfer*; volume 52, issues 13-14, pages 3297-3304, https://doi.org/ 10.1016/j.ijheatmasstrnsfer. 2009.02.002

Pérez Guérrero JS, Pimentel LCG, Skaggs TH, Van Genuchten MTh., 1 April 2013 “Analytical solution of the one-dimensional advection–dispersion solute transport equation subject to time boundary conditions". *Chemical Engineering Journal*; volume 221, pages 487-491, https://doi.org/10.1016/j.ijheatmasstransfer . 2009.02.002

Sun P, Chen L, Xu J, 1 April 2013, "Numerical studies of adaptive finite element methods for two-dimensional convection–dominated problems". *Journal of Scientific Computing*, volume 43, pages 24–43, https://doi.org/10.1007/s10915-009-9337-6

Mohammad Farrukh N. Mohsen and Mohammed H. Baluch, August 1983, Paper "An analytical solution of the diffusion convection equation over a finite domain". *Journal of Applied Mathematics Modelling*, Volume 7, Issue 4, pages 285-287, , https://doi.org/10.1016/0307-904X(83)90084-7

C. Zoppou and J.H. Knight, 1999. "Analytical solution of a spatially variable coefficient advection diffusion equation in up to three dimensions", *Journal of Applied Mathematical Modelling*, volume 23, page no.s 667-685, https://doi.org/10.1016/S0307-904X(99) 00005-0

Chen, Y. and Falconer, R. A., 1994, "Modified forms of the third-order convection, second-order diffusion equation". *Journal of Advances in Water Resources*, volume 17, page no.s 147–170, https://doi.org/10.1016/0309-1708 (94)90038-8.

Gentry, R. A., Martin, R. E. and Daly, B. J, 1996, "A Eulerian differencing method for unsteady compressible flow problems". *Journal of Computational Physics*, volume 8, page no.s 55–76, https://doi.org/10.1016/0021-9991 (66) 90014-3

Hindmarsh, A. C., Gresho, P. and Griffiths, D. F, 1984, "The stability of explicit Euler integration for certain finite difference approximations of the multidimensional advection–diffusion equation". *International Journal for Numerical Methods in Fluids*, volume 4, page nos 853–897, https://doi.org/10.1002/fld. 1650040905

Leonard, B. P, 1979, "A stable and accurate convective modelling procedure based on upstream formulation". *Journal of Computer Methods in Applied Mechanics and Engineering*, volume 19, pages 59–98. https://doi.org/10.1016/0045-7825(79) 90034 -3.

Leonard, B. P, 1988, "Simple high accuracy resolution program for convective modeling of discontinuities". *International Journal for Numerical Methods in Fluids*, volume 8, pages 1291–1318, https://doi.org/10.1002/ fld. 1650081013

Li, Y. S. and Chen, C. P., 1989, "An efficient split operator scheme for 2D advection diffusion equation using finite elements and characteristics". *Journal of Applied Mathematical Modeling*, volume 13, pages 248–253, https://doi.org/10.1016/0307-904X (8)90083-8

Noye, B. J. and Tan, H. H. 1988: "A third-order semi-implicit finite difference method for solving the one-dimensional convection–diffusion equation". *International Journal for Numerical Methods in Engineering*, volume 26, pages 1615–1629. https://doi.org/10.1002/nme.1620260711

Sobey, R. J, 1983, "Fractional step algorithm for estuarine mass transport", *International Journal for Numerical Methods in Fluids*, volume 3, pages 567–581, https://doi.org/10.1002 /fld.1650030604

Sommeijer, B. P. and Kok, J, 1995 "Implementation and performance of the time integration of a 3D numerical transport model", *International Journal for Numerical Methods in Fluids*, volume 21, pages 349–367, https://doi.org/10.1002/fld.1650200303.

Abdelkader Mojtabi, Michel O. Deville, 2015 "One-Dimensional linear advection-diffusion equation: Analytical and finite element solutions" *International Journal of Computers and Fluids*, Volume 107, pages 189-195, http://dx.doi.org/ 10.1016/j.compfluid.2014. 11. 006.

Sadiq Akter Lima, Md.Kamrujjaman and Md.Shafiqul Islam; 2021, “Numerical solution of convection-Diffusion reaction equations by a finite element method with error correlation”, *Journal of American Institute of Physics advances*, volume 11, 085225: http://doi.org/10.1063/5.0050792