Original Research Article

HERMITE POLYNOMIAL-BASED METHODS FOR OPTIMAL ORDER APPROXIMATION OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Abstract

This study investigates the continuous linear multistep techniques utilized for solving first-order initial value problems in ordinary differential equations. Specifically, the study focuses on step k = 9, utilizing Hermite polynomials as basis functions. This study effectively constructs the Adams-Bashforth, Adams-Moulton, and optimal order methods by applying collocation and interpolation methodologies. The methods are thoroughly examined using various numerical instances to demonstrate their efficacy and validity. Notably, the optimal order method exhibits superior accuracy and efficiency compared to the traditional Adams-Bashforth and Adams-Moulton methods. The research results contribute novel and improved methodologies for solving initial value problems in differential equations, which have extensive applications across diverse mathematical and scientific domains.

Keywords: *Continuous linear multistep methods, Hermite polynomials, Adams-Bashforth, Adams-Moulton,q-step method.*

1. Introduction

Linear multistep methods (LMMs) are widely used for modeling dynamic systems in various scientific and engineering fields. However, they face challenges in determining an appropriate step size and addressing stiffness. To overcome these challenges, researchers have developed adaptive step size control and combined linear multistep methods with other numerical techniques.

To solve initial value problems (IVPs) of ordinary differential equations (ODEs), several approaches have been developed to derive linear multistep methods in discrete form. Continuous collocation and interpolation techniques, such as block and hybrid methods, have gained popularity in deriving LMMs. However, the accuracy of these methods can be limited.

Previous studies have been conducted to address the challenges of LMMs and improve their accuracy. For instance, Alabi (2008), Mohammed (2011), Odekunle *et al.* (2012), Adesanya *et al.* (2012), and Anake (2011) have all made contributions in this area. Additionally, Olaiya *et al.* (2019) proposed a new method for solving the Black-Scholes Partial Differential Equation that showed better accuracy than existing methods. Aboiyar *et al.* (2015) developed continuous linear multistep methods for solving first-order IVPs of ODEs using Hermite polynomials as basis functions.

Despite these efforts, there is still a gap in the literature regarding the development of continuous linear multistep methods that can effectively address the challenges faced by traditional LMMs and improve their accuracy. This research aims to fill this gap by deriving and evaluating a continuous linear multistep method using Hermite polynomials of degree 8 as basis functions, extending the degree 4 scheme developed by Aboiyar *et al.* (2015).

The proposed method aims to increase accuracy and address ambiguity in solving ambiguous IVPs using linear multistep methods. The method's effectiveness and validity will be evaluated using numerical examples. The research will also highlight the potential of Hermite polynomials as basis functions for improving the accuracy of continuous linear multistep methods, leading to better solutions for dynamic systems in various scientific and engineering fields.

To achieve this aim, this paper is organized as follows. Section 2 introduces the derivation of the linear multistep methods, while Section 3 presents the numerical solutions and details of the investigation into first-order differential equation problems. Finally, Section 4 summarizes the research's findings.

2. Derivation of the Linear Multistep Methods

The characteristic feature of one step methods is that they need, for computing y_{k+1}, only the value from the previous approximation of the solutiony_k. Methods that use, for computing y_{k+1}, more than one of the previous approximations $y_k, y_{k-1}, \ldots$ are called multi-step methods.

2.1 Definition: q-step method, linear q-step method.

A q-step method with $q \geq 1$ is a numerical method for approximately solving

$$y' = f(x, y(x)), y(x_0) = y_0, \tag{2.1}$$

where y_{k+1} depends on y_{k+1-q} but not on y_i with $i < k + 1 - q$.

A q-step method is called linear if it has the form

$$y_{k+1} = \sum_{j=0}^{q-1} a_j y_{k-j} + h \sum_{j=0}^{q-1} b_j f(x_{k-j}, y_{k-j}) + hb_{-1} f(x_{k+1}, y_{k+1}), k = q, q + 1, \ldots. \tag{2.2}$$

with $q \geq 1, a_0, \ldots, a_{q-1}, b_{-1}, \ldots\ldots, b_{q-1} \in \mathbb{R}, a_{q-1} \neq 0$ and $b_{q-1} \neq 0$. For $q = 1$, the method is called a one-step method. If $b_{-1} \neq 0$, then the linear q-step method is implicit, otherwise it is an explicit method. Some continuous LMM of the type in (2.2) was developed using a collocation function of the form:

$$y(x) = \sum_{j=0}^{k} \alpha_j x^j \tag{2.3}$$

Awoyemi *et al.* (2014) proposed a similar function of the type in (2.3)

$$y(x) = \sum_{j=0}^{k} \alpha_j (x - x_k)^j \tag{2.4}$$

to develop LMM for the solution of third-order IVPs. Adeniyi and Alabi (2006) used the Chebyshev polynomial function of the form:

$$y(x) = \sum_{j=0}^{M} \alpha_j T_j\left(\frac{x-x_k}{h}\right) \quad (2.5)$$

where $T_j(x)$ are some Chebyshev functions to develop continuous LMM.

In this research, we will apply the Probabilists' Hermite polynomial of the form (Koornwinder, *et al.*, 2010), introduced by Aboiyar *et al.* (2015):

$$y(x) = \sum_{j=0}^{k} \alpha_j \, H_j(x - x_k) \quad (2.6)$$

where $H_j(x)$ are probabilists' Hermite polynomials generated by the formula:

$$H_n(x) = (-1)^n e^{\left(\frac{x^2}{2}\right)} \frac{d^n}{dx^n} e^{\left(-\frac{x^2}{2}\right)} = \left(x - \frac{d}{dx}\right)^n \quad (2.7)$$

and whose recursive relation is

$$H_{n+1}(x) = xH_n(x) - H_n(x) \quad (2.8)$$

to develop continuous LMMs for the solution of first-order IVPs of ODEs of the form:

$$y' = f(x, y(x)), y(x_0) = y_0 \quad (2.9)$$

The first nine probabilists' Hermite polynomials are

$$H_0 = 1,$$

$$H_1 = x,$$

$$H_2 = x^2 - 1,$$

$$H_3 = x^3 - 3x,$$

$$H_4 = x^4 - 6x^2 + 3,$$

$$H_5 = x^5 - 10x^3 + 15x,$$

$$H_6 = x^6 - 15x^4 + 45x^2 - 15,$$

$$H_7 = x^7 - 21x^5 + 105x^3 - 105x,$$

$$H_8 = x^8 - 28x^6 + 210x^4 - 420x^2 + 105$$

We wish to approximate the exact solution $y(x)$ to the IVP in (2.9) by a polynomial of degree n of the form:

$$y(x) = \sum_{j=0}^{n} \alpha_j H_j(x - x_k), x_k \le x \le x_{k+p} \quad (2.10)$$

which satisfies

$$\left.\begin{aligned} y'(x) &= f\big(x, y(x)\big), x_k \le x \le x_{k+p} \\ y(x_k) &= y_k \end{aligned}\right\} \quad (2.11)$$

2.2 Derivation of Eight - step Adams-Bash forth method

To derive the eight-step Adams – Bashforth method using Hermite polynomials, we set n=8, in (2.10) yielding

$$\begin{aligned} y(x) = a_0 &+ a_1(x - x_k) + a_2((x - x_k)^2 - 1) + a_3\big((x - x_k)^3 - 3(x - x_k)\big) \\ &+ a_4((x - x_k)^4 - 6(x - x_k)^2 + 3) \\ &+ a_5\big((x - x_k)^5 - 10(x - x_k)^3 + 15(x - x_k)\big) \\ &+ a_6((x - x_k)^6 - 15(x - x_k)^4 + 45(x - x_k)^2 - 15) \\ &+ a_7\big((x - x_k)^7 - 21(x - x_k)^5 + 105(x - x_k)^3 - 105(x - x_k)\big) \\ &+ a_8((x - x_k)^8 - 28(x - x_k)^6 + 210(x - x_k)^4 - 420(x - x_k)^2 + 105); \end{aligned} \quad (2.12)$$

Differentiating (2.12) once gives:

$$\dot{y}(x) = a_1 + 2a_2(x - x_k) + 3a_3[(x - x_k)^2 - 1] + 4a_4[(x - x_k)^3 - 3(x - x_k)] + 5a_5[(x - x_k)^4 - 6(x - x_k)^2 + 3] + 6a_6[(x - x_k)^5 - 10(x - x_k)^3 + 15(x - x_k)] + 7a_7[(x - x_k)^6 - 15(x - x_k)^4 + 45(x - x_k)^2 - 15] + 8a_8[(x - x_k)^7 - 21(x - x_k)^5 + 105(x - x_k)^3 - 105(x - x_k)] \quad (2.13)$$

Interpolating (2.12) at $x = x_{k+7}$gives

$$y(x_{k+7}) = a_0 + 7a_1(x_{k+7} - x_k) + a_2\big(49(x_{k+7} - x_k)^2 - 1\big) + 7a_3\left(49(x_{k+7} - x_k)^3 - 3(x_{k+7} - x_k)\right) + a_4\big(2401(x_{k+7} - x_k)^4 - 294(x_{k+7} - x_k)^2 + 3\big) + a_5\left(16807(x_{k+7} - x_k)^5 - 3430(x_{k+7} - x_k)^3 + 105(x_{k+7} - x_k)\right) + a_6\big(117649(x_{k+7} - x_k)^6 - 36015(x_{k+7} - x_k)^4 + 2205(x_{k+7} - x_k)^2 - 15\big) + a_7\left(823543(x_{k+7} - x_k)^7 - 352947(x_{k+7} - x_k)^5 + 36015(x_{k+7} - x_k)^3 - 735(x_{k+7} - x_k)\right) + a_8\big(5764801(x_{k+7} - x_k)^8 - 3294172(x_{k+7} - x_k)^6 + 504210(x_{k+7} - x_k)^4 - 20580(x_{k+7} - x_k)^2 + 105\big) \quad (2.14)$$

Collocating (2.13) at $x = x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}$, we have (2.15 – 2.22):

$$f_k = a_1 - 3a_3 + 15a_5 - 105a_7 = \dot{y}(x_k) \tag{2.15}$$

$$f_{k+1} = a_1 + 2a_2(x_{k+1} - x_k) + 3a_3[(x_{k+1} - x_k)^2 - 1] + 4a_4[(x_{k+1} - x_k)^3 - 3(x_{k+1} - x_k)] + 5a_5[(x_{k+1} - x_k)^4 - 6(x_{k+1} - x_k)^2 + 3] + 6a_6[(x_{k+1} - x_k)^5 - 10(x_{k+1} - x_k)^3 + 15(x_{k+1} - x_k)] + 7a_7[(x_{k+1} - x_k)^6 - 15(x_{k+1} - x_k)^4 + 45(x_{k+1} - x_k)^2 - 15] + 8a_8[(x_{k+1} - x_k)^7 - 21(x_{k+1} - x_k)^5 + 105(x_{k+1} - x_k)^3 - 105(x_{k+1} - x_k)] = \dot{y}(x_{k+1}) \tag{2.16}$$

$$f_{k+2} = a_1 + 4a_2(x_{k+2} - x_k) + 3a_3[4(x_{k+2} - x_k)^2 - 1] + 8a_4[4(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)] + 5a_5[16(x_{k+2} - x_k)^4 - 24(x_{k+2} - x_k)^2 + 3] + 12a_6[16(x_{k+2} - x_k)^5 - 40(x_{k+2} - x_k)^3 + 15(x_{k+2} - x_k)] + 7a_7[64(x_{k+2} - x_k)^6 - 240(x_{k+2} - x_k)^4 + 180(x_{k+2} - x_k)^2 - 15] + 16a_8[64(x_{k+2} - x_k)^7 - 336(x_{k+2} - x_k)^5 + 420(x_{k+2} - x_k)^3 - 105(x_{k+2} - x_k)] = \dot{y}(x_{k+2}) \tag{2.17}$$

$$f_{k+3} = a_1 + 6a_2(x_{k+3} - x_k) + 3a_3[9(x_{k+3} - x_k)^2 - 1] + 36a_4[6(x_{k+3} - x_k)^3 - (x_{k+3} - x_k)] + 15a_5[27(x_{k+3} - x_k)^4 - 18(x_{k+3} - x_k)^2 + 1] + 54a_6[27(x_{k+3} - x_k)^5 - 30(x_{k+3} - x_k)^3 + 5(x_{k+3} - x_k)] + a_7[5103(x_{k+3} - x_k)^6 - 8505(x_{k+3} - x_k)^4 + 2835(x_{k+3} - x_k)^2 - 105] + a_8[17496(x_{k+3} - x_k)^7 - 40824(x_{k+3} - x_k)^5 + 22680(x_{k+3} - x_k)^3 - 2520(x_{k+3} - x_k)] = \dot{y}(x_{k+3}) \tag{2.18}$$

$$f_{k+4} = a_1 + 8a_2(x_{k+4} - x_k) + 3a_3[16(x_{k+4} - x_k)^2 - 1] + a_4[256(x_{k+4} - x_k)^3 - 48(x_{k+4} - x_k)] + a_5[1280(x_{k+4} - x_k)^4 - 480(x_{k+4} - x_k)^2 + 15] + a_6[6144(x_{k+4} - x_k)^5 - 3840(x_{k+4} - x_k)^3 + 360(x_{k+4} - x_k)] + a_7[28672(x_{k+4} - x_k)^6 - 26880(x_{k+4} - x_k)^4 + 5040(x_{k+4} - x_k)^2 - 105] + a_8[1.31072 \times 10^5(x_{k+4} - x_k)^7 - 1.72032 \times 10^5(x_{k+4} - x_k)^5 + 53760(x_{k+4} - x_k)^3 - 3360(x_{k+4} - x_k)] = \dot{y}(x_{k+4}) \tag{2.19}$$

$$f_{k+5} = a_1 + 10a_2(x_{k+5} - x_k) + 3a_3[25(x_{k+5} - x_k)^2 - 1] + a_4[500(x_{k+5} - x_k)^3 - 60(x_{k+5} - x_k)] + a_5[3125(x_{k+5} - x_k)^4 - 750(x_{k+5} - x_k)^2 + 15] + a_6[18750(x_{k+5} - x_k)^5 - 7500(x_{k+5} - x_k)^3 + 450(x_{k+5} - x_k)] + a_7[1.09375 \times 10^5(x_{k+5} - x_k)^6 - 65625(x_{k+5} - x_k)^4 + 7875(x_{k+5} - x_k)^2 - 105] + a_8[6.25000 \times 10^5(x_{k+5} - x_k)^7 - 5.25000 \times 10^5(x_{k+5} - x_k)^5 + 1.05000 \times 10^5(x_{k+5} - x_k)^3 - 4200(x_{k+5} - x_k)] = \dot{y}(x_{k+5}) \tag{2.20}$$

$$f_{k+6} = a_1 + 12a_2(x_{k+6} - x_k) + 3a_3[36(x_{k+6} - x_k)^2 - 1] + a_4[864(x_{k+6} - x_k)^3 - 72(x_{k+6} - x_k)] + a_5[6480(x_{k+6} - x_k)^4 - 1080(x_{k+6} - x_k)^2 + 15] + a_6[46656(x_{k+6} - x_k)^5 - 12960(x_{k+6} - x_k)^3 + 540(x_{k+5} - x_k)] + a_7[3.26592 \times 10^5(x_{k+6} - x_k)^6 - 1.36080 \times 10^5(x_{k+6} - x_k)^4 + 11340(x_{k+5} - x_k)^2 - 105] + a_8[2.239488 \times 10^6(x_{k+6} - x_k)^7 - 1.306368 \times 10^6(x_{k+6} - x_k)^5 + 1.81440 \times 10^5(x_{k+6} - x_k)^3 - 5040(x_{k+6} - x_k)] = \dot{y}(x_{k+6}) \tag{2.21}$$

$$f_{k+7} = a_1 + 14a_2(x_{k+7} - x_k) + a_3[147(x_{k+7} - x_k)^2 - 3] + a_4[1372(x_{k+7} - x_k)^3 - 84(x_{k+7} - x_k)] + a_5[12005(x_{k+7} - x_k)^4 - 1470(x_{k+7} - x_k)^2 + 15] + a_6[1.00842 \times 10^5(x_{k+7} - x_k)^5 - 20580(x_{k+7} - x_k)^3 + 630(x_{k+7} - x_k)] + a_7[8.23543 \times 10^5(x_{k+7} - x_k)^6 - 2.52105 \times 10^5(x_{k+7} - x_k)^4 + 15435(x_{k+7} - x_k)^2 - 105] + a_8[6.588344 \times 10^6(x_{k+7} - x_k)^7 - 2.823576 \times 10^6(x_{k+7} - x_k)^5 + 2.88120 \times 10^5(x_{k+7} - x_k)^3 - 5880(x_{k+7} - x_k)] = \dot{y}(x_{k+7}) \tag{2.22}$$

In matrix form, we have:

$$
\left[\begin{pmatrix}
1 & 7h & (49h^2-1) & (343h^3-21h) & (2401h^4-294h^2+3) & (16807h^5-3430h^3+105h) & (117649h^6-36015h^4+2205h^2-15) \\
0 & 1 & 0 & -3 & 0 & 15 & 0 \\
0 & 1 & 2h & (3h^2-3) & (4h^3-12h) & (5h^4-30h^2+15) & (6h^5-60h^3+90h) \\
0 & 1 & 4h & (12h^2-3) & (32h^3-24h) & (80h^4-120h^2+15) & (192h^5-480h^3+180h) \\
0 & 1 & 6h & (27h^2-3) & (108h^3-36h) & (405h^4-270h^2+15) & (1485h^5-1620h^3+270h) \\
0 & 1 & 8h & (48h^2-3) & (256h^3-48h) & (1280h^4-480h^2+15) & (6144h^5-3840h^3+360h) \\
0 & 1 & 10h & (75h^2-3) & (500h^3-60h) & (3125h^4-750h^2+15) & (18750h^5-7500h^3+450h) \\
0 & 1 & 12h & (108h^2-3) & (864h^3-72h) & (6480h^4-1080h^2+15) & (46656h^5-12960h^3+540h) \\
0 & 1 & 14h & (147h^2-3) & (1372h^3-84h) & (12005h^4-1470h^2+15) & (100842h^5-20580h^3+630h)
\end{pmatrix} +
\begin{pmatrix}
(823543h^7-352947h^5+36015h^3-735h) & (5764801h^8-3294172h^6+504210h^4-20580h^2+105) \\
-105 & 0 \\
(7h^6-105h^4+315h^2-105) & (8h^7-168h^5+840h^3-840h) \\
(448h^6-1680h^4+1260h^2-105) & (1024h^7-5376h^5+6720h^3-1680h) \\
(5103h^6-8505h^4+2835h^2-105) & (17496h^7-40824h^5+22680h^3-2520h) \\
(28672h^6-26880h^4+5040h^2-105) & (131072h^7-172032h^5+53760h^3-3360h) \\
(109375h^6-65625h^4+7875h^2-105) & (625000h^7-525000h^5+105000h^3-4200h) \\
(326592h^6-136080h^4+11340h^2-105) & (2239488h^7-1306368h^5+181440h^3-5040h) \\
(823543h^6-252105h^4+15435h^2-105) & (6588344h^7-2823576h^5+288120h^3-5880h)
\end{pmatrix}\right]
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}
=
\begin{bmatrix} y_{k+7} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \end{bmatrix}
\tag{2.23}
$$

Solving the system of (2.23) above the by Gaussian elimination method, we have

$$
\begin{aligned}
a_0 = {} & y_{k+7} - h\bigg(\left(\frac{5257}{17280}\right) f_k + \left(\frac{25039}{17280}\right) f_{k+1} + \left(\frac{343}{640}\right) f_{k+2} + \left(\frac{20923}{17280}\right) f_{k+3} + \left(\frac{20923}{17280}\right) f_{k+4} + \left(\frac{343}{640}\right) f_{k+5} + \\
& \left(\frac{25039}{17280}\right) f_{k+6} + \left(\frac{5257}{17280}\right) f_{k+7}\bigg) - \frac{1}{h}\bigg(\left(\frac{363}{280}\right) f_k - \left(\frac{7}{2}\right) f_{k+1} + \left(\frac{21}{4}\right) f_{k+2} - \left(\frac{35}{6}\right) f_{k+3} + \left(\frac{35}{8}\right) f_{k+4} - \\
& \left(\frac{21}{10}\right) f_{k+5} + \left(\frac{7}{12}\right) f_{k+6} - \left(\frac{1}{14}\right) f_{k+7}\bigg) - \frac{1}{h^3}\bigg(\left(\frac{967}{960}\right) f_k - \left(\frac{319}{60}\right) f_{k+1} + \left(\frac{3929}{320}\right) f_{k+2} - \left(\frac{389}{24}\right) f_{k+3} + \\
& \left(\frac{2545}{192}\right) f_{k+4} - \left(\frac{67}{10}\right) f_{k+5} + \left(\frac{1849}{960}\right) f_{k+6} - \left(\frac{29}{120}\right) f_{k+7}\bigg) - \frac{1}{h^5}\bigg(\left(\frac{23}{144}\right) f_k - \left(\frac{295}{288}\right) f_{k+1} + \left(\frac{45}{16}\right) f_{k+2} - \\
& \left(\frac{1235}{288}\right) f_{k+3} + \left(\frac{565}{144}\right) f_{k+4} - \left(\frac{69}{32}\right) f_{k+5} + \left(\frac{95}{144}\right) f_{k+6} - \left(\frac{25}{288}\right) f_{k+7}\bigg) - \frac{1}{h^7}\bigg(\left(\frac{7}{384}\right) f_{k+6} + \left(\frac{1}{384}\right) f_{k+7} + \\
& \left(\frac{7}{128}\right) f_{k+5} + \left(\frac{7}{384}\right) f_{k+1} + \left(\frac{35}{384}\right) f_{k+3} - \left(\frac{7}{128}\right) f_{k+2} - \left(\frac{1}{384}\right) f_k - \left(\frac{35}{384}\right) f_{k+4}\bigg)
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
a_1 = {} & f_k + \frac{1}{h^2}\bigg(\left(\frac{469}{180}\right) f_k - \left(\frac{223}{20}\right) f_{k+1} + \left(\frac{879}{40}\right) f_{k+2} - \left(\frac{949}{36}\right) f_{k+3} + \left(\frac{41}{2}\right) f_{k+4} - \left(\frac{201}{20}\right) f_{k+5} + \\
& \left(\frac{1019}{360}\right) f_{k+6} - \left(\frac{7}{20}\right) f_{k+7}\bigg) + \frac{1}{h^4}\bigg(\left(\frac{7}{6}\right) f_k - \left(\frac{111}{16}\right) f_{k+1} + \left(\frac{71}{4}\right) f_{k+2} - \left(\frac{1219}{48}\right) f_{k+3} + 22 f_{k+4} - \\
& \left(\frac{185}{16}\right) f_{k+5} + \left(\frac{41}{12}\right) f_{k+6} - \left(\frac{7}{16}\right) f_{k+7}\bigg) + \frac{1}{h^6}\bigg(\left(\frac{1}{12}\right) f_k - \left(\frac{9}{16}\right) f_{k+1} + \left(\frac{13}{8}\right) f_{k+2} - \left(\frac{125}{48}\right) f_{k+3} + \\
& \left(\frac{5}{2}\right) f_{k+4} - \left(\frac{23}{16}\right) f_{k+5} + \left(\frac{11}{24}\right) f_{k+6} - \left(\frac{1}{16}\right) f_{k+7}\bigg)
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
a_2 = {} & \frac{1}{h}\bigg(\left(\frac{1}{14}\right) f_{k+7} - \left(\frac{7}{12}\right) f_{k+6} + \left(\frac{21}{10}\right) f_{k+5} - \left(\frac{35}{8}\right) f_{k+4} + \left(\frac{35}{6}\right) f_{k+3} - \left(\frac{21}{4}\right) f_{k+2} + \left(\frac{7}{2}\right) f_{k+1} - \\
& \left(\frac{363}{280}\right) f_k\bigg) + \frac{1}{h^3}\bigg(\left(\frac{29}{60}\right) f_{k+7} - \left(\frac{1849}{480}\right) f_{k+6} + \left(\frac{67}{5}\right) f_{k+5} - \left(\frac{2545}{96}\right) f_{k+4} + \left(\frac{389}{12}\right) f_{k+3} - \left(\frac{3929}{160}\right) f_{k+2} + \\
& \left(\frac{319}{30}\right) f_{k+1} - \left(\frac{967}{480}\right) f_k\bigg) + \frac{1}{h^5}\bigg(\left(\frac{25}{96}\right) f_{k+7} - \left(\frac{95}{48}\right) f_{k+6} + \left(\frac{207}{32}\right) f_{k+5} - \left(\frac{565}{48}\right) f_{k+4} + \left(\frac{1235}{96}\right) f_{k+3} - \\
& \left(\frac{135}{16}\right) f_{k+2} + \left(\frac{295}{96}\right) f_{k+1} - \left(\frac{23}{48}\right) f_k\bigg) + \frac{1}{h^7}\bigg(\left(\frac{1}{96}\right) f_{k+7} - \left(\frac{7}{96}\right) f_{k+6} + \left(\frac{7}{32}\right) f_{k+5} - \left(\frac{35}{96}\right) f_{k+4} + \\
& \left(\frac{35}{96}\right) f_{k+3} - \left(\frac{7}{32}\right) f_{k+2} + \left(\frac{7}{96}\right) f_{k+1} - \left(\frac{1}{96}\right) f_k\bigg)
\end{aligned}
\tag{2.26}
$$

$$a_3 = \frac{1}{h^2}\left(\left(\frac{469}{540}\right) f_k - \left(\frac{223}{60}\right) f_{k+1} + \left(\frac{293}{40}\right) f_{k+2} - \left(\frac{949}{108}\right) f_{k+3} + \left(\frac{41}{6}\right) f_{k+4} - \left(\frac{67}{20}\right) f_{k+5} + \left(\frac{1019}{1080}\right) f_{k+6} - \left(\frac{7}{60}\right) f_{k+7}\right) + \frac{1}{h^4}\left(\left(\frac{7}{9}\right) f_k - \left(\frac{37}{8}\right) f_{k+1} + \left(\frac{71}{6}\right) f_{k+2} - \left(\frac{1219}{72}\right) f_{k+3} + \left(\frac{44}{3}\right) f_{k+4} - \left(\frac{185}{24}\right) f_{k+5} + \left(\frac{41}{18}\right) f_{k+6} - \left(\frac{7}{24}\right) f_{k+7}\right) + \frac{1}{h^6}\left(\left(\frac{1}{12}\right) f_k - \left(\frac{9}{16}\right) f_{k+1} + \left(\frac{13}{8}\right) f_{k+2} - \left(\frac{125}{48}\right) f_{k+3} + \left(\frac{5}{2}\right) f_{k+4} - \left(\frac{23}{16}\right) f_{k+5} + \left(\frac{11}{24}\right) f_{k+6} - \left(\frac{1}{16}\right) f_{k+7}\right) \quad (2.27)$$

$$a_4 = \frac{1}{h^3}\left(\left(\frac{29}{360}\right) f_{k+7} - \left(\frac{1849}{2880}\right) f_{k+6} + \left(\frac{67}{30}\right) f_{k+5} - \left(\frac{2545}{576}\right) f_{k+4} + \left(\frac{389}{72}\right) f_{k+3} - \left(\frac{3929}{960}\right) f_{k+2} + \left(\frac{319}{180}\right) f_{k+1} - \left(\frac{967}{2880}\right) f_k\right) + \frac{1}{h^5}\left(\left(\frac{25}{288}\right) f_{k+7} + \left(\frac{69}{32}\right) f_{k+5} - \left(\frac{95}{144}\right) f_{k+6} - \left(\frac{565}{144}\right) f_{k+4} + \left(\frac{1235}{288}\right) f_{k+3} - \left(\frac{45}{16}\right) f_{k+2} + \left(\frac{295}{288}\right) f_{k+1} - \left(\frac{23}{144}\right) f_k\right) + \frac{1}{h^7}\left(\left(\frac{1}{192}\right) f_{k+7} - \left(\frac{7}{192}\right) f_{k+6} + \left(\frac{7}{64}\right) f_{k+5} - \left(\frac{35}{192}\right) f_{k+4} + \left(\frac{35}{192}\right) f_{k+3} - \left(\frac{7}{64}\right) f_{k+2} + \left(\frac{7}{192}\right) f_{k+1} - \left(\frac{1}{192}\right) f_k\right) \quad (2.28)$$

$$a_5 = \frac{1}{h^4}\left(\left(\frac{7}{90}\right) f_k - \left(\frac{37}{80}\right) f_{k+1} + \left(\frac{71}{60}\right) f_{k+2} - \left(\frac{1219}{720}\right) f_{k+3} + \left(\frac{22}{15}\right) f_{k+4} - \left(\frac{37}{48}\right) f_{k+5} + \left(\frac{41}{180}\right) f_{k+6} - \left(\frac{7}{240}\right) f_{k+7}\right) + \frac{1}{h^6}\left(\left(\frac{1}{60}\right) f_k - \left(\frac{9}{80}\right) f_{k+1} + \left(\frac{13}{40}\right) f_{k+2} - \left(\frac{25}{48}\right) f_{k+3} + \left(\frac{1}{2}\right) f_{k+4} - \left(\frac{23}{80}\right) f_{k+5} + \left(\frac{11}{120}\right) f_{k+6} - \left(\frac{1}{80}\right) f_{k+7}\right) \quad (2.29)$$

$$a_6 = \frac{1}{h^5}\left(\left(\frac{5}{864}\right) f_{k+7} - \left(\frac{19}{432}\right) f_{k+6} + \left(\frac{23}{160}\right) f_{k+5} - \left(\frac{113}{432}\right) f_{k+4} + \left(\frac{247}{864}\right) f_{k+3} - \left(\frac{3}{16}\right) f_{k+2} + \left(\frac{59}{864}\right) f_{k+1} - \left(\frac{23}{2160}\right) f_k\right) + \left(\left(\frac{1}{1440}\right) f_{k+7} - \left(\frac{7}{1440}\right) f_{k+6} + \left(\frac{7}{480}\right) f_{k+5} - \left(\frac{7}{288}\right) f_{k+4} + \left(\frac{7}{288}\right) f_{k+3} - \left(\frac{7}{480}\right) f_{k+2} + \left(\frac{7}{1440}\right) f_{k+1} - \left(\frac{1}{1440}\right) f_k\right) \quad (2.30)$$

$$a_7 = \frac{1}{h^6}\left(\left(\frac{1}{1260}\right) f_k - \left(\frac{3}{560}\right) f_{k+1} + \left(\frac{13}{840}\right) f_{k+2} - \left(\frac{25}{1008}\right) f_{k+3} + \left(\frac{1}{42}\right) f_{k+4} - \left(\frac{23}{1680}\right) f_{k+5} + \left(\frac{11}{2520}\right) f_{k+6} - \left(\frac{1}{1680}\right) f_{k+7}\right) \quad (2.31)$$

$$a_8 = \frac{1}{h^7}\left(+\left(\frac{1}{40320}\right) f_{k+7} - \left(\frac{1}{5760}\right) f_{k+6} + \left(\frac{1}{1920}\right) f_{k+5} - \left(\frac{1}{1152}\right) f_{k+4} + \left(\frac{1}{1152}\right) f_{k+3} - \left(\frac{1}{1920}\right) f_{k+2} + \left(\frac{1}{5760}\right) f_{k+1} - \left(\frac{1}{40320}\right) f_k\right) \quad (2.32)$$

Substituting for a_j = j =0,1,2,3,4,5,6,7,8 in (2.12) yields the continuous method

$$y(x) = y_{k+7} - h\left(\left(\frac{5257}{17280}\right) f_{k+7} + \left(\frac{25039}{17280}\right) f_{k+6} + \left(\frac{343}{640}\right) f_{k+5} + \left(\frac{20923}{17280}\right) f_{k+4} + \left(\frac{20923}{17280}\right) f_{k+3} + \left(\frac{343}{640}\right) f_{k+2} + \left(\frac{25039}{17280}\right) f_{k+1} + \left(\frac{5257}{17280}\right) f_k\right) + (x - x_k) f_k + \frac{(x-x_k)^2}{h}\left(\left(\frac{1}{14}\right) f_{k+7} - \left(\frac{7}{12}\right) f_{k+6} +\right.$$

$$\left(\frac{21}{10}\right) f_{k+5} - \left(\frac{35}{8}\right) f_{k+4} + \left(\frac{35}{6}\right) f_{k+3} - \left(\frac{21}{4}\right) f_{k+2} + \left(\frac{7}{2}\right) f_{k+1} - \left(\frac{363}{280}\right) f_k \Big) - \frac{(x-x_k)^3}{h^2} \Big(\left(\frac{7}{60}\right) f_{k+7} -$$
$$\left(\frac{1019}{1080}\right) f_{k+6} + \left(\frac{67}{20}\right) f_{k+5} - \left(\frac{41}{6}\right) f_{k+4} + \left(\frac{949}{108}\right) f_{k+3} - \left(\frac{293}{40}\right) f_{k+2} + \left(\frac{223}{60}\right) f_{k+1} - \left(\frac{469}{540}\right) f_k \Big) +$$
$$\frac{(x-x_k)^4}{h^3} \Big(\left(\frac{29}{360}\right) f_{k+7} - \left(\frac{1849}{2880}\right) f_{k+6} + \left(\frac{67}{30}\right) f_{k+5} - \left(\frac{2545}{576}\right) f_{k+4} + \left(\frac{389}{72}\right) f_{k+3} - \left(\frac{3929}{960}\right) f_{k+2} +$$
$$\left(\frac{319}{180}\right) f_{k+1} - \left(\frac{967}{2880}\right) f_k \Big) - \frac{(x-x_k)^5}{h^4} \Big(\left(\frac{7}{240}\right) f_{k+7} - \left(\frac{41}{180}\right) f_{k+6} + \left(\frac{37}{48}\right) f_{k+5} - \left(\frac{22}{15}\right) f_{k+4} +$$
$$\left(\frac{1219}{720}\right) f_{k+3} - \left(\frac{71}{60}\right) f_{k+2} + \left(\frac{37}{80}\right) f_{k+1} - \left(\frac{7}{90}\right) f_k \Big) + \frac{(x-x_k)^6}{h^5} \Big(+ \left(\frac{5}{864}\right) f_{k+7} - \left(\frac{19}{432}\right) f_{k+6} +$$
$$\left(\frac{23}{160}\right) f_{k+5} - \left(\frac{113}{432}\right) f_{k+4} + \left(\frac{247}{864}\right) f_{k+3} - \left(\frac{3}{16}\right) f_{k+2} + \left(\frac{59}{864}\right) f_{k+1} - \left(\frac{23}{2160}\right) f_k \Big) -$$
$$\frac{(x-x_k)^7}{h^6} \Big(\left(\frac{1}{1680}\right) f_{k+7} - \left(\frac{11}{2520}\right) f_{k+6} + \left(\frac{23}{1680}\right) f_{k+5} - \left(\frac{1}{42}\right) f_{k+4} + \left(\frac{25}{1008}\right) f_{k+3} - \left(\frac{13}{840}\right) f_{k+2} +$$
$$\left(\frac{3}{560}\right) f_{k+1} - \left(\frac{1}{1260}\right) f_k \Big) + \frac{(x-x_k)^8}{h^7} \Big(+ \left(\frac{1}{40320}\right) f_{k+7} - \left(\frac{1}{5760}\right) f_{k+6} + \left(\frac{1}{1920}\right) f_{k+5} - \left(\frac{1}{1152}\right) f_{k+4} +$$
$$\left(\frac{1}{1152}\right) f_{k+3} - \left(\frac{1}{1920}\right) f_{k+2} + \left(\frac{1}{5760}\right) f_{k+1} - \left(\frac{1}{40320}\right) f_k \Big) \qquad (2.32)$$

Evaluating (2.32) at $x = y_{k+8}$, we have obtained the discrete form as:

$$y_{k+8} = y_{k+7} + h\Big(\left(\frac{16083}{4480}\right) f_{k+7} - \left(\frac{1152169}{120960}\right) f_{k+6} + \left(\frac{242653}{13440}\right) f_{k+5} - \left(\frac{296053}{13440}\right) f_{k+4} +$$
$$\left(\frac{2102243}{120960}\right) f_{k+3} - \left(\frac{115747}{13440}\right) f_{k+2} + \left(\frac{32863}{13440}\right) f_{k+1} - \left(\frac{5257}{17280}\right) f_k\Big) \qquad (2.33)$$

The equation (2.33) is the eight-step Adams-Bashforth method.

2.3 Derivation of Eight-Step Adams-Moulton Method

To derive the eight-step Adams – Moulton method using Hermite polynomials, we set n=9, in (2.10) yielding

$$\begin{aligned} y(x) = a_0 &+ a_1(x-x_k) + a_2((x-x_k)^2 - 1) + a_3((x-x_k)^3 - 3(x-x_k)) \\ &+ a_4((x-x_k)^4 - 6(x-x_k)^2 + 3) \\ &+ a_5((x-x_k)^5 - 10(x-x_k)^3 + 15(x-x_k)) \\ &+ a_6((x-x_k)^6 - 15(x-x_k)^4 + 45(x-x_k)^2 - 15) \\ &+ a_7((x-x_k)^7 - 21(x-x_k)^5 + 105(x-x_k)^3 - 105(x-x_k)) \\ &+ a_8((x-x_k)^8 - 28(x-x_k)^6 + 210(x-x_k)^4 - 420(x-x_k)^2 + 105) \\ &+ a_9((x-x_k)^9 - 36(x-x_k)^7 + 378(x-x_k)^5 - 1260(x-x_k)^2 \\ &+ 945(x-x_k)) \end{aligned}$$

(2.34)

Differentiating (2.34) once gives:

$$
\begin{aligned}
\dot{y}(x) = {} & a_1 + 2a_2(x-x_k) + 3a_3[(x-x_k)^2-1] + 4a_4[(x-x_k)^3-3(x-x_k)] \\
& + 5a_5[(x-x_k)^4-6(x-x_k)^2+3] + 6a_6[(x-x_k)^5-10(x-x_k)^3+15(x-x_k)] \\
& + 7a_7[(x-x_k)^6-15(x-x_k)^4+45(x-x_k)^2-15] \\
& + 8a_8[(x-x_k)^7-21(x-x_k)^5+105(x-x_k)^3-105(x-x_k)] \\
& + a_9[9(x-x_k)^8-252(x-x_k)^6+1390(x-x_k)^4-2520(x-x_k)+945]
\end{aligned}
\tag{2.35}
$$

Interpolating (2.34) at $x = x_{k+7}$and collocating (2.35) at $x = x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}, x_{k+8}$ gives

$$
\begin{aligned}
y(x_{k+7}) = {} & a_0 + a_1(x_{k+7}-x_k) + a_2((x_{k+7}-x_k)^2-1) + a_3\left((x_{k+7}-x_k)^3-3(x_{k+7}-x_k)\right) + \\
& a_4((x_{k+7}-x_k)^4-6(x_{k+7}-x_k)^2+3) + a_5\left((x_{k+7}-x_k)^5-10(x_{k+7}-x_k)^3+15(x_{k+7}-\right. \\
& \left.x_k)\right) + a_6((x_{k+7}-x_k)^6-15(x_{k+7}-x_k)^4+45(x_{k+7}-x_k)^2-15) + a_7\left((x_{k+7}-x_k)^7-\right. \\
& \left.21(x_{k+7}-x_k)^5+105(x_{k+7}-x_k)^3-105(x_{k+7}-x_k)\right) + a_8((x_{k+7}-x_k)^8-28(x_{k+7}-x_k)^6+ \\
& 210(x_{k+7}-x_k)^4-420(x_{k+7}-x_k)^2+105) + a_9\left((x_{k+7}-x_k)^9-36(x_{k+7}-x_k)^7+\right. \\
& \left.378(x_{k+7}-x_k)^5-1260(x_{k+7}-x_k)^2+945(x_{k+7}-x_k)\right)
\end{aligned}
\tag{2.36}
$$

$$f_k = a_1 - 3a_3 + 15a_5 - 105a_7 + 945 = \dot{y}(x_k) \tag{2.37}$$

$$
\begin{aligned}
f_{k+1} = {} & a_1 + 2a_2(x_{k+1}-x_k) + 3a_3[(x_{k+1}-x_k)^2-1] + 4a_4[(x_{k+1}-x_k)^3-3(x_{k+1}-x_k)] + \\
& 5a_5[(x_{k+1}-x_k)^4-6(x_{k+1}-x_k)^2+15] + 6a_6[(x_{k+1}-x_k)^5-10(x_{k+1}-x_k)^3+15(x_{k+1}-x_k)] + \\
& 7a_7[(x_{k+1}-x_k)^6-15(x_{k+1}-x_k)^4+45(x_{k+1}-x_k)^2-15] + 8a_8[(x_{k+1}-x_k)^7-21(x_{k+1}- \\
& x_k)^5+105(x_{k+1}-x_k)^3-105(x_{k+1}-x_k)] + a_9[9(x_{k+1}-x_k)^8-252(x_{k+1}-x_k)^6+1390(x_{k+1}- \\
& x_k)^4-2520(x_{k+1}-x_k)+945] = \dot{y}(x_{k+1})
\end{aligned}
\tag{2.38}
$$

$$
\begin{aligned}
f_{k+2} = {} & a_1 + 2a_2(x_{k+2}-x_k) + 3a_3[(x_{k+2}-x_k)^2-1] + 4a_4[(x_{k+2}-x_k)^3-3(x_{k+2}-x_k)] + \\
& 5a_5[(x_{k+2}-x_k)^4-6(x_{k+2}-x_k)^2+3] + 6a_6[(x_{k+2}-x_k)^5-10(x_{k+2}-x_k)^3+15(x_{k+2}-x_k)] + \\
& 7a_7[(x_{k+2}-x_k)^6-15(x_{k+2}-x_k)^4+45(x_{k+2}-x_k)^2-15] + 8a_8[(x_{k+2}-x_k)^7-21(x_{k+2}- \\
& x_k)^5+105(x_{k+2}-x_k)^3-105(x_{k+2}-x_k)] + a_9[9(x_{k+2}-x_k)^8-252(x_{k+2}-x_k)^6+1890(x_{k+2}- \\
& x_k)^4-2520(x_{k+2}-x_k)+945] = \dot{y}(x_{k+2})
\end{aligned}
\tag{2.39}
$$

$$
\begin{aligned}
f_{k+3} = {} & a_1 + 2a_2(x_{k+3}-x_k) + 3a_3[(x_{k+3}-x_k)^2-1] + 4a_4[(x_{k+3}-x_k)^3-3(x_{k+3}-x_k)] + \\
& 5a_5[(x_{k+3}-x_k)^4-6(x_{k+3}-x_k)^2+3] + 6a_6[(x_{k+3}-x_k)^5-10(x_{k+3}-x_k)^3+15(x_{k+3}-x_k)] + \\
& 7a_7[(x_{k+3}-x_k)^6-15(x_{k+3}-x_k)^4+45(x_{k+3}-x_k)^2-15] + 8a_8[(x_{k+3}-x_k)^7-21(x_{k+3}- \\
& x_k)^5+105(x_{k+3}-x_k)^3-105(x_{k+3}-x_k)] + a_9[9(x_{k+3}-x_k)^8-252(x_{k+3}-x_k)^6+1890(x_{k+3}- \\
& x_k)^4-2520(x_{k+3}-x_k)+945] = \dot{y}(x_{k+3})
\end{aligned}
\tag{2.40}
$$

$$
\begin{aligned}
f_{k+4} = {} & a_1 + 2a_2(x_{k+4}-x_k) + 3a_3[(x_{k+4}-x_k)^2-1] + 4a_4[(x_{k+4}-x_k)^3-3(x_{k+4}-x_k)] + \\
& 5a_5[(x_{k+4}-x_k)^4-6(x_{k+4}-x_k)^2+3] + 6a_6[(x_{k+4}-x_k)^5-10(x_{k+4}-x_k)^3+15(x_{k+4}-x_k)] + \\
& 7a_7[(x_{k+4}-x_k)^6-15(x_{k+4}-x_k)^4+45(x_{k+4}-x_k)^2-15] + 8a_8[(x_{k+4}-x_k)^7-21(x_{k+4}-
\end{aligned}
$$

$x_k)^5 + 105(x_{k+4} - x_k)^3 - 105(x_{k+4} - x_k)] + a_9[9(x_{k+4} - x_k)^8 - 252(x_{k+4} - x_k)^6 + 1890(x_{k+4} - x_k)^4 - 2520(x_{k+4} - x_k) + 945] = \dot{y}(x_{k+4})$ (2.41)

$f_{k+5} = a_1 + 2a_2(x_{k+5} - x_k) + 3a_3[(x_{k+5} - x_k)^2 - 1] + 4a_4[(x_{k+5} - x_k)^3 - 3(x_{k+5} - x_k)] + 5a_5[(x_{k+5} - x_k)^4 - 6(x_{k+5} - x_k)^2 + 3] + 6a_6[(x_{k+5} - x_k)^5 - 10(x_{k+5} - x_k)^3 + 15(x_{k+5} - x_k)] + 7a_7[(x_{k+5} - x_k)^6 - 15(x_{k+5} - x_k)^4 + 45(x_{k+5} - x_k)^2 - 15] + 8a_8[(x_{k+5} - x_k)^7 - 21(x_{k+5} - x_k)^5 + 105(x_{k+5} - x_k)^3 - 105(x_{k+5} - x_k)] + a_9[9(x_{k+5} - x_k)^8 - 252(x_{k+5} - x_k)^6 + 1890(x_{k+5} - x_k)^4 - 2520(x_{k+5} - x_k) + 945] = \dot{y}(x_{k+5})$ (2.42)

$f_{k+6} = a_1 + 2a_2(x_{k+6} - x_k) + 3a_3[(x_{k+6} - x_k)^2 - 1] + 4a_4[(x_{k+6} - x_k)^3 - 3(x_{k+6} - x_k)] + 5a_5[(x_{k+6} - x_k)^4 - 6(x_{k+6} - x_k)^2 + 3] + 6a_6[(x_{k+6} - x_k)^5 - 10(x_{k+6} - x_k)^3 + 15(x_{k+6} - x_k)] + 7a_7[(x_{k+6} - x_k)^6 - 15(x_{k+6} - x_k)^4 + 45(x_{k+6} - x_k)^2 - 15] + 8a_8[(x_{k+6} - x_k)^7 - 21(x_{k+6} - x_k)^5 + 105(x_{k+6} - x_k)^3 - 105(x_{k+6} - x_k)] + a_9[9(x_{k+6} - x_k)^8 - 252(x_{k+6} - x_k)^6 + 1890(x_{k+6} - x_k)^4 - 2520(x_{k+6} - x_k) + 945] = \dot{y}(x_{k+6})$ (2.43)

$f_{k+7} = a_1 + 2a_2(x_{k+7} - x_k) + 3a_3[(x_{k+7} - x_k)^2 - 1] + 4a_4[(x_{k+7} - x_k)^3 - 3(x_{k+7} - x_k)] + 5a_5[(x_{k+7} - x_k)^4 - 6(x_{k+7} - x_k)^2 + 3] + 6a_6[(x_{k+7} - x_k)^5 - 10(x_{k+7} - x_k)^3 + 15(x_{k+7} - x_k)] + 7a_7[(x_{k+7} - x_k)^6 - 15(x_{k+7} - x_k)^4 + 45(x_{k+7} - x_k)^2 - 15] + 8a_8[(x_{k+7} - x_k)^7 - 21(x_{k+7} - x_k)^5 + 105(x_{k+7} - x_k)^3 - 105(x_{k+7} - x_k)] + a_9[9(x_{k+7} - x_k)^8 - 252(x_{k+7} - x_k)^6 + 1890(x_{k+7} - x_k)^4 - 2520(x_{k+7} - x_k) + 945] = \dot{y}(x_{k+7})$ (2.44)

$f_{k+8} = a_1 + 2a_2(x_{k+8} - x_k) + 3a_3[(x_{k+8} - x_k)^2 - 1] + 4a_4[(x_{k+8} - x_k)^3 - 3(x_{k+8} - x_k)] + 5a_5[(x_{k+8} - x_k)^4 - 6(x_{k+8} - x_k)^2 + 3] + 6a_6[(x_{k+8} - x_k)^5 - 10(x_{k+8} - x_k)^3 + 15(x_{k+8} - x_k)] + 7a_7[(x_{k+8} - x_k)^6 - 15(x_{k+8} - x_k)^4 + 45(x_{k+8} - x_k)^2 - 15] + 8a_8[(x_{k+8} - x_k)^7 - 21(x_{k+8} - x_k)^5 + 105(x_{k+8} - x_k)^3 - 105(x_{k+8} - x_k)] + a_9[9(x_{k+8} - x_k)^8 - 252(x_{k+8} - x_k)^6 + 1890(x_{k+8} - x_k)^4 - 2520(x_{k+8} - x_k) + 945] = \dot{y}(x_{k+8})$ (2.45)

In matrix form, we have:

$$\left[\begin{pmatrix} 1 & 7h & (49h^2-1) & (343h^3-21h) & (2401h^4-294h^2+3) & (16807h^5-3430h^3+105h) & (117649h^6-36015h^4+2205h^2-15) \\ 0 & 1 & 0 & -3 & 0 & 15 & 0 \\ 0 & 1 & 2h & (3h^2-3) & (4h^3-12h) & (5h^4-30h^2+15) & (6h^5-60h^3+90h) \\ 0 & 1 & 4h & (12h^2-3) & (32h^3-24h) & (80h^4-120h^2+15) & (192h^5-480h^3+180h) \\ 0 & 1 & 6h & (27h^2-3) & (108h^3-36h) & (405h^4-270h^2+15) & (145h^5-1620h^3+270h) \\ 0 & 1 & 8h & (48h^2-3) & (256h^3-48h) & (1280h^4-480h^2+15) & (6144h^5-3840h^3+360h) \\ 0 & 1 & 10h & (75h^2-3) & (500h^3-60h) & (3125h^4-750h^2+15) & (18750h^5-7500h^3+450h) \\ 0 & 1 & 12h & (108h^2-3) & (864h^3-72h) & (6480h^4-1080h^2+15) & (46656h^5-12960h^3+540h) \\ 0 & 1 & 14h & (147h^2-3) & (1372h^3-84h) & (12005h^4-1470h^2+15) & (100842h^5-20580h^3+630h) \\ 0 & 1 & 16h & (192h^2-3) & (2048h^3-96h) & (20480h^4-1920h^2+15) & (196608h^5-30720h^3+720h) \end{pmatrix} + \begin{pmatrix} (823543h^7-352947h^5+36015h^3-735h) & (5764801h^8-3294172h^6+504210h^4-20580h^2+105) & (40353607h^9-29647548h^7+6353046h^5-61740h^2+6615h) \\ -105 & 0 & 945 \\ (7h^6-105h^4+315h^2-105) & (8h^7-168h^5+840h^3-840h) & (9h^8-252h^6+1890h^4-2520h+945) \\ (448h^6-1680h^4+1260h^2-105) & (1024h^7-5376h^5+6720h^3-1680h) & (2304h^8-16128h^6+30240h^4-5040h+945) \\ (5103h^6-8505h^4+2835h^2-105) & (17496h^7-40824h^5+22680h^3-2520h) & (59049h^8-183708h^6+112590h^4-7560h+945) \\ (28672h^6-26880h^4+5040h^2-105) & (131072h^7-172032h^5+53760h^3-3360h) & (589824h^8-1032192h^6+355840h^4-10080h+945) \\ (109375h^6-65625h^4+7875h^2-105) & (625000h^7-525000h^5+105000h^3-4200h) & (3515625h^8-3937500h^6+868750h^4-12600h+945) \\ (326592h^6-136080h^4+11340h^2-105) & (2239488h^7-1306368h^5+181440h^3-5040h) & (15116544h^8-11757312h^6+1801440h^4-15120h+945) \\ (823543h^6-252105h^4+15435h^2-105) & (6588344h^7-2823576h^5+288120h^3-5880h) & (51883209h^8-29647548h^6+3337390h^4-17640h+945) \\ (1835008h^6-430080h^4+20160h^2-105) & (16777216h^7-5505024h^5+4.30080h^3-6720h) & (150994944h^8-66060288h^6+5693440h^4-20160h+945) \end{pmatrix}\right]$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{k+7} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \\ f_{k+8} \end{bmatrix} \tag{2.46}$$

Solving the system of equationsin (2.46) above by Gaussian elimination method

$$a_0 = y_{k+7} - h\left(\left(\frac{149527}{518400}\right) f_k + \left(\frac{408317}{259200}\right) f_{k+1} + \left(\frac{542969}{259200}\right) f_{k+3} - \left(\frac{368039}{259200}\right) f_{k+5} - \left(\frac{343}{3240}\right) f_{k+4} - \left(\frac{24353}{259200}\right) f_{k+2} - \left(\frac{111587}{259200}\right) f_{k+7} - \left(\frac{261023}{259200}\right) f_{k+6} + \left(\frac{8183}{518400}\right) f_{k+8}\right) - \frac{1}{h}\left(\left(\frac{761}{560}\right) f_k - 4f_{k+1} + 7f_{k+2} - \left(\frac{28}{3}\right) f_{k+3} + \left(\frac{35}{4}\right) f_{k+4} - \left(\frac{28}{5}\right) f_{k+5} + \left(\frac{7}{3}\right) f_{k+6} - \left(\frac{4}{7}\right) f_{k+7} + \left(\frac{1}{16}\right) f_{k+8}\right) - \frac{1}{h^3}\left(\left(\frac{801}{640}\right) f_k - \left(\frac{349}{48}\right) f_{k+1} + \left(\frac{18353}{960}\right) f_{k+2} - \left(\frac{2391}{80}\right) f_{k+3} + \left(\frac{1457}{48}\right) f_{k+4} - \left(\frac{4891}{240}\right) f_{k+5} + \left(\frac{561}{64}\right) f_{k+6} - \left(\frac{527}{240}\right) f_{k+7} + \left(\frac{469}{1920}\right) f_{k+8}\right) - \frac{1}{h^5}\left(\left(\frac{9}{32}\right) f_k - \left(\frac{575}{288}\right) f_{k+1} + \left(\frac{895}{144}\right) f_{k+2} - \left(\frac{355}{32}\right) f_{k+3} + \left(\frac{895}{72}\right) f_{k+4} - \left(\frac{2581}{288}\right) f_{k+5} + \left(\frac{65}{16}\right) f_{k+6} - \left(\frac{305}{288}\right) f_{k+7} + \left(\frac{35}{288}\right) f_{k+8}\right) - \frac{1}{h^7}\left(\left(\frac{3}{256}\right) f_k - \left(\frac{35}{384}\right) f_{k+1} + \left(\frac{119}{384}\right) f_{k+2} - \left(\frac{77}{128}\right) f_{k+3} + \left(\frac{35}{48}\right) f_{k+4} - \left(\frac{217}{384}\right) f_{k+5} + \left(\frac{35}{128}\right) f_{k+6} - \left(\frac{29}{384}\right) f_{k+7} + \left(\frac{7}{768}\right) f_{k+8}\right) + \frac{1}{h^8}\left(\left(\frac{1}{288}\right) f_k - \left(\frac{1}{36}\right) f_{k+1} + \left(\frac{7}{72}\right) f_{k+2} - \left(\frac{7}{36}\right) f_{k+3} + \left(\frac{35}{144}\right) f_{k+4} - \left(\frac{7}{36}\right) f_{k+5} + \left(\frac{7}{72}\right) f_{k+6} - \left(\frac{1}{36}\right) f_{k+7} + \left(\frac{1}{288}\right) f_{k+8}\right) \tag{2.47}$$

$$a_1 = f_k + \frac{1}{h^2}\left(\left(\frac{363}{1120}\right) f_{k+8} - \left(\frac{103}{35}\right) f_{k+7} + \left(\frac{2143}{180}\right) f_{k+6} - \left(\frac{141}{5}\right) f_{k+5} + \left(\frac{691}{16}\right) f_{k+4} - \left(\frac{2003}{45}\right) f_{k+3} + \left(\frac{621}{20}\right) f_{k+2} - \left(\frac{481}{35}\right) f_{k+1} + \left(\frac{29531}{10080}\right) f_k\right) + \frac{1}{h^4}\left(\left(\frac{967}{1920}\right) f_{k+8} - \left(\frac{67}{15}\right) f_{k+7} + \left(\frac{2803}{160}\right) f_{k+6} - \left(\frac{1193}{30}\right) f_{k+5} + \left(\frac{10993}{192}\right) f_{k+4} - \left(\frac{268}{5}\right) f_{k+3} + \left(\frac{15289}{480}\right) f_{k+2} - \left(\frac{329}{30}\right) f_{k+1} + \left(\frac{1069}{640}\right) f_k\right) + \frac{1}{h^6}\left(\left(\frac{23}{192}\right) f_{k+8} - \left(\frac{49}{48}\right) f_{k+7} + \left(\frac{61}{16}\right) f_{k+6} - \left(\frac{391}{48}\right) f_{k+5} + \left(\frac{1045}{96}\right) f_{k+4} - \left(\frac{149}{16}\right) f_{k+3} + \left(\frac{239}{48}\right) f_{k+2} - \left(\frac{73}{48}\right) f_{k+1} + \left(\frac{13}{64}\right) f_k\right) - \frac{1}{h^8}\left(\left(\frac{89}{24192}\right) f_{k+8} - \left(\frac{89}{3024}\right) f_{k+7} + \left(\frac{89}{864}\right) f_{k+6} - \left(\frac{89}{432}\right) f_{k+5} + \left(\frac{445}{1728}\right) f_{k+4} - \left(\frac{89}{432}\right) f_{k+3} + \left(\frac{89}{864}\right) f_{k+2} - \left(\frac{89}{3024}\right) f_{k+1} + \left(\frac{89}{24192}\right) f_k\right) \tag{2.48}$$

$$a_2 = -\frac{1}{h}\left(\left(\frac{1}{16}\right) f_{k+8} - \left(\frac{4}{7}\right) f_{k+7} + \left(\frac{7}{3}\right) f_{k+6} - \left(\frac{28}{5}\right) f_{k+5} + \left(\frac{35}{4}\right) f_{k+4} - \left(\frac{28}{3}\right) f_{k+3} + 7f_{k+2} - 4f_{k+1} + \left(\frac{761}{560}\right) f_k\right) - \frac{1}{h^3}\left(\left(\frac{469}{960}\right) f_{k+8} - \left(\frac{527}{120}\right) f_{k+7} + \left(\frac{561}{32}\right) f_{k+6} - \left(\frac{4891}{120}\right) f_{k+5} + \left(\frac{1457}{24}\right) f_{k+4} - \left(\frac{2391}{40}\right) f_{k+3} + \left(\frac{18353}{480}\right) f_{k+2} - \left(\frac{349}{24}\right) f_{k+1} + \left(\frac{801}{320}\right) f_k\right) - \frac{1}{h^5}\left(\left(\frac{35}{96}\right) f_{k+8} - \left(\frac{305}{96}\right) f_{k+7} +\right.$$

$$\left(\frac{195}{16}\right) f_{k+6} - \left(\frac{2581}{96}\right) f_{k+5} + \left(\frac{895}{24}\right) f_{k+4} - \left(\frac{1065}{32}\right) f_{k+3} + \left(\frac{895}{48}\right) f_{k+2} - \left(\frac{575}{96}\right) f_{k+1} + \left(\frac{27}{32}\right) f_k \Big) - \frac{1}{h^7}\Big(\left(\frac{7}{192}\right) f_{k+8} - \left(\frac{29}{96}\right) f_{k+7} + \left(\frac{35}{32}\right) f_{k+6} - \left(\frac{217}{96}\right) f_{k+5} + \left(\frac{35}{12}\right) f_{k+4} - \left(\frac{77}{32}\right) f_{k+3} + \left(\frac{119}{96}\right) f_{k+2} - \left(\frac{35}{96}\right) f_{k+1} + \left(\frac{3}{64}\right) f_k \Big) + \frac{1}{h^8}\Big(\left(\frac{1}{288}\right) f_{k+8} - \left(\frac{1}{36}\right) f_{k+7} + \left(\frac{7}{72}\right) f_{k+6} - \left(\frac{7}{36}\right) f_{k+5} + \left(\frac{35}{144}\right) f_{k+4} - \left(\frac{7}{36}\right) f_{k+3} + \left(\frac{7}{72}\right) f_{k+2} - \left(\frac{1}{36}\right) f_{k+1} + \left(\frac{1}{288}\right) f_k \Big) \tag{2.49}$$

$$a_3 = \frac{1}{h^2}\Big(\left(\frac{121}{1120}\right) f_{k+8} - \left(\frac{103}{105}\right) f_{k+7} + \left(\frac{2143}{540}\right) f_{k+6} - \left(\frac{47}{5}\right) f_{k+5} + \left(\frac{691}{48}\right) f_{k+4} - \left(\frac{2003}{135}\right) f_{k+3} + \left(\frac{207}{20}\right) f_{k+2} - \left(\frac{481}{105}\right) f_{k+1} + \left(\frac{29531}{30240}\right) f_k \Big) + \frac{1}{h^4}\Big(\left(\frac{967}{2880}\right) f_{k+8} - \left(\frac{134}{45}\right) f_{k+7} + \left(\frac{2803}{240}\right) f_{k+6} - \left(\frac{1193}{45}\right) f_{k+5} + \left(\frac{10993}{288}\right) f_{k+4} - \left(\frac{536}{15}\right) f_{k+3} + \left(\frac{15289}{720}\right) f_{k+2} - \left(\frac{329}{45}\right) f_{k+1} + \left(\frac{1069}{960}\right) f_k \Big) + \frac{1}{h^6}\Big(\left(\frac{23}{192}\right) f_{k+8} - \left(\frac{49}{48}\right) f_{k+7} + \left(\frac{61}{16}\right) f_{k+6} - \left(\frac{391}{48}\right) f_{k+5} + \left(\frac{1045}{96}\right) f_{k+4} - \left(\frac{149}{16}\right) f_{k+3} + \left(\frac{239}{48}\right) f_{k+2} - \left(\frac{73}{48}\right) f_{k+1} + \left(\frac{13}{64}\right) f_k \Big) + \frac{1}{h^8}\Big(\left(\frac{25}{9072}\right) f_{k+8} - \left(\frac{25}{1134}\right) f_{k+7} + \left(\frac{25}{324}\right) f_{k+6} - \left(\frac{25}{162}\right) f_{k+5} + \left(\frac{125}{648}\right) f_{k+4} - \left(\frac{25}{162}\right) f_{k+3} + \left(\frac{25}{324}\right) f_{k+2} - \left(\frac{25}{1134}\right) f_{k+1} + \left(\frac{25}{9072}\right) f_k \Big) \tag{2.50}$$

$$a_4 = -\frac{1}{h^3}\Big(\left(\frac{469}{5760}\right) f_{k+8} - \left(\frac{527}{720}\right) f_{k+7} + \left(\frac{187}{64}\right) f_{k+6} - \left(\frac{4891}{720}\right) f_{k+5} + \left(\frac{1457}{144}\right) f_{k+4} - \left(\frac{797}{80}\right) f_{k+3} + \left(\frac{18353}{2880}\right) f_{k+2} - \left(\frac{349}{144}\right) f_{k+1} + \left(\frac{267}{640}\right) f_k \Big) - \frac{1}{h^5}\Big(\left(\frac{35}{288}\right) f_{k+8} - \left(\frac{305}{288}\right) f_{k+7} + \left(\frac{65}{16}\right) f_{k+6} - \left(\frac{2581}{288}\right) f_{k+5} + \left(\frac{895}{72}\right) f_{k+4} - \left(\frac{355}{32}\right) f_{k+3} + \left(\frac{895}{144}\right) f_{k+2} - \left(\frac{575}{288}\right) f_{k+1} + \left(\frac{9}{32}\right) f_k \Big) - \frac{1}{h^7}\Big(\left(\frac{7}{384}\right) f_{k+8} - \left(\frac{29}{192}\right) f_{k+7} + \left(\frac{35}{64}\right) f_{k+6} - \left(\frac{217}{192}\right) f_{k+5} + \left(\frac{35}{24}\right) f_{k+4} - \left(\frac{77}{64}\right) f_{k+3} + \left(\frac{119}{192}\right) f_{k+2} - \left(\frac{35}{192}\right) f_{k+1} + \left(\frac{3}{128}\right) f_k \Big) \tag{2.51}$$

$$a_5 = \frac{1}{h^4}\Big(\left(\frac{967}{28800}\right) f_{k+8} - \left(\frac{67}{225}\right) f_{k+7} + \left(\frac{2803}{2400}\right) f_{k+6} - \left(\frac{1193}{450}\right) f_{k+5} + \left(\frac{10993}{2880}\right) f_{k+4} - \left(\frac{268}{75}\right) f_{k+3} + \left(\frac{15289}{7200}\right) f_{k+2} - \left(\frac{329}{450}\right) f_{k+1} + \left(\frac{1069}{9600}\right) f_k \Big) + \frac{1}{h^6}\Big(\left(\frac{23}{960}\right) f_{k+8} - \left(\frac{49}{240}\right) f_{k+7} + \left(\frac{61}{80}\right) f_{k+6} - \left(\frac{391}{240}\right) f_{k+5} + \left(\frac{209}{96}\right) f_{k+4} - \left(\frac{149}{80}\right) f_{k+3} + \left(\frac{239}{240}\right) f_{k+2} - \left(\frac{73}{240}\right) f_{k+1} + \left(\frac{13}{320}\right) f_k \Big) + \frac{1}{h^8}\Big(\left(\frac{239}{181440}\right) f_{k+8} - \left(\frac{239}{22680}\right) f_{k+7} + \left(\frac{239}{6480}\right) f_{k+6} - \left(\frac{239}{3240}\right) f_{k+5} + \left(\frac{239}{2592}\right) f_{k+4} - \left(\frac{239}{3240}\right) f_{k+3} + \left(\frac{239}{6480}\right) f_{k+2} - \left(\frac{239}{22680}\right) f_{k+1} + \left(\frac{239}{181440}\right) f_k \Big) \tag{2.52}$$

$$a_6 = -\frac{1}{h^5}\left(\left(\frac{3}{160}\right)f_k - \left(\frac{115}{864}\right)f_{k+1} + \left(\frac{179}{432}\right)f_{k+2} - \left(\frac{71}{96}\right)f_{k+3} + \left(\frac{179}{216}\right)f_{k+4} - \left(\frac{2581}{4320}\right)f_{k+5} + \left(\frac{13}{48}\right)f_{k+6} - \left(\frac{61}{864}\right)f_{k+7} + \left(\frac{7}{864}\right)f_{k+8}\right) - \frac{1}{h^7}\left(\left(\frac{1}{320}\right)f_k - \left(\frac{7}{288}\right)f_{k+1} + \left(\frac{119}{1440}\right)f_{k+2} - \left(\frac{77}{480}\right)f_{k+3} + \left(\frac{7}{36}\right)f_{k+4} - \left(\frac{217}{1440}\right)f_{k+5} + \left(\frac{7}{96}\right)f_{k+6} - \left(\frac{29}{1440}\right)f_{k+7} + \left(\frac{7}{2880}\right)f_{k+8}\right) \quad (2.53)$$

$$a_7 = \frac{1}{h^6}\left(\left(\frac{13}{6720}\right)f_k - \left(\frac{73}{5040}\right)f_{k+1} + \left(\frac{239}{5040}\right)f_{k+2} - \left(\frac{149}{1680}\right)f_{k+3}\left(\frac{209}{2016}\right)f_{k+4} - \left(\frac{391}{5040}\right)f_{k+5} + \left(\frac{61}{1680}\right)f_{k+6} - \left(\frac{7}{720}\right)f_{k+7} + \left(\frac{23}{20160}\right)f_{k+8}\right) + \frac{1}{h^8}\left(+\left(\frac{1}{10080}\right)f_k - \left(\frac{1}{1260}\right)f_{k+1} + \left(\frac{1}{360}\right)f_{k+2} - \left(\frac{1}{180}\right)f_{k+3} + \left(\frac{1}{144}\right)f_{k+4} - \left(\frac{1}{180}\right)f_{k+5} + \left(\frac{1}{360}\right)f_{k+6} - \left(\frac{1}{1260}\right)f_{k+7} + \left(\frac{1}{10080}\right)f_{k+8}\right) \quad (2.54)$$

$$a_8 = -\frac{1}{h^7}\left(\left(\frac{1}{8960}\right)f_k - \left(\frac{1}{1152}\right)f_{k+1} + \left(\frac{17}{5760}\right)f_{k+2} - \left(\frac{11}{1920}\right)f_{k+3} + \left(\frac{1}{144}\right)f_{k+4} - \left(\frac{31}{5760}\right)f_{k+5} + \left(\frac{1}{384}\right)f_{k+6} - \left(\frac{29}{40320}\right)f_{k+7} + \left(\frac{1}{11520}\right)f_{k+8}\right) \quad (2.55)$$

$$a_9 = \frac{1}{h^8}\left(\left(\frac{1}{362880}\right)f_k - \left(\frac{1}{45360}\right)f_{k+1} + \left(\frac{1}{12960}\right)f_{k+2} - \left(\frac{1}{6480}\right)f_{k+3} + \left(\frac{1}{5184}\right)f_{k+4} - \left(\frac{1}{6480}\right)f_{k+5} + \left(\frac{1}{12960}\right)f_{k+6} - \left(\frac{1}{45360}\right)f_{k+7} + \left(\frac{1}{362880}\right)f_{k+8}\right) \quad (2.56)$$

Substituting for a_j = j =0,1,2,3,4,5,6,7,8,9 in (2.34) yields the continuous method

$$y(x) = y_{k+7} - h\left(-\left(\frac{24353}{259200}\right)f_{k+2} - \left(\frac{542969}{259200}\right)f_{k+3} - \left(\frac{343}{3240}\right)f_{k+4} - \left(\frac{368039}{259200}\right)f_{k+5} - \left(\frac{261023}{259200}\right)f_{k+6} - \left(\frac{111587}{259200}\right)f_{k+7} + \left(\frac{8183}{518400}\right)f_{k+8} - \left(\frac{149527}{518400}\right)f_k - \left(\frac{408317}{259200}\right)f_{k+1}\right) + (x-x_k)f_k + \frac{(x-x_k)^2}{h}\left(\left(\frac{28}{3}\right)f_{k+3} + \left(\frac{28}{5}\right)f_{k+5} - \left(\frac{1}{16}\right)f_{k+8} - \left(\frac{35}{4}\right)f_{k+4} - \left(\frac{7}{3}\right)f_{k+6} - 7f_{k+2} + 4f_{k+1} - \left(\frac{761}{560}\right)f_k + \left(\frac{4}{7}\right)f_{k+7}\right) - \frac{(x-x_k)^3}{h^2}\left(-\left(\frac{47}{5}\right)f_{k+5} - \left(\frac{103}{105}\right)f_{k+7} + \left(\frac{2143}{540}\right)f_{k+6} + \left(\frac{29531}{30240}\right)f_k - \left(\frac{2003}{135}\right)f_{k+3} + \left(\frac{121}{1120}\right)f_{k+8} - \left(\frac{481}{105}\right)f_{k+1} + \left(\frac{207}{20}\right)f_{k+2} + \left(\frac{691}{48}\right)f_{k+4}\right) + \frac{(x-x_k)^4}{h^3}\left(-\left(\frac{187}{64}\right)f_{k+6} - \left(\frac{18353}{2880}\right)f_{k+2} - \left(\frac{267}{640}\right)f_k + \left(\frac{4891}{720}\right)f_{k+5} + \left(\frac{527}{720}\right)f_{k+7} + \left(\frac{349}{144}\right)f_{k+1} - \left(\frac{1457}{144}\right)f_{k+4} + \left(\frac{797}{80}\right)f_{k+3} - \left(\frac{469}{5760}\right)f_{k+8}\right) - \frac{(x-x_k)^5}{h^4}\left(\left(\frac{15289}{7200}\right)f_{k+2} + \left(\frac{1069}{9600}\right)f_k + \left(\frac{967}{28800}\right)f_{k+8} - \left(\frac{67}{225}\right)f_{k+7} - \left(\frac{1193}{450}\right)f_{k+5} + \left(\frac{10993}{2880}\right)f_{k+4} + \left(\frac{2803}{2400}\right)f_{k+6} - \left(\frac{329}{450}\right)f_{k+1} - \left(\frac{268}{75}\right)f_{k+3}\right) + \frac{(x-x_k)^6}{h^5}\left(\left(\frac{71}{96}\right)f_{k+3} - \left(\frac{179}{216}\right)f_{k+4} + \left(\frac{2581}{4320}\right)f_{k+5} - \left(\frac{13}{48}\right)f_{k+6} + \left(\frac{61}{864}\right)f_{k+7} - \left(\frac{7}{864}\right)f_{k+8} - \left(\frac{3}{160}\right)f_k + \left(\frac{115}{864}\right)f_{k+1} - \left(\frac{179}{432}\right)f_{k+2}\right) - \frac{(x-x_k)^7}{h^6}\left(\left(\frac{209}{2016}\right)f_{k+4} + \left(\frac{23}{20160}\right)f_{k+8} - \left(\frac{391}{5040}\right)f_{k+5} + \left(\frac{61}{1680}\right)f_{k+6} - \left(\frac{7}{720}\right)f_{k+7} - \left(\frac{149}{1680}\right)f_{k+3} + \left(\frac{13}{6720}\right)f_k - \left(\frac{73}{5040}\right)f_{k+1} + \left(\frac{239}{5040}\right)f_{k+2}\right) + \frac{(x-x_k)^8}{h^7}\left(-\left(\frac{1}{8960}\right)f_k + \left(\frac{1}{1152}\right)f_{k+1} - \left(\frac{17}{5760}\right)f_{k+2} + \right.$$

$\left(\frac{11}{1920}\right) f_{k+3} + \left(\frac{29}{40320}\right) f_{k+7} - \left(\frac{1}{384}\right) f_{k+6} - \left(\frac{1}{11520}\right) f_{k+8} - \left(\frac{1}{144}\right) f_{k+4} + \left(\frac{31}{5760}\right) f_{k+5}\Bigg) +$
$\frac{(x-x_k)^9}{h^8}\left(\left(\frac{1}{362880}\right) f_k - \left(\frac{1}{45360}\right) f_{k+1} + \left(\frac{1}{12960}\right) f_{k+2} - \left(\frac{1}{6480}\right) f_{k+3} + \left(\frac{1}{362880}\right) f_{k+8} -\right.$
$\left.\left(\frac{1}{45360}\right) f_{k+7} + \left(\frac{1}{5184}\right) f_{k+4} - \left(\frac{1}{6480}\right) f_{k+5} + \left(\frac{1}{12960}\right) f_{k+6}\right)$ (2.57)

Evaluating (2.57) at $x = y_{k+8}$, we have obtained the discrete form as:

$$y_{k+8} = y_{k+7} + h\left(-\left(\frac{33953}{3628800}\right) f_k + \left(\frac{156437}{1814400}\right) f_{k+1} - \left(\frac{645607}{1814400}\right) f_{k+2} + \left(\frac{1573169}{1814400}\right) f_{k+3} - \left(\frac{31457}{22680}\right) f_{k+4} + \left(\frac{2797679}{1814400}\right) f_{k+5} - \left(\frac{2302297}{1814400}\right) f_{k+6} + \left(\frac{2233547}{1814400}\right) f_{k+7} + \left(\frac{1070017}{3628800}\right) f_{k+8}\right) \quad (2.58)$$

The equation (2.58) is the eight-step Adams-Moulton method.

2.4 Eight-step optimal order method

The optimal scheme is an implicit multi-step method similar to the Adams-Moulton method. To derive the eight-step optimal method, we shall consider the system of equations in (2.34 - 2.45) except for $y(x_{k+7})$. Interpolating (2.34) at $x = x_{k+6}$, we have

$$y(x_{k+6}) = a_0 + a_1(x_{k+6} - x_k) + a_2((x_{k+6} - x_k)^2 - 1) + a_3\big((x_{k+6} - x_k)^3 - 3(x_{k+6} - x_k)\big) + a_4((x_{k+6} - x_k)^4 - 6(x_{k+6} - x_k)^2 + 3) + a_5\big((x_{k+6} - x_k)^5 - 10(x_{k+6} - x_k)^3 + 15(x_{k+6} - x_k)\big) + a_6((x_{k+6} - x_k)^6 - 15(x_{k+6} - x_k)^4 + 45(x_{k+6} - x_k)^2 - 15) + a_7\big((x_{k+6} - x_k)^7 - 21(x_{k+6} - x_k)^5 + 105(x_{k+6} - x_k)^3 - 105(x_{k+6} - x_k)\big) + a_8((x_{k+6} - x_k)^8 - 28(x_{k+6} - x_k)^6 + 210(x_{k+6} - x_k)^4 - 420(x_{k+6} - x_k)^2 + 105) + a_9\big((x_{k+6} - x_k)^9 - 36(x_{k+6} - x_k)^7 + 378(x_{k+6} - x_k)^5 - 1260(x_{k+6} - x_k)^2 + 945(x_{k+6} - x_k)\big) \quad (2.59)$$

The corresponding matrix is

$$\left[\begin{pmatrix}
1 & 6h & (36h^2-1) & (216h^3-18h) & (1296h^4-216h^2+3) & (7776h^5-2160h^3+90h) & (46656h^6-19440h^4+1620h^2-15) \\
0 & 1 & 0 & -3 & 0 & 15 & 0 \\
0 & 1 & 2h & (3h^2-3) & (4h^3-12h) & (5h^4-30h^2+15) & (6h^5-60h^3+90h) \\
0 & 1 & 4h & (12h^2-3) & (32h^3-24h) & (80h^4-120h^2+15) & (192h^5-480h^3+180h) \\
0 & 1 & 6h & (27h^2-3) & (108h^3-36h) & (405h^4-270h^2+15h) & (145h^5-1620h^3+270h) \\
0 & 1 & 8h & (48h^2-3) & (256h^3-48h) & (1280h^4-480h^2+15h) & (6144h^5-3840h^3+360h) \\
0 & 1 & 10h & (75h^2-3) & (500h^3-60h) & (3125h^4-750h^2+15h) & (18750h^5-7500h^3+450h) \\
0 & 1 & 12h & (108h^2-3) & (864h^3-72h) & (6480h^4-1080h^2+15h) & (46656h^5-12960h^3+540h) \\
0 & 1 & 14h & (147h^2-3) & (1372h^3-84h) & (12005h^4-1470h^2+15h) & (100842h^5-20580h^3+630h) \\
0 & 1 & 16h & (192h^2-3) & (2048h^3-96h) & (20480h^4-1920h^2+15h) & (196608h^5-30720h^3+720h)
\end{pmatrix} + \begin{pmatrix}
(279936h^7-163296h^5+22680h^3-630h) & (1679616h^8-1306368h^6+272160h^4-15120h^2+105) & (10077696h^9-10077696h^7+2161728h^5-45360h^2+5670h) \\
-105 & 0 & 945 \\
(7h^6-105h^4+315h^2-105) & (8h^7-168h^5+840h^3-840h) & (9h^8-252h^6+1390h^4-2520h+945) \\
(448h^6-1680h^4+1260h^2-105) & (1024h^7-5376h^5+6720h^3-1680h) & (2304h^8-16128h^6+22240h^4-5040h+945) \\
(5103h^6-8505h^4+2835h^2-105) & (17496h^7-40824h^5+22680h^3-2520h) & (59049h^8-183708h^6+112590h^4-7560h+945) \\
(28672h^6-26880h^4+5040h^2-105) & (131072h^7-172032h^5+53760h^3-3360h) & (589824h^8-1032192h^6+355840h^4-10080h+945) \\
(109375h^6-65625h^4+7875h^2-105) & (625000h^7-525000h^5+105000h^3-4200h) & (3515625h^8-3937500h^6+868750h^4-12600h+945) \\
(326592h^6-136080h^4+11340h^2-105) & (2239488h^7-1306368h^5+181440h^3-5040h) & (15116544h^8-11757312h^6+1801440h^4-15120h+945) \\
(823543h^6-252105h^4+15435h^2-105) & (6588344h^7-2823576h^5+288120h^3-5880h) & (51883209h^8-29647548h^6+3337390h^4-17640h+945) \\
(1835008h^6-430080h^4+20160h^2-105) & (16777216h^7-5505024h^5+4.30080h^3-6720h) & (150994944h^8-66060288h^6+5693440h^4-20160h+945)
\end{pmatrix}\right]$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{k+6} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \\ f_{k+8} \end{bmatrix} \tag{2.60}$$

Solving the system of equations by Gaussian's elimination methods gives the same result in (2.46 – 2.59) above except for a_0. Now, we have a_0 to be

$$a_0 = y_{k+6} - h\left(-\left(\frac{333}{175}\right)f_{k+5} - \left(\frac{79}{700}\right)f_{k+6} - \left(\frac{9}{175}\right)f_{k+7} + \left(\frac{9}{1400}\right)f_{k+8} - \left(\frac{401}{1400}\right)f_k - \left(\frac{279}{175}\right)f_{k+1} - \left(\frac{9}{700}\right)f_{k+2} - \left(\frac{403}{175}\right)f_{k+3} + \left(\frac{9}{35}\right)f_{k+4}\right) - \frac{1}{h}\left(\left(\frac{4}{7}\right)f_{k+7} + 4f_{k+1} - 7f_{k+2} + \left(\frac{28}{3}\right)f_{k+3} - \left(\frac{761}{560}\right)f_k + \left(\frac{28}{5}\right)f_{k+5} - \left(\frac{7}{3}\right)f_{k+6} - \left(\frac{1}{16}\right)f_{k+8} - \left(\frac{35}{4}\right)f_{k+4}\right) - \frac{1}{h^3}\left(-\left(\frac{561}{64}\right)f_{k+6} + \left(\frac{527}{240}\right)f_{k+7} + \left(\frac{2391}{80}\right)f_{k+3} + \left(\frac{4891}{240}\right)f_{k+5} - \left(\frac{801}{640}\right)f_k - \left(\frac{469}{1920}\right)f_{k+8} + \left(\frac{349}{48}\right)f_{k+1} - \left(\frac{1457}{48}\right)f_{k+4} - \left(\frac{18353}{960}\right)f_{k+2}\right) - \frac{1}{h^5}\left(-\left(\frac{895}{144}\right)f_{k+2} + \left(\frac{2581}{288}\right)f_{k+5} - \left(\frac{35}{288}\right)f_{k+8} + \left(\frac{575}{288}\right)f_{k+1} + \left(\frac{355}{32}\right)f_{k+3} - \left(\frac{65}{16}\right)f_{k+6} + \left(\frac{305}{288}\right)f_{k+7} - \left(\frac{9}{32}\right)f_k - \left(\frac{895}{72}\right)f_{k+4}\right) - \frac{1}{h^7}\left(-\left(\frac{35}{128}\right)f_{k+6} + \left(\frac{29}{384}\right)f_{k+7} - \left(\frac{119}{384}\right)f_{k+2} + \left(\frac{217}{384}\right)f_{k+5} - \left(\frac{7}{768}\right)f_{k+8} + \left(\frac{35}{384}\right)f_{k+1} + \left(\frac{77}{128}\right)f_{k+3} - \left(\frac{35}{48}\right)f_{k+4} - \left(\frac{3}{256}\right)f_k\right) + \frac{1}{h^8}\left(-\left(\frac{7}{36}\right)f_{k+3} + \left(\frac{7}{72}\right)f_{k+6} + \left(\frac{1}{288}\right)f_{k+8} + \left(\frac{7}{72}\right)f_{k+2} + \left(\frac{35}{144}\right)f_{k+4} - \left(\frac{1}{36}\right)f_{k+7} - \left(\frac{1}{36}\right)f_{k+1} + \left(\frac{1}{288}\right)f_k - \left(\frac{7}{36}\right)f_{k+5}\right) \tag{2.61}$$

Substituting for $a_j, j = 0,1,2,3,4,5,6,7,8,9$ in (2.61) yields the continuous eight-step optimal order method:

$$y(x) = y_{k+6} - h\left(\left(\frac{9}{1400}\right)f_{k+8} - \left(\frac{401}{1400}\right)f_k - \left(\frac{279}{175}\right)f_{k+1} - \left(\frac{9}{700}\right)f_{k+2} - \left(\frac{403}{175}\right)f_{k+3} + \left(\frac{9}{35}\right)f_{k+4} - \left(\frac{333}{175}\right)f_{k+5} - \left(\frac{79}{700}\right)f_{k+6} - \left(\frac{9}{175}\right)f_{k+7}\right) + (x - x_k)f_k + \frac{(x-x_k)^2}{h}\left(-7f_{k+2} + \left(\frac{28}{3}\right)f_{k+3} - \left(\frac{761}{560}\right)f_k + 4f_{k+1} + \left(\frac{4}{7}\right)f_{k+7} - \left(\frac{35}{4}\right)f_{k+4} + \left(\frac{28}{5}\right)f_{k+5} - \left(\frac{7}{3}\right)f_{k+6} - \left(\frac{1}{16}\right)f_{k+8}\right) - \frac{(x-x_k)^3}{h^2}\left(-\left(\frac{2003}{135}\right)f_{k+3} + \left(\frac{207}{20}\right)f_{k+2} + \left(\frac{29531}{30240}\right)f_k - \left(\frac{481}{105}\right)f_{k+1} + \left(\frac{2143}{540}\right)f_{k+6} - \left(\frac{103}{105}\right)f_{k+7} + \left(\frac{691}{48}\right)f_{k+4} - \left(\frac{47}{5}\right)f_{k+5} + \left(\frac{121}{1120}\right)f_{k+8}\right) + \frac{(x-x_k)^4}{h^3}\left(-\left(\frac{187}{64}\right)f_{k+6} + \left(\frac{797}{80}\right)f_{k+3} - \left(\frac{1457}{144}\right)f_{k+4} + \left(\frac{349}{144}\right)f_{k+1} - \left(\frac{267}{640}\right)f_k + \left(\frac{527}{720}\right)f_{k+7} - \left(\frac{469}{5760}\right)f_{k+8} + \left(\frac{4891}{720}\right)f_{k+5} - \left(\frac{18353}{2880}\right)f_{k+2}\right) - \frac{(x-x_k)^5}{h^4}\left(-\left(\frac{67}{225}\right)f_{k+7} + \left(\frac{1069}{9600}\right)f_k - \left(\frac{329}{450}\right)f_{k+1} + \left(\frac{15289}{7200}\right)f_{k+2} + \left(\frac{2803}{2400}\right)f_{k+6} - \left(\frac{268}{75}\right)f_{k+3} + \left(\frac{10993}{2880}\right)f_{k+4} - \left(\frac{1193}{450}\right)f_{k+5} + \left(\frac{967}{28800}\right)f_{k+8}\right) + \frac{(x-x_k)^6}{h^5}\left(\left(\frac{71}{96}\right)f_{k+3} - \left(\frac{179}{432}\right)f_{k+2} - \left(\frac{3}{160}\right)f_k + \left(\frac{115}{864}\right)f_{k+1} - \left(\frac{13}{48}\right)f_{k+6} + \left(\frac{61}{864}\right)f_{k+7} - \left(\frac{179}{216}\right)f_{k+4} + \left(\frac{2581}{4320}\right)f_{k+5} - \left(\frac{7}{864}\right)f_{k+8}\right) -$$

$$\frac{(x-x_k)^7}{h^6}\left(\left(\frac{61}{1680}\right)f_{k+6}-\left(\frac{149}{1680}\right)f_{k+3}+\left(\frac{209}{2016}\right)f_{k+4}-\left(\frac{73}{5040}\right)f_{k+1}+\left(\frac{13}{6720}\right)f_k-\left(\frac{7}{720}\right)f_{k+7}+\right.$$
$$\left.\left(\frac{23}{20160}\right)f_{k+8}-\left(\frac{391}{5040}\right)f_{k+5}+\left(\frac{239}{5040}\right)f_{k+2}\right)+\frac{(x-x_k)^8}{h^7}\left(-\left(\frac{17}{5760}\right)f_{k+2}-\left(\frac{1}{8960}\right)f_k+\left(\frac{1}{1152}\right)f_{k+1}-\right.$$
$$\left.\left(\frac{1}{11520}\right)f_{k+8}+\left(\frac{29}{40320}\right)f_{k+7}+\left(\frac{11}{1920}\right)f_{k+3}+\left(\frac{31}{5760}\right)f_{k+5}-\left(\frac{1}{384}\right)f_{k+6}-\left(\frac{1}{144}\right)f_{k+4}\right)+$$
$$\frac{(x-x_k)^9}{h^8}\left(-\left(\frac{1}{6480}\right)f_{k+3}+\left(\frac{1}{362880}\right)f_{k+8}-\left(\frac{1}{45360}\right)f_{k+1}+\left(\frac{1}{12960}\right)f_{k+2}-\left(\frac{1}{45360}\right)f_{k+7}+\right.$$
$$\left.\left(\frac{1}{5184}\right)f_{k+4}-\left(\frac{1}{6480}\right)f_{k+5}+\left(\frac{1}{12960}\right)f_{k+6}+\left(\frac{1}{362880}\right)f_k\right) \qquad (2.62)$$

Evaluating (2.62) at $x = x_{k+8}$, we obtain the discrete form as

$$y_{k+8} = y_{k+6} + h\left(-\left(\frac{119}{16200}\right)f_k+\left(\frac{953}{14175}\right)f_{k+1}-\left(\frac{15577}{56700}\right)f_{k+2}+\left(\frac{9341}{14175}\right)f_{k+3}-\left(\frac{2903}{2835}\right)f_{k+4}+\right.$$
$$\left.\left(\frac{15011}{14175}\right)f_{k+5}-\left(\frac{21247}{56700}\right)f_{k+6}+\left(\frac{22823}{14175}\right)f_{k+7}+\left(\frac{32377}{113400}\right)f_{k+8}\right) \qquad (2.63)$$

Thus, (2.63) is the eight-step optimal order method.

3. Numerical Solutions: An application of the Linear Multistep Scheme

In this study, we have developed continuous multistep collocation methods for solving first-order initial value problems (IVPs) of ordinary differential equations (ODEs). We have used probabilists' Hermite polynomials as the basis functions for our methods. Additionally, we have obtained corresponding discrete schemes.

In this section, we will apply the derived eight-step methods of Adams-Bashforth, Adams-Moulton, and the proposed optimal order methods to solve non-stiff IVPs of ODEs. We do this by interpolating the continuous solution at both grid and off-grid points, and then collocating them using the chosen basis functions. We will also obtain the errors associated with these methods. To obtain starting values, we use the fourth-order Runge-Kutta method, which is known to be the most efficient method for generating starting values for linear multistep methods. We then use the fourth-order Adams-Bashforth method as a predictor for the implicit schemes.

The results and errors obtained from the numerical experiments will be tabulated for clarity. These experiments involve evaluating the starting values y_n, where $n = 0, 1, \dots, 5$, using the fourth-order Runge-Kutta method. We will then apply the continuous multistep collocation methods to solve the IVPs. This approach offers a practical and efficient way to approximate the solutions of ODEs, and we believe that the use of probabilists' Hermite polynomials as basis functions will enhance the accuracy and efficiency of the numerical methods developed in this study.

Test Problem 1: Consider the IVP,

$$\frac{dy}{dx} = -y;\ y(0) = 1$$

Exact Solution:

$$y(x) = e^{-x}$$

Table 1: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 1, with stepsize h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	0.904837418035960	0.904837500000000	0.904837500000000	0.904837500000000
0.2	0.818730753077982	0.818730901406250	0.818730901406250	0.818730901406250
0.3	0.740818220681718	0.740818422001178	0.740818422001178	0.740818422001178
0.4	0.670320046035639	0.670320288917491	0.670320288917491	0.670320288917491
0.5	0.606530659712633	0.606530934423380	0.606530934423380	0.606530934423380
0.6	0.548811636094026	0.548811934376315	0.548811934376315	0.548811934376315
0.7	0.496585303791409	0.496585618671229	0.496585618671229	0.496585618671229
0.8	0.449328964117222	0.449329247126416	0.449329248184475	0.449329200835643
0.9	0.406569659740599	0.406569925334822	0.406569917974354	0.406569925854396
1	0.367879441171442	0.367879656723068	0.367879673352084	0.367879622283663

Table 2: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 1, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	8.20E-08	8.20E-08	8.20E-08
0.2	1.48E-07	1.48E-07	1.48E-07
0.3	2.01E-07	2.01E-07	2.01E-07
0.4	2.43E-07	2.43E-07	2.43E-07
0.5	2.75E-07	2.75E-07	2.75E-07
0.6	2.98E-07	2.98E-07	2.98E-07
0.7	3.15E-07	3.15E-07	3.15E-07
0.8	2.83E-07	2.84E-07	2.37E-07

0.9	2.66E-07	2.58E-07	2.66E-07
1	2.16E-07	2.32E-07	1.81E-07

Test Problem 2: Consider the IVP,

$$\frac{dy}{dx} = 1 - x + 4y; \quad y(0) = 1$$

Exact Solution:

$$y(x) = \frac{1}{16}(4x + 19e^{4x} - 3)$$

Table 3: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 2, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.609041828449010	1.608933333333330	1.608933333333330	1.608933333333330
0.2	2.505329852584810	2.505006151111110	2.505006151111110	2.505006151111110
0.3	3.830138845749650	3.829414509150810	3.829414509150810	3.829414509150810
0.4	5.794226003969200	5.792785270450580	5.792785270450580	5.792785270450580
0.5	8.712004117480150	8.709317547440140	8.709317547440140	8.709317547440140
0.6	13.052521952011900	13.047712629434700	13.047712629434700	13.047712629434700
0.7	19.515518040677800	19.507147853082100	19.507147853082100	19.507147853082100
0.8	29.144879609067300	29.131606335987000	29.132202731569100	29.133183299790600
0.9	43.497903401867600	43.477870180375300	43.478995629506700	43.479729037231900
1	64.897803164358800	64.866065719536600	64.869044474654400	64.870397343912000

Table 4: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 2, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.08E-04	1.08E-04	1.08E-04
0.2	3.24E-04	3.24E-04	3.24E-04
0.3	7.24E-04	7.24E-04	7.24E-04
0.4	1.44E-03	1.44E-03	1.44E-03
0.5	2.69E-03	2.69E-03	2.69E-03
0.6	4.81E-03	4.81E-03	4.81E-03
0.7	8.37E-03	8.37E-03	8.37E-03
0.8	1.33E-02	1.27E-02	1.17E-02
0.9	2.00E-02	1.89E-02	1.82E-02
1	3.17E-02	2.88E-02	2.74E-02

Test Problem 3: Consider the IVP,

$$\frac{dy}{dx} = 5y + \frac{e^{-2x}}{y^2}, \quad y(0) = 2$$

Exact Solution:

$$y(x) = \sqrt[3]{\frac{(139e^{15x} - 3e^{-2x})}{17}}$$

Table 5: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 3, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	3.317141203207850	3.316705869531850	3.316705869531850	3.316705869531850
0.2	5.474932869673380	5.473315121398090	5.473315121398090	5.473315121398090
0.3	9.028410539856130	9.024203136693940	9.024203136693940	9.024203136693940
0.4	14.885866168811800	14.876372621469500	14.876372621469500	14.876372621469500
0.5	24.542804917158000	24.522932513230200	24.522932513230200	24.522932513230200
0.6	40.464292920687700	40.424570380692300	40.424570380692300	40.424570380692300
0.7	66.714355023308700	66.637392445034900	66.637392445034900	66.637392445034900
0.8	109.993380579720000	109.852968811974000	109.863396523534000	109.872374058953000
0.9	181.348427520768000	181.101308915549000	181.130798254438000	181.139931713824000
1	298.993010259922000	298.552128950694000	298.620412667443000	298.638430344442000

Table 6: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 3, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	4.35E-04	4.35E-04	4.35E-04
0.2	1.62E-03	1.62E-03	1.62E-03
0.3	4.21E-03	4.21E-03	4.21E-03
0.4	9.49E-03	9.49E-03	9.49E-03
0.5	1.99E-02	1.99E-02	1.99E-02
0.6	3.97E-02	3.97E-02	3.97E-02
0.7	7.70E-02	7.70E-02	7.70E-02
0.8	1.40E-01	1.30E-01	1.21E-01
0.9	2.47E-01	2.18E-01	2.08E-01
1	4.41E-01	3.73E-01	3.55E-01

Test Problem 4: Consider the IVP,

$$\frac{dy}{dx} - y = -\frac{1}{2}e^{\frac{x}{2}}\sin(5x) + 5e^{\frac{x}{2}}\cos(5x), \quad y(0) = 0$$

Exact Solution:

$$y(x) = e^{\frac{x}{2}} \sin(5x)$$

Table 7: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 4, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	0.504006211599106	0.504014759887403	0.504014759887403	0.504014759887403
0.2	0.929969260814162	0.929983373647291	0.929983373647291	0.929983373647291
0.3	1.158923832386380	1.158938590752240	1.158938590752240	1.158938590752240
0.4	1.110618385112830	1.110627991859330	1.110627991859330	1.110627991859330
0.5	0.768453444209089	0.768452618751726	0.768452618751726	0.768452618751726
0.6	0.190492085804808	0.190477469066620	0.190477469066620	0.190477469066620
0.7	-0.497785095005140	-0.497813889877322	-0.497813889877322	-0.497813889877322
0.8	-1.129016653736910	-1.129352840215060	-1.129066324157410	-1.129052476475610
0.9	-1.533072395217810	-1.533494108291160	-1.533148367354660	-1.533126945181480
1	-1.580998848627810	-1.580895807556460	-1.581085845119980	-1.581047653893670

Table 8: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 4, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	8.55E-06	8.55E-06	8.55E-06
0.2	1.41E-05	1.41E-05	1.41E-05
0.3	1.48E-05	1.48E-05	1.48E-05
0.4	9.61E-06	9.61E-06	9.61E-06
0.5	8.25E-07	8.25E-07	8.25E-07
0.6	1.46E-05	1.46E-05	1.46E-05
0.7	2.88E-05	2.88E-05	2.88E-05
0.8	3.36E-04	4.97E-05	3.58E-05
0.9	4.22E-04	7.60E-05	5.45E-05
1	1.03E-04	8.70E-05	4.88E-05

Test Problem 5: Consider the IVP,

$$\frac{dy}{dx} - \frac{y\ln y}{x+1} = (x+1)y, \quad y(0) = 1$$

Exact Solution:

$$y(x) = e^{x(x+1)}$$

Table 9: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 5, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.116278070458870	1.116276566958480	1.116276566958480	1.116276566958480
0.2	1.271249150321400	1.271244993844780	1.271244993844780	1.271244993844780
0.3	1.476980793882640	1.476971897098690	1.476971897098690	1.476971897098690
0.4	1.750672500296100	1.750655088565740	1.750655088565740	1.750655088565740
0.5	2.117000016612680	2.116967258713190	2.116967258713190	2.116967258713190
0.6	2.611696473423120	2.611635968023550	2.611635968023550	2.611635968023550
0.7	3.287081207383120	3.286970339191010	3.286970339191010	3.286970339191010
0.8	4.220695816996550	4.220471385469230	4.220537241735280	4.220563216666740
0.9	5.528961477624000	5.528507197783100	5.528723529870500	5.528742731240950
1	7.389056098930650	7.388193146884370	7.388665357756570	7.388716640390750

Table 10: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 5, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.50E-06	1.50E-06	1.50E-06
0.2	4.16E-06	4.16E-06	4.16E-06
0.3	8.90E-06	8.90E-06	8.90E-06
0.4	1.74E-05	1.74E-05	1.74E-05
0.5	3.28E-05	3.28E-05	3.28E-05
0.6	6.05E-05	6.05E-05	6.05E-05
0.7	1.11E-04	1.11E-04	1.11E-04
0.8	2.24E-04	1.59E-04	1.33E-04
0.9	4.54E-04	2.38E-04	2.19E-04
1	8.63E-04	3.91E-04	3.39E-04

Conclusion

This study aimed to derive both continuous and discrete linear multistep methods (LMMs) using collocation and interpolation techniques with the probabilists' Hermite polynomials serving as basis functions. The results suggest that LMMs can be derived using any polynomial function and approach. Among the Adams-Moulton and Adams-Bashforth methods examined, the proposed optimal order method demonstrated the highest accuracy.

LMMs play a significant role in numerically solving various types of ordinary differential equations. Although the approach and basis functions in this study differ from those in previous research, the derived LMMs were found to be identical. The proposed optimal order method, which was built from the Adams-Moulton method, demonstrated efficient convergence when evaluating initial value problems (IVPs) and was shown to be superior in terms of accuracy compared to the standard Adams-Bashforth and Adams-Moulton methods with the same step number.

The accuracy of the proposed method was demonstrated by solving numerous differential equations using the numerical method. The results obtained were comparable to those of other similar methods, indicating that the proposed method is accurate. These findings are significant for researchers in the field of numerical analysis and mathematical modeling.

Future research could explore the application of the proposed optimal order method to other types of differential equations, including partial differential equations. Additionally, investigating the effect of different polynomial functions and approaches on the accuracy and efficiency of LMMs could provide useful insights. Further research could also consider the implementation of the proposed optimal order method in parallel computing to improve computational efficiency. These future directions could enhance the practical utility and applicability of the proposed method in various scientific and engineering fields.

References

Aboiyar. T, Luga. T, and Iyorter, B.V. (2015). Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions. American Journal of Applied Mathematics and Statistics, vol. 3(6) pp220-225. doi: 10.12691/ajams-3-6-2.

Adeniyi, R. B. and Alabi, M. O. (2006). Derivation of Continuous Multistep Methods Using Chebyshev polynomial Basis Function. *Abacus*, Vol 33 pp 351-361.

Adesanya, A. O., Momoh, A. A., Alkali, M. A. and Tahir, A. (2012). Five Steps Block Method for the Solution of Forth Order ODEs. *International Journal of Engineering Research and Applications*, Vol. 2(5) pp 991-998.

Alabi, M. O. (2008). A Continuous Formulation of Initial Value Solvers with Chebyshev Basis Function in a Multistep Collocation Technique. PhD Thesis. Department of Mathematics, University of Ilorin. 136pp.

Anake, T. A. (2011). Continuous Implicit Hybrid One-Step Methods for the Solution of Initial Value Problems of General Second Order ODEs. PhD Thesis. School of Postgraduate Studies, Covenant University, Ota, Nigeria. 170pp.

Awoyemi, D. O., Kayode, J. S. and Adoghe, L. O. (2014). A Four-Point Fully Implicit Method for the Numerical Integration of Third-Order Ordinary Differential Equations. *International Journal of Physical Sciences*, Vol 9(1) pp 7-12.

Koornwinder, T. H., Wong, R. S. C., Koekoek, R., and Swarttouw, R. F. (2010). Orthogonal Polynomials. In: Olver, F. W. J., Lozier, D. M., Boisvert, R. F. and Clark, C. W. (2010). NIST Handbook

Mohammed, U. (2011). A Linear Multistep Method with Continuous Coefficients for Solving First-Order ODEs. *Journal of the Nigerian Association of Mathematical Physics*, Vol. 19 pp159-166.

Odekunle, M. R., Adesanya, A. O. and Sunday, J. (2012). A New Block Integrator for the Solution of Initial Value Problems of First Order ODEs. *International Journal of Pure and Applied Sciences and Technology*, Vol. 11(1) pp92-100.

Olaiya, O.O., Oduwole, H. K. and J. K. Odeyemi, J.K. (2019). Numerical solution of Black–Scholes Partial Differential Equation using Direct solution of second-order Ordinary Differential Equation with two-step hybrid Block Method of Order seven. Science World Journal, Vol. 14(2) pp 23-29.