Research on Equation $\varphi(x)+2=\varphi(x+2)$

Abstract：In this paper, we use the properties of Euler's function,, elementary methods and the idea of classification discussion to study the solvability of equation $\varphi(x)+2=\varphi(x+2)$ related to Euler's functions and find positive integer solutions.

Key Words：Euler function；Equation；Positive integer solution

1 Introduction

Research on Euler's function is a very important and meaningful topic in number theory. Many scholars have studied its properties and obtained many interesting results.

Euler's function is defined as the number of positive integers that are less than or equal to n and relatively prime to n, denoted as $\varphi(n)$ **Error! Reference source not found.**. From the definition, we can see that $\varphi(1)=1,\varphi(2)=1,\varphi(3)=2,\cdots$. For a prime number p, all positive integers less than p are relatively prime to p, so $\varphi(p)=p-1$. If $n>1$, let canonical form of n be $n=p_1^{r_1}p_2^{r_2}\dots p_k^{r_k}$, where $p_1,p_2,\dots,p_k$ are different primes, $r_i\ge 1$ $(1\le i\le k)$, then

$$\varphi(n)=n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k}).$$ [2]

R. D. Carmichael[3] proof that if $\varphi(n)=2j$ ($j>1$ and j is odd), then $n=p^\alpha$ or $2p^\alpha$, p is an odd prime .

In 1945, Paul Erdös[4] conjectured that the equation

$$\varphi(n)=\varphi(n+1)=\varphi(n+2)=\cdots=\varphi(n+q)$$

is solvable for arbitrary positive integer q.

M.Lal and P.Gillard[5] provided the number of solutions for the equation $\varphi(n)=\varphi(n+k)$ for $k<30$ and n in the range of 10^4 to 10^5.

Makowski[6] considered the solution of equation $\varphi(x)+\varphi(k)=\varphi(x+k)$.

Patricia Jones[7] proof that if $\varphi(x)+\varphi(3)=\varphi(x+3)$, then

(i) $x=2p^\alpha$ or $x=2p^\alpha-3$, and prime $p>3$.

(ii) Either x or $x+3$ has at least 33 distinct prime factors.

(iii) $x=2p^{\alpha}$, where α is odd, prime $p\equiv 2(\mathrm{mod}\,3)$, $x>10^{11}$, and $x+3$ has at least 9 distinct prime factors.

V.L.klee[8] listed the values of the Euler function for $n<3000$, and find that the equation $\varphi(n)+2=\varphi(n+2)$ holds when both n and $n+2$ are prime, or n is the form of $4p$ and both p and $2p+1$ are prime.

Moser Leo [9] proof that if $\varphi(n)+2=\varphi(n+2)$, then at least one of n and $n+2$ is of the form p^{α} or $2p^{\alpha}$, and p is a prime number in the form of $4r+3$.

When $x>2$, $\varphi(x)$ must be even. When $k=1$, the equation $\varphi(x)+k=\varphi(x+k)$ has only one solution, which is $x=2$ obviously. For odd k, it is easy to show that the equation $\varphi(x)+k=\varphi(x+k)$ has only one solution $x=2$ when $k+2$ is prime .For even k, it is more difficult, we study the equation $\varphi(x)+k=\varphi(x+k)$ due to $k=2$, and get the following results.

Theorem 1 If prime $p\equiv 3(\mathrm{mod}\,4)$ and positive integer α satisfying

$$\varphi\left(2p^{\alpha}-2\right)+2=\varphi\left(2p^{\alpha}\right),$$

then α = 1 and both p and $\frac{p-1}{2}$ are primes.

Theorem 2 If prime $p\equiv 3(\mathrm{mod}\,4)$ and positive integer α satisfying $\varphi\left(2p^{\alpha}\right)+2=\varphi\left(2p^{\alpha}+2\right)$, except for the cases when $\alpha=1$ and p is a Mersenne prime, or when $\alpha=2$ and $p=3$, any other solutions must satisfy the following $\alpha=2^{a}\,(a>1)$ and $\omega\left(\frac{p^{2^{a}}+1}{2}\right)>1$, where $\omega(n)$ denotes the number of distinct prime factors of n.

Theorem 3 If prime $p\equiv 3(\mathrm{mod}\,4)$ and positive integer α satisfying $\varphi\left(p^{\alpha}\right)+2=\varphi\left(p^{\alpha}+2\right)$, except for the cases when $\alpha=1$, both p and $p+2$ are prime, any other solutions must satisfy the following conditions, α is odd and $\alpha>1$ and $p\equiv 11(\mathrm{mod}\,12)$ and p^{α} has one prime factor $q\equiv 1(\mathrm{mod}\,3)$ at least.

Theorem 4 If prime $p\equiv 3(\mathrm{mod}\,4)$ and positive integer α satisfying $\varphi\left(p^{\alpha}-2\right)+2=\varphi\left(p^{\alpha}\right)$, except for the cases when $\alpha=1$, both p and $p-2$ are twin primes, any other solutions must satisfy the following conditions:

(1) $p=3$ and $3^{\alpha}-2$ has even number of prime factors $q\equiv 2(\bmod 3)$ or

(2) $p\equiv 1(\bmod 3)$ and $p^{\alpha}-2$ has odd number of prime factors $q\equiv 2(\bmod 3)$ or

(3) $p\equiv 2(\bmod 3)$, α is even and $p^{\alpha}-2$ has one prime factor $q\equiv 1(\bmod 3)$ at least .

Theorem 5 Except for $x=18$ or $x=2M_p$ where M_p is a Mersenne prime, or x and $x+2$ are twin primes, or $x=2p-2$ and both p and $\frac{p-1}{2}$ are primes, other solutions of the equation $\varphi(x)+2=\varphi(x+2)$ must satisfy the following conditions:

(i) $x=2p^{\alpha}$,where $\alpha=2^{a}(a>1)$ and $\omega\left(\frac{p^{2^{a}}+1}{2}\right)>1$.

(ii) $x=p^{\alpha}$, where α is odd and $\alpha>1$, $p\equiv 11(\bmod 12)$ and p^{α} has one prime factor $q\equiv 1(\bmod 3)$ at least .

(iii) $x=3^{\alpha}-2$ and all the prime factors of x must be the form $3r+2$.

(iv) $x=p^{\alpha}-2$, where either $p\equiv 1(\bmod 3)$ and all the prime factors of x must be the form $3r+2$ or $p\equiv 2(\bmod 3)$, α is even and $p^{\alpha}-2$ has one prime factor $q\equiv 1(\bmod 3)$ at least.

2 Preliminaries

Lemma 1[10] If n is an odd integer, then $\varphi(2n)=\varphi(n)$. If n is an even integer, then $\varphi(2n)=2\varphi(n)$.

Lemma 2[10] If $\varphi(n)=n-1$, then n is a prime.

Lemma 3[9] If $\varphi(n)+2=\varphi(n+2)$ then at least one of n and $n+2$ has the form p^{α} or $2p^{\alpha}$, where p is a prime of the form $4r+3$.

Lemma 4 If $\varphi(n)=\frac{n}{2}$, then $n=2^{\alpha}(\alpha>0)$.

Proof Let $n=2^{\alpha}n_1(\alpha>0,(2,n_1)=1)$. Then

$$2^{\alpha-1}n_1=\frac{n}{2}=\varphi(n)=\varphi(2^{\alpha})\varphi(n_1)=2^{\alpha-1}\varphi(n_1).$$

So $\varphi(n_1)=n_1$, we have $n_1=1$. Thus $n=2^{\alpha}(\alpha>0)$.

3 Proof of the Theorems

3.1 Proof of theorem 1

For the equation $\varphi(2p^\alpha - 2) + 2 = \varphi(2p^\alpha)$, since $p \equiv 3 \pmod 4$ and α is positive integer, then $p^\alpha - 1$ is even. By Lemma 1, we have

$$p^\alpha - p^{\alpha-1} - 2 = 2\varphi(p^\alpha - 1).$$

(1) When $\alpha = 1$, $p - 3 = 2\varphi(p-1)$, it is obviously that $p \neq 3$. Since p is a prime number of the form $4r+3$, we have $\frac{p-1}{2}$ is an odd prime. Therefore, by Lemma 1, we have

$$\varphi\left(\frac{p-1}{2}\right) = \frac{p-3}{2} = \frac{p-1}{2} - 1.$$

By Lemma 2, we have $\frac{p-1}{2} = q$ is prime.

(2) When $\alpha > 1$ and is odd, there exists a positive integer M such that $p^\alpha - 1 = (p-1)M$. Also, since p is a prime of the form $4r+3$, it follows that $\frac{p-1}{2}$ is odd.Thus

$$2p^\alpha - 2 = 2(p-1)M = 4\cdot\frac{p-1}{2}\cdot M,$$

Therefore, $2p^\alpha - 2$ must have an odd prime factor not exceeding $\frac{p-1}{2}$, so

$$\varphi(2p^\alpha - 2) \le (2p^\alpha - 2)\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{p-1}\right).$$

Furthermore $\varphi(2p^\alpha - 2) = \varphi(2p^\alpha) - 2 = p^\alpha - p^{\alpha-1} - 2$, so

$$p^\alpha - p^{\alpha-1} - 2 \le (2p^\alpha - 2)\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{p-1}\right).$$

Hence $(p^{\alpha-1} - 1)(p+1) \le 0$, it is impossible.

(3) When α is even, if $p > 3$, as $p^\alpha \equiv 1 \pmod 2$, $p^\alpha \equiv 1 \pmod 3$, so $p^\alpha - 1$must have factor 2 and 3. So

$$p^\alpha - p^{\alpha-1} - 2 = 2\varphi(p^\alpha - 1) \le 2(p^\alpha - 1)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right).$$

Hence $p^{\alpha}-3p^{\alpha-1}-4<0$, $p<4$ is contradictory to condition $p>3$.

If $p=3$, then $\varphi\left(2\cdot 3^{\alpha}\right)=\varphi\left(2\cdot 3^{\alpha}-2\right)+2$, as 3^{α} is odd, $3^{\alpha}-1$ is even, by Lemma 1, we have $\varphi\left(3^{\alpha}\right)=2\varphi\left(3^{\alpha}-1\right)+2$. As $\varphi\left(3^{\alpha}\right)=3^{\alpha-1}\times 2$, so $3^{\alpha-1}-1=\varphi\left(3^{\alpha}-1\right)$. As $3^{\alpha}-1\equiv 0\left(\bmod 8\right)$, according to the computation and properties of Euler's totient function, we can obtain $\varphi\left(3^{\alpha}-1\right)\equiv 0\left(\bmod 4\right)$, but $3^{\alpha-1}-1\equiv 2\left(\bmod 4\right)$, it is contradictory.

Combining with (1)、(2) and (3), we obtain the conclusion of Theorem 1.

3.2 Proof of theorem 2

For the equation $\varphi\left(2p^{\alpha}\right)+2=\varphi\left(2p^{\alpha}+2\right)$, since $\varphi\left(2p^{\alpha}\right)=p^{\alpha}-p^{\alpha-1}$, we have

$$\varphi\left(2p^{\alpha}+2\right)=p^{\alpha}-p^{\alpha-1}+2. \tag{3-1}$$

(1) When $\alpha=1$, by Lemma 1, we obtain

$$\varphi\left(p+1\right)=\frac{p+1}{2}.$$

By Lemma 4, we have $p+1=2^{\beta}$,that is $p=2^{\beta}-1$ is a Mersenne prime.

(2) When $\alpha>1$ and α is odd, there exists a positive integer M such that

$2p^{\alpha}+2=4\cdot\frac{p+1}{2}\cdot M$. If $p+1=2^{k}$, then $2p^{\alpha}+2=2^{k+1}\cdot\frac{p^{\alpha}+1}{p+1}$. And

$$\frac{p^{\alpha}+1}{p+1}=p^{\alpha-1}-p^{\alpha-2}+p^{\alpha-3}-p^{\alpha-4}+\cdots+1>1$$

is odd and greater than 1. Therefore, the left side of (3-1) is

$$\varphi\left(2p^{\alpha}+2\right)=\varphi(2^{k+1})\varphi\left(\frac{p^{\alpha}+1}{p+1}\right)=2^{k}\varphi\left(\frac{p^{\alpha}+1}{p+1}\right)\equiv 0\left(\bmod 2^{k+1}\right).$$

But, the right side of (3-1) is

$$\begin{aligned}
&p^{\alpha}-p^{\alpha-1}+2=p^{\alpha-1}\left(p-1\right)+2=\left(2^{k}-2\right)\left(2^{k}-1\right)^{\alpha-1}+2\\
&=\left(2^{k}-2\right)\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}+1\right)+2\\
&=2^{k}\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}\right)\\
&-2\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}\right)+2^{k}\\
&\equiv 2^{k}\left(\bmod 2^{k+1}\right).
\end{aligned}$$

Contradictory, thus $p+1 \neq 2^k$. So $p^\alpha + 1$ must have an odd prime factor not exceeding $\frac{p+1}{4}$. Thus

$$p^\alpha - p^{\alpha-1} + 2 = \varphi\left(2p^\alpha + 2\right) \le \left(2p^\alpha + 2\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{p+1}{4}\right),$$

hence $3p^\alpha - p^{\alpha-1} + p + 5 \le 0$, it is impossible.

(3) When $\alpha = 2^a b$, $(2,b) = 1$, if $b > 1$, then

$$p^{2^a b} - p^{2^a b-1} + 2 = \varphi\left(2p^{2^a b} + 2\right),$$

because $2p^{2^a b} + 2$ must have factors 2 and $p^{2^a} + 1$,

$$\varphi\left(2p^{2^a b} + 2\right) \le \left(2p^{2^a b} + 2\right)(1 - \frac{1}{2})\left(1 - \frac{2}{p^{2^a} + 1}\right) = \left(p^{2^a b} + 1\right)\frac{p^{2^a} - 1}{p^{2^a} + 1}.$$

Hence $p^{2^a} + p^{2^a b} - p^{2^a b-1} + 3 < 0$, it is impossible. Therefore, if such an even number α exists, then it must be $\alpha = 2^a (a \ge 1)$. In this case (3-1) is

$$\varphi\left(2p^{2^a} + 2\right) = p^{2^a} - p^{2^a - 1} + 2.$$

Since p is an odd prime, $p^{2^a} + 1 \equiv 2 \pmod 8$, thus $\frac{p^{2^a} + 1}{2}$ is odd, by Lemma 1 we have

$$\varphi\left(\frac{p^{2^a} + 1}{2}\right) = \frac{p^{2^a} - p^{2^a - 1} + 2}{2}. \qquad \text{(3-2)}$$

Because

$$\left(\frac{p^{2^a} + 1}{2}, \frac{p^{2^a} - p^{2^a - 1} + 2}{2}\right) = \left(\frac{p^{2^a} + 1}{2}, \frac{p^{2^a - 1} - 1}{2}\right) = \left(\frac{p+1}{2}, \frac{p^{2^a - 1} + 1}{2} - 1\right) = \left(\frac{p+1}{2}, 1\right) = 1,$$

$\frac{p^{2^a} + 1}{2}$ is square-free.

(i) When $\omega\left(\frac{p^{2^a} + 1}{2}\right) = 1$, (3-2) is

$$\frac{p^{2^a} + 1}{2} - 1 = \frac{p^{2^a} - p^{2^a - 1} + 2}{2},$$

it leads to $p^{2^a - 1} = 3$, so $p = 3, a = 1$, in this case $x = 2p^{2^a} = 2 \times 3^2 = 18$.

(ii) When $\omega\left(\frac{p^{2^a}+1}{2}\right)>1$, then $\frac{p^{2^a}+1}{2}$ must have an odd prime factor that is less than $\sqrt{\frac{p^{2^a}+1}{2}}$, so

$$\varphi\left(\frac{p^{2^a}+1}{2}\right)<\left(\frac{p^{2^a}+1}{2}\right)\left(1-\frac{1}{\sqrt{\frac{p^{2^a}+1}{2}}}\right)=\frac{p^{2^a}+1}{2}-\sqrt{\frac{p^{2^a}+1}{2}}.$$

By (3-2)

$$\frac{p^{2^a}-p^{2^a-1}+2}{2}<\frac{p^{2^a}+1}{2}-\sqrt{\frac{p^{2^a}+1}{2}},$$

it leads $2p^{2^a}-p^{2^{a+1}-2}+2p^{2^a-1}+1<0$. When $a=1$, $p^2+2p+1<0$, it is impossible.

Therefore, if such an α exists, then it must be $\alpha=2^a\,(a>1)$ and $\omega\left(\frac{p^{2^a}+1}{2}\right)>1$.

Combining with (1)、(2) and (3), we obtain the conclusion of Theorem 2.

3.3 Proof of theorem 3

For the equation $\varphi\left(p^\alpha+2\right)=\varphi\left(p^\alpha\right)+2=p^\alpha-p^{\alpha-1}+2$, since

$$\left(p^\alpha+2,\ p^\alpha-p^{\alpha-1}+2\right)=\left(p^\alpha+2,-p^{\alpha-1}\right)=1,$$

$p^\alpha+2$ is square-free.

(1) When $\alpha=1$, $\varphi(p+2)=p+1$, by Lemma 2, we have $p+2$ is prime, that is when $p,\ p+2$ form a pair of twin primes, the equation $\varphi\left(p^\alpha+2\right)=\varphi\left(p^\alpha\right)+2$ holds.

(2) When $\alpha>1$ and $p=3$, $\varphi\left(3^\alpha+2\right)=3^\alpha-3^{\alpha-1}+2$, let $3^\alpha+2=q_1q_2\cdots q_i$, since $3^\alpha+2\equiv 2(\bmod 3)$, $3^\alpha+2$ must have an odd number of prime factors $q_j\equiv 2(\bmod 3)\ (1\le j\le i)$.

(i) If all $q_j\equiv 2(\bmod 3)\ (1\le j\le i)$, then

$$\varphi\left(3^\alpha+2\right)=\left(q_1-1\right)\left(q_2-1\right)\cdots\left(q_i-1\right)\equiv 1(\bmod 3),$$

(ii) If there exists $q_j\equiv 1(\bmod 3)\ (1\le j\le i)$, then

$$\varphi\left(3^\alpha+2\right)=\left(q_1-1\right)\left(q_2-1\right)\cdots\left(q_i-1\right)\equiv 0(\bmod 3),$$

but $3^\alpha-3^{\alpha-1}+2\equiv 2(\bmod 3)$, Contradiction.

(3) When $p>3$ and α is even, since $3\mid(p^\alpha+2)$, we have

$$p^{\alpha}-p^{\alpha-1}+2=\varphi\left(p^{\alpha}+2\right)<\left(p^{\alpha}+2\right)\left(1-\frac{1}{3}\right).$$

That is $p^{\alpha-1}(p-3)+2<0$, contradiction.

(4) When $p>3$ and $\alpha>1$ and is odd

(i) If $p\equiv 1(\bmod 3)$, then $3 \mid p^{\alpha}+2$, so $p^{\alpha}+2$ must have a factor 3, thus

$$p^{\alpha}-p^{\alpha-1}+2=\varphi\left(p^{\alpha}+2\right)<\left(p^{\alpha}+2\right)\left(1-\frac{1}{3}\right),$$

Simplifying gives $p^{\alpha-1}(p-3)+2<0$, contradiction.

(ii) If $p\equiv 2(\bmod 3)$, then $p^{\alpha}+2\equiv 1(\bmod 3)$. Let $p^{\alpha}+2=q_1q_2\cdots q_i$.

If all $q_j\equiv 2(\bmod 3)$ $(1\le j\le i)$, then

$$\varphi\left(p^{\alpha}+2\right)=(q_1-1)(q_2-1)\cdots(q_i-1)\equiv 1(\bmod 3).$$

But $p^{\alpha}-p^{\alpha-1}+2\equiv 0(\bmod 3)$, contradiction. Thus there must at least exist one prime $q_j\equiv 1(\bmod 3)$ $(1\le j\le i)$. Furthermore $p\equiv 3(\bmod 4)$, by Chinese Remainder Theorem, we have $p\equiv 11(\bmod 12)$ and $p^{\alpha}+2$ has at least one prime factor $q\equiv 1(\bmod 3)$.

Combining with (1)、(2)、(3) and (4), we obtain the conclusion of Theorem 3.

3.4 Proof of theorem 4

For the equation $\varphi\left(p^{\alpha}-2\right)+2=\varphi\left(p^{\alpha}\right)$ we have

$$\varphi\left(p^{\alpha}-2\right)=p^{\alpha}-p^{\alpha-1}-2. \tag{3-3}$$

(1) When $\alpha=1$, (3-3) is $\varphi(p-2)=p-3$. By Lemma 2, we have $p-2$ is prime, so when $\alpha=1$, both p and $p-2$ is a pair of twin primes, (3-3) holds.

(2) When $\alpha>1$ and $p=3$, (3-3) is $\varphi\left(3^{\alpha}-2\right)=3^{\alpha}-3^{\alpha-1}-2$.

Let $3^{\alpha}-2=q_1q_2\cdots q_i$. If there exists a prime factor $q_j\equiv 1(\bmod 3)$ $(1\le j\le i)$, then $\varphi\left(3^{\alpha}-2\right)\equiv 0(\bmod 3)$, but

$$3^{\alpha}-3^{\alpha-1}-2\equiv 1(\bmod 3),$$

contradiction.

Since $3^{\alpha}-2\equiv 1(\bmod 3)$, then we have all prime factors $q_j\equiv 2(\bmod 3)$ $(1\le j\le i)$ and i is even.

(3) When $\alpha>1$ and $p\equiv 1(\bmod 3)$, we have $p^{\alpha}-2\equiv 2(\bmod 3)$, let $p^{\alpha}-2=\prod_{j=1}^{i}q_j$.

If there exists a prime factor $q_j\equiv 1(\bmod 3)$ $(1\le j\le i)$, then $\varphi\left(p^{\alpha}-2\right)\equiv 0(\bmod 3)$, but

$$p^{\alpha}-p^{\alpha-1}-2\equiv 1(\bmod 3),$$

contradiction.

Thus all prime factors of $p^{\alpha}-2$ satisfying $q_j\equiv 2(\bmod 3)\ (1\le j\le i)$ and i is odd.

(4) When $\alpha>1$ and $p\equiv 2(\bmod 3)$, (i) If α is odd, then $3\mid(p^{\alpha}-2)$, therefore

$$p^{\alpha}-p^{\alpha-1}-2=\varphi(p^{\alpha}-2)<(p^{\alpha}-2)\left(1-\frac{1}{3}\right),$$

it gives $p^{\alpha-1}(p-3)<2$, contradiction.

(ii) If α is even, then $p^{\alpha}-2\equiv 2(\bmod 3)$, let $p^{\alpha}-2=q_1q_2\cdots q_i$.

If all $q_j\equiv 2(\bmod 3)\ (1\le j\le i)$, then $\varphi(p^{\alpha}-2)\equiv 1(\bmod 3)$, but

$$p^{\alpha}-p^{\alpha-1}-2\equiv 0(\bmod 3),$$

contradiction. Thus there exists one prime factor of $p^{\alpha}-2$ satisfying $q_j\equiv 1(\bmod 3)$.

Combining with (1)、(2)、(3) and (4), we obtain the conclusion of Theorem 4.

3.5 Proof of theorem 5

By Lemma 3, we know that the solution of equation $\varphi(x)+2=\varphi(x+2)$ satisfies $x=2p^{\alpha}-2,\ p^{\alpha}-2,\ p^{\alpha}$ or $2p^{\alpha}$ and $p\equiv 3(\bmod 4)$, α is a positive integer . Based on Theorem 1-4, we can conclude that the solutions to the equation

$$\varphi(x)+2=\varphi(x+2)$$

satisfying the following:

(1) $x=2p-2$, both p and $\dfrac{p-1}{2}$ are primes;

(2) $x, x+2$ is a pair of twin primes;

(3) $x=18$ or $x=2M_p$, where M_p is a Mersenne prime

(4) The other solutions x must satisfy

(i) $x=2p^{\alpha}$, where $\alpha=2^{a}\ (a>1)$ and $\omega\left(\dfrac{p^{2^{a}}+1}{2}\right)>1$.

(ii) $x=p^{\alpha}$, where $\alpha>1$ and α is odd, $p\equiv 11(\bmod 12)$ and p^{α} has at least one prime factor $q\equiv 1(\bmod 3)$.

(iii) $x=3^{\alpha}-2$, and all the prime factors of x must be the form $3r+2$.

(iv) $x=p^{\alpha}-2$, where either $p\equiv 1(\bmod 3)$ and all the prime factors of x must be the form $3r+2$ or $p \equiv 2 \pmod 3$, α is even and $p^{\alpha}-2$ has at least one prime factor $q \equiv 1 \pmod 3$.

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