Original Research Article

ROTATION WITHOUT IMAGINARY NUMBERS, TRANSCENDENTAL FUNCTIONS, OR INFINITE SUMS

Abstract. Algebraic equations are derived to approximate the relationship between the rectangular components and the polar coordinates of a vector. These equations can then be used without recourse to imaginary numbers, transcendental functions, orinfnite sums. This is done with an error of approximately one-half percent.

Key words and phrases. vector, angle, rotation, polar, rectangular, coordinates.

1. Introduction:

DesCartes [1] invented Analytic Geometry by relating equations utilizing constants and variables to diagrams in the Cartesian plane. Since that time, mathematicians have developed many methods to convert back and forth between the rectangular and polar coordinates of a point in the plane. Coolidge [2] documents the history of polar coordinates. De Moivre's formula [3] and Euler's formula [4] used the complex plane to make such calculations easier. Taylor [5] pioneered the use of infinite sums. These were then used to provide approximations of the trigonometric functions to whatever desired degree of accuracy.

This paper will first present an alternative method of describing rotation that is based upon the Manhattan distance. This new type of angle measure will then be used in its own appropriate type of polar coordinate system. Describing polar coordinates in this way has an advantage. The advantage is that one may then convert between polar and rectangular forms using only algebraic expressions. In other words, no imaginary or complex numbers are needed. No transcendental functions are needed. No infinite sums are needed. All of the trigonometric functions become algebraic rather than transcendental.

2. Method

This section introduces the method and uses it first to provide an approximation. Later sections of this paper will then develop this idea further such that the method will produce exact values.

Place a vector of magnitude 1 with its initial point at the origin, and its terminal point at $(1,0)$, rotating counter-clockwise such that a positive angle θ is formed between the vector and the x-axis. As the terminal point of the vector completes one rotation around the origin, three different distances are summed:

1: The distance traced along the unit circle will be $D_c = 2\pi$.
2: The total distance traced in the x dimension, from $x = 1$, to $x = -1$, and back again, will be $D_x = 4$.
3: Likewise, the total distance traced in the y dimension,from $y = 0$, to $y = 1$, to $y = -1$, and then back to 0, will be $D_y = 4$.

The sum of the total distance travelled in the x direction plus the total distance travelled in the y direction has been called in the literature by many different names: Some examples are the taxicab distance, the rectilinear distance, and the Manhattan distance.

Assume that the proportion $\frac{D_c}{D_x+D_y} = \frac{\pi}{4}$ stays approximately constant over $0 \leq \theta \leq 2\pi$. $\int_0^\theta |\cos\theta| + |-\sin\theta| \, d\theta = \sin\theta - \cos\theta + 1$ for $0 \leq \theta \leq \frac{\pi}{2}$ will deliver the sum $D_y + D_x$ as a function of θ in the first quadrant. For any given point on the unit circle within the first quadrant, θ can be approximated using

$$\theta = \frac{\pi}{4}(y - x + 1) \tag{1}$$

For right triangles, and in the first quadrant, for all distances, 1 can be modified thus:

$$\theta = \frac{\pi}{4}(\frac{y-x}{r} + 1) \tag{2}$$

3. Simple form: Within right triangles and the first quadrant

In this section the simplest form represented by 1 will be used to conduct transformations back and forth between rectangular and polar coordinates, restricting the analysis to angles within right triangles, i.e. quadrant 1, upon the unit circle. This simple form will give θ to within 3 percent; a later section will provide a more detailed and accurate version that gives θ to within one-half percent.

Once the principles are established, following sections will present the general form, valid at all distances, and in all 4 quadrants.

Substituting $\sqrt{1-y^2}$ for x, and then solving 1 for y gives

$$y = r\frac{\sqrt{-16\theta^2 + 8\pi\theta + \pi^2} + 4\theta - \pi}{2\pi} \tag{3}$$

By a similar process, the x coordinate is given by

$$x = r\frac{\sqrt{-16\theta^2 + 8\pi\theta + \pi^2} - 4\theta + \pi}{2\pi} \tag{4}$$

When $r = 1$, These are the x and y coordinates on the unit circle, or $\sin\theta$ and $\cos\theta$. Multiplying by r will thus give the x and y coordinates of a vector of any distance in the first quadrant.

4. General Forms

A major issue in applying these principles to the other three quadrants is determining an algebraic method to govern the changes in the signs of the coefficients with step functions so as to be continuous while giving the correct sign at the correct time. In prior works this was accomplished using the imaginary plane. This method accomplishes the same thing by first defining a function E that is used to reduce the error in θ, and also defining the following algebraic step functions to govern the coefficients in the general equations:

5. The helper functions

The error correction function, E. Because the error in θ given by 1 is continuous and periodic, this periodic function is used to reduce the error from 3 percent down to one-half percent.

$$E = \frac{.1324xy(x^2 - y^2)}{(x^2 + y^2)^2} \tag{5}$$

Step function: the θ constant function, T, adjusts the constant of 1 to the appropriate quadrant.

$$T = |4\frac{y}{|y|} - \frac{x}{|x|}| - 2\frac{y}{|y|} \tag{6}$$

Step function: the sign of the input, ς, for the coordinate functions gives the correct sign for the input adjusted to the quadrant

$$\varsigma = (\frac{\frac{\pi}{2} - \theta}{|\frac{\pi}{2} - \theta|})(\frac{\pi - \theta}{|\pi - \theta|})(\frac{\frac{3\pi}{2} - \theta}{|\frac{3\pi}{2} - \theta|}) \tag{7}$$

Step function: the constant of π, P, for the coordinates functions

$$P = ((\frac{\theta - \frac{\pi}{2}}{|\theta - \frac{\pi}{2}|}) + 1)((\frac{\theta - \frac{3\pi}{2}}{|\theta - \frac{3\pi}{2}|}) + \frac{3}{4}) \tag{8}$$

Step function: the constant, K, of the denominator function for x

$$K = (\frac{\theta - \frac{\pi}{2}}{|\theta - \frac{\pi}{2}|})(\frac{\theta - \frac{3\pi}{2}}{|\theta - \frac{3\pi}{2}|}) \tag{9}$$

Step function: the constant, Γ of the denominator function for y

$$\Gamma = \frac{\pi - \theta}{|\pi - \theta|} \tag{10}$$

6. The General Conversion Functions

These functions are continuous; they work uniformly, unambiguously, and universally in all four quadrants, and at all distances:

To extract θ, given rectangular coordinates, with an error of less than one-half percent:

$$\theta = \frac{\pi}{4}(\frac{\frac{x}{|x|}y - \frac{y}{|y|}x}{r} + T) + E \tag{11}$$

To extract x, for a vector of any magnitude r, given θ; (When $r = 1$, this is the $\cos\theta$ function):

$$x = r\frac{\sqrt{-16(\varsigma(\theta - P\pi))^2 + 8\pi(\varsigma(\theta - P\pi)) + \pi^2} - 4(\varsigma(\theta - P\pi)) + \pi}{2K\pi} \quad (12)$$

To extract y, for a vector of any magnitude r, given θ; (When $r = 1$, this is the $\sin\theta$ function):

$$y = r\frac{\sqrt{-16(\varsigma(\theta - P\pi))^2 + 8\pi(\varsigma(\theta - P\pi)) + \pi^2} + 4(\varsigma(\theta - P\pi)) - \pi}{2\Gamma\pi} \quad (13)$$

7. Graphic comparison

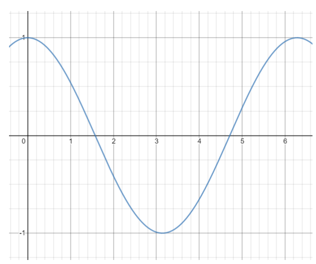

Figure 1. True Cosine, $0 \leq \theta \leq 2\pi$

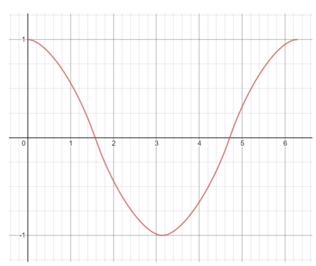

Figure 2. Approximate Cosine, according to equation 12

8. An alternative approach to rotation

A polar/rectangular coordinate system might be used, based on 2, wherein angles are defined as "quaterns" instead of radians. 2 would then be written as:

$$ϟ = \frac{y - x}{r} + 1 \quad (14)$$

3 would become:

$$y = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} + ϟ - 1) \quad (15)$$

4 would become:

$$x = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} - ϟ + 1) \tag{16}$$

This may be useful in certain contexts:
1: The equations and calculations described in this paper would then be exact. There would be an exact relationship between polar and rectangular coordinates.
2: Under this approach, the relationship between rectangular and polar coordinates would then be real and algebraic, rather than imaginary or transcendental.

9. Conclusion

The archaic Greek symbol for koppa, ϟ, is suggested for use representing quaterns. There are several reasons for this:
1: Quaternions are the best way to describe rotations, so an allusion is made to that, as well as the phrase "quarter turn."
2: ϟ was the archaic Greek letter "q". It also represented the number "90", which recalls 90 degrees.

References

[1] René Descartes. *A discourse on method.* Aladdin Book Company, 1901.
[2] Julian Lowell Coolidge. “The origin of polar coordinates”. In: *The American Mathematical Monthly* 59.2 (1952), pp. 78–85.
[3] Abraham De Moivre. “De Sectione Anguli, Autore A. de Moivre, RSS”. In: *Philosophical Transactions (1683-1775)* 32 (1722), pp. 228–230.
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[5] Lenore Feigenbaum. *BROOK TAYLOR'S" METHODUS INCREMENTORUM": A TRANSLATION WITH MATHEMATICAL AND HISTORICAL COMMENTARY.* Yale University, 1981.