ROTATION WITHOUT IMAGINARY NUMBERS, TRANSCENDENTAL FUNCTIONS, OR INFINITE SUMS

Abstract. Quaterns are introduced as a new measure of rotation. Rotation in quaterns has an advantage in that only simple algebra is required to convert back and forth between rectangular and polar coordinates that use quaterns as the angle measure. All analogue trigonometric functions also become algebraic when angles are expressed in quaterns. This paper will show how quatern measure can be easily used to approximate trigonometric functions in the first quadrant without recourse to technology, infinite sums, imaginary numbers, or transcendental functions. Using technology, these approximations can be applied to all four quadrants to any degree of accuracy. This will also be shown by approximating π to any degree of accuracy desired without reference to any traditional angle measure at all.

1. Introduction

DesCartes [1] invented Analytic Geometry by relating equations utilizing constants and variables to diagrams in the cartesian plane. Since that time, mathematicians have developed many methods to convert back and forth between the rectangular and polar coordinates of a point in the plane. Coolidge [2] documents the history of polar coordinates. De Moivre's formula [3] and Euler's formula [4] used the complex plane to make such calculations easier. Taylor [5] pioneered the use of infinite sums. These were then used to provide approximations of the trigonometric functions to whatever desired degree of accuracy.

This paper will first present an alternative method of describing rotation that is based upon the Manhattan distance. This new type of angle measure will then be used in its own appropriate type of polar coordinate system. Describing polar coordinates in this way has an advantage. The advantage is that one may then convert between polar and rectangular forms using only algebraic expressions. In other words, no imaginary or complex numbers are needed. No transcendental functions are needed. No infinite sums are needed. All of the trigonometric functions become algebraic rather than transcendental.

2. Method

This section introduces the method and uses it first to provide an approximation. Later sections of this paper will then develop this idea further such that the method will produce exact values.

Key words and phrases. Vector, angle, rotation, polar, rectangular, coordinates.

Place a vector of magnitude 1 with its initial point at the origin, and its terminal point at $(1,0)$, rotating counter-clockwise such that a positive angle θ is formed between the vector and the x-axis. As the terminal point of the vector completes one rotation around the origin, three different distances are summed:

1: The distance traced along the unit circle will be $D_c = 2\pi$.
2: The total distance traced in the x dimension, from $x = 1$, to $x = -1$, and back again, will be $D_x = 4$.
3: Likewise, the total distance traced in the y dimension, from $y = 0$, to $y = 1$, to $y = -1$, and then back to 0, will be $D_y = 4$.

The sum of the total distance travelled in the x direction plus the total distance travelled in the y direction has been called in the literature by many different names: Some examples are the taxicab distance, the rectilinear distance, and the Manhattan distance.

Assume that the proportion in equation 1 stays approximately constant over $0 \leq \theta \leq 2\pi$. This assumption will later be used in various approximations.

$$\frac{D_c}{D_x + D_y} \approx \frac{\pi}{4} \tag{1}$$

On the other hand, the integral in equation 2, for $0 \leq \theta \leq \frac{\pi}{2}$, will deliver the exact sum $D_y + D_x$ as a function of θ in the first quadrant.

$$\int_0^\theta |\cos\theta| + |-\sin\theta| \, d\theta = \sin\theta - \cos\theta + 1 \tag{2}$$

In the unit circle, $\sin\theta = y$ and $\cos\theta = x$. This means that the Manhattan distance travelled in the y direction plus the Manhattan distance travelled in the x direction along the unit circle from $(1,0)$ is simply $y - x + 1$. This distance is defined to be a new measure of rotation that has units of length. The ancient letter koppa ϟ will be used to signify this "angle". ϟ will be measured in quaterns where one quatern has a length of one unit, and one quatern also signifies a rotation $= 45°$, or $\frac{\pi}{4}$ radians. Thus, 8 quaterns would be a rotation of exactly $360°$, or 2π radians.

For any given point on the unit circle within the first quadrant, ϟ is exactly calculated using equation 3.

$$\text{ϟ} = y - x + 1 \tag{3}$$

In the first quadrant for all distances, 3 can be modified thus:

$$\text{ϟ} = \frac{y - x}{r} + 1 \tag{4}$$

Given the rectangular coordinates (x, y), $\theta \approx 45\text{ϟ}°$ or $\theta \approx \frac{\pi}{4}\text{ϟ}$ radians. EXAMPLE: If given the point $(15, 23)$ in rectangular coordinates, what is the approximation for θ?

$$r = \sqrt{15^2 + 23^2} \approx 27.46 \tag{5}$$

$$ϟ \approx \frac{23 - 15}{27.46} + 1 \approx 1.29 \tag{6}$$

$$\theta \approx 45ϟ \approx 58.05^\circ \text{ or } \theta \approx \frac{\pi}{4}ϟ \approx 1.01 \text{ radians} \tag{7}$$

The true value for θ associated with the point $(15, 23)$ is $56.89°$, about a 2% error.

3. Simple form within the first quadrant

In this section the simplest form represented by 4 will be used to conduct easily calculated approximations back and forth between rectangular and polar coordinates for any point in the first quadrant. Note that values converting between quatern-analogue polar and rectangular coordinates are exact; they only become approximations when converting to degrees or radians. When used as approximations, these simple formulas have errors of less than 3 percent.

Once the principles are established, following sections will present the general form, valid at all distances, and in all 4 quadrants.

Substituting $\sqrt{1-y^2}$ for x, and then solving 4 for y gives equation 8. When $r = 1$, this would be the $sine$ function.

$$y = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} + ϟ - 1) \tag{8}$$

By a similar process, solving equation 4 for x gives equation 9. When $r = 1$, this would be the $cosine$ function.

$$x = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} - ϟ + 1) \tag{9}$$

EXAMPLE: Given the standard polar point $(r, \theta) = (15, 35°$, what would be the $ϟ$ approximation for the x and y coordinates? First, equation 4 would be used to find $ϟ$:

$$ϟ \approx \frac{35}{45} \approx .778 \tag{10}$$

Using $.778$ in equation 9 will give the approximate x coordinate:

$$x \approx \frac{15}{2}(\sqrt{-(.778)^2 + 2(.778) + 1} - (.778) + 1) \approx 12.14 \tag{11}$$

The true value for the x coordinate is ≈ 12.29, which is an error of about 1%.

Using equation 8 in the same way to find the y coordinate yields an approximation of 8.81. Since the true value is about 8.60, this is an error of about 2.4%.

4. General Forms

A major issue in applying these principles to the other three quadrants is determining an algebraic method to govern the changes in the signs of the coefficients with step functions so as to be continuous while giving the correct sign at the correct time. In prior works this was accomplished using the imaginary plane, as in [3] and [4]. This method shown here accomplishes all of this using following algebraic step functions to govern the coefficients in the general equations:

4.1. The helper functions.

The I function adjusts the constant of the M term, and provides the I^2 term beneath the radical.

$$I = |4H - C| - 2H \tag{12}$$

The M step function adjusts the $ϟ$ input by quadrant:

$$M = ϟ - 8\left\lfloor \frac{ϟ}{8} \right\rfloor \tag{13}$$

The next three functions govern sign changes in all the quadrants:
Step function: the constant of π, P, for the coordinates functions

$$H = -1^{\left\lfloor \frac{ϟ}{4} \right\rfloor} \tag{14}$$

$$T = -1^{\left\lfloor \frac{ϟ}{2} \right\rfloor} \tag{15}$$

$$C = -(-1)^{\left\lfloor \frac{ϟ-2}{4} \right\rfloor} \tag{16}$$

4.2. The General Conversion Functions.

These functions are continuous; they work uniformly, unambiguously, and universally in all four quadrants, and at all distances:

Equation 17 gives $ϟ$ from rectangular coordinates:

$$ϟ = \left(\frac{\frac{x}{|x|}y - \frac{y}{|y|}x}{r} \right) + \left| \frac{4y}{|y|} - \frac{x}{|x|} \right| - \frac{2y}{|y|} \tag{17}$$

Equation 18 gives the x coordinate. If $r = 1$, then this is the analogue $cosine$ function.

$$x = \left(\frac{r}{2}\right)\left(\frac{\sqrt{-M^2 + 2IM + 2 - I^2} + T(I - M)}{C} \right) \tag{18}$$

Equation 19 gives the y coordinate. If $r = 1$, then this is the analogue $sine$ function.

$$y = \left(\frac{r}{2}\right)\left(\frac{\sqrt{-M^2 + 2IM + 2 - I^2} + T(M - I)}{H} \right) \tag{19}$$

4.3. **Other approaches approximating the trigonometric functions.** The approach to polar/rectangular conversion approximation presented in this paper is interesting in that one can work with vectors in a purely algebraic way that is 100% accurate so long as all angles remain in quaterns. These conversions only become approximations when one converts back into degrees or radians.

The desire for an algebraic approach is not new; this is related to quests like squaring the circle. Here are some other examples of various attempts at approximating the trigonometric functions.

Monks [6] explains an ancient method for approximating the $sine$ function. Azim [7] presents a more modern approach wherein one can also do the approximation without technology. Kusaka [8] seeks to increase computation speeds for low performing computers, while Kumar [9] does the same thing for faster computers.

Because quaterns can be used with technology to produce approximations to any degree of accuracy, it is possible that researching this method further might supplant some other methods.

5. Websites to demonstrate conversions

This link, rotation in quaterns, uses the Desmos online graphing calculator. At this site, one can calculate ϟ exactly given (x, y), and calculate (x, y) exactly if given $(r, ϟ)$. One can also approximate θ or approximate (x, y) using only algebraic expressions.

This paper thus far has shown that quatern measure can be used to easily approximate the trig and inverse trig functions. Using another link, Koppa π approximation, one can show the validity of quatern measure by showing that an infinite sum using quaterns can provide exact values. At this site, one can approximate π to any degree of accuracy **without referencing traditional angle measure in any way.**

The author demonstrates the Desmos calculations above, in the video accessed with this link: A new type of angle measure.

6. Conclusion

The archaic Greek symbol for koppa, ϟ, is suggested for use representing quaterns. There are several reasons for this:
1: Quaternions are the best way to describe rotations, so an allusion is made to that, as well as the phrase "quarter turn."
2: ϟ was the archaic Greek letter "q". It also represented the number "90", which recalls 90 degrees.

References

[1] René Descartes. *A discourse on method.* Aladdin Book Company, 1901.
[2] Julian Lowell Coolidge. "The origin of polar coordinates". In: *The American Mathematical Monthly* 59.2 (1952), pp. 78–85.
[3] Abraham De Moivre. "De Sectione Anguli, Autore A. de Moivre, RSS". In: *Philosophical Transactions (1683-1775)* 32 (1722), pp. 228–230.
[4] Leonhard Euler. *Introductio in analysin infinitorum.* Vol. 2. Apud Marcum-Michaelem Bousquet & Socios, 1748.

[5] Lenore Feigenbaum. *BROOK TAYLOR'S" METHODUS INCREMENTORUM": A TRANSLATION WITH MATHEMATICAL AND HISTORICAL COMMENTARY*. Yale University, 1981.

[6] Kenneth M Monks. "Bhaskara's Approximation to and Madhava's Series for Sine". In: (2020).

[7] Mashrur Azim, Asma Akter Akhi, and Md Kamrujjaman. "A Series of New Formulas To Approximate the Sine and Cosine Functions". In: (2023).

[8] Takashi Kusaka and Takayuki Tanaka. "Fast and Accurate Approximation Methods for Trigonometric and Arctangent Calculations for Low-Performance Computers". In: *Electronics* 11.15 (2022), p. 2285.

[9] Puli Anil Kumar. "FPGA implementation of the trigonometric functions using the CORDIC algorithm". In: *2019 5th International Conference on Advanced Computing & Communication Systems (ICACCS)*. IEEE. 2019, pp. 894–900.

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